Dynamics on Ahlfors quasi-circles

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Abstract. The celebrated theory of Denjoy introduced a topological invariant distinguishing $C^1$ and $C^2$ diffeomorphisms of the circle. A $C^3$ diffeomorphism of the circle cannot have an infinite minimal set other than the circle itself. However, this is possible for $C^1$ diffeomorphisms. In dimension two there is a related invariant distinguishing $C^2$ and $C^3$ diffeomorphisms.

Theorem. Let $Q$ be a quasi-circle contained in a surface. If $F$ is an infinite minimal isometry set in $Q$ for a $C^3$ diffeomorphism, then $F$ equals $Q$. There exists a $C^2$ diffeomorphism of the annulus with a minimal Cantor set contained in a quasi-circle.

Keywords. Quasi-circle; minimal set; rotation number; Cantor sets; Denjoy counterexample.

1. Introduction

Poincaré and Birkhoff proved that a measure preserving homeomorphism of the two-dimensional annulus which twists the two boundary components in opposite directions must have fixed points in the interior of $A$. What is known as KAM theory emerged from this and is currently being developed and refined. (See [7], [8] and [9], for example.) The theory produces global topological and dynamical conclusions from local assumptions—$C^3$ differentiability and infinitesimal 'twisting'. The $C^3$ hypothesis is sharp; there exist counterexamples in the $C^2 + \delta$ category. (See [6].) In this paper we consider a problem with some rudimentary resemblance to twist theory.

Let $f$ be a $C^r$ diffeomorphism of the two-dimensional annulus $A = S^1 \times [-1, 1]$ to itself which is repelling at one boundary component $A^+ = S^1 \times \{1\}$ and attracting at the other $A^- = S^1 \times \{-1\}$. Suppose $f$ has no periodic points in int$(A_0)$. If one orbit 'gets across', must they all? That is, if the $\omega$-limit set of $x_0$ is in $A^+$ and its $\alpha$-limit set is in $A^-$ for some $X_0 \in A$, then is this true for every $X \in A$? There is growing evidence that the answer is in the affirmative for $r = 3$. We pose an equivalent version of this question.

2. The north pole, south pole conjecture

Suppose $f : S^2 \to S^2$ is a $C^3$ diffeomorphism with the north pole $N$ a repeller, the south pole $S$ an attractor and no other periodic points. If one orbit is asymptotic to both

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S and N then $f$ is dynamically equivalent to the standard north pole, south pole diffeomorphism.

Formally, the conclusion means that $f$ is the time one map of the gradient flow on $S^2$. A $C^r$ diffeomorphism $f: M \to M$ is of class $C^{r,\delta}$ if the $r$th derivative satisfies a $\delta$-Hölder condition. That is, there exists $C > 0$ such that $\|D^r f_x - D^r f_y\| < C \|x - y\|^{\delta}$ for $x, y \in M$.

There exist counter-examples to the NP-SP conjecture if $f$ is $C^{2+\delta}$. (See [5], [4] and [3].) We show in this paper that $C^3$ is a natural bound to these examples. Hence the NP-SP conjecture concerns the topological-dynamical invariants that might distinguish $C^{2+\delta}$ and $C^3$.

We find the coordinates of the annulus more convenient to work with and put this spherical formulation aside.

A non-empty, closed, invariant set $\Gamma$ of a homeomorphism $f$ is said to be **minimal** if it is closed and contains no smaller non-empty, closed, invariant sets.

If there is a counter-example to the NP-SP conjecture then there exists a $C^r$ diffeomorphism $f$ of the annulus without periodic points which has one orbit asymptotic to both boundary components and one orbit whose closure $\Gamma$ stays bounded away from $\partial A$. Furthermore, $\Gamma$ may be taken to be a minimal set. In [4] it is shown that the existence of such a diffeomorphism implies the existence of a $C^r$ Seifert counter-example. That is, there exists a $C^r$ vector field on the three-sphere $S^3$ with neither zeroes nor closed integral curves. Hence the NP-SP conjecture is 'contained' in the Seifert conjecture.

A recent theorem of John Franks is useful in analyzing the dynamics of $f$.

**Theorem** (Franks). Let $f: A \to A$ be a homeomorphism of the open annulus $A$ and $x \in A$. Let $g$ be a lift off to the universal cover $\mathbb{R}x[-1, 1]$ of $A$ and $y$ a lift of $x$. Let $y_n$ denote the first component of $g^n(y)$. If there exists a rational number $p/q$ with

\[
\lim y_n \leq \frac{p}{q} \leq \lim y_n
\]

then there exists a point $z \in A$ with $f^q(z) = z$.

By the theorem of Franks, $\Gamma$ must be an infinite, perfect minimal set which has irrational rotation number—the cyclic order is preserved by $f$. Certainly $\Gamma$ could not be a circle, otherwise no orbit would get across it; however, it is not known if $\Gamma$ must be a Cantor set.

### 3. The Denjoy Cantor sets

The reader might be reminded of the Denjoy's theory where the critical degree of differentiability is 2 and the dimension is 1. Denjoy [1] found that the degree of differentiability of a circle diffeomorphism $f$ influences its topological type. If $f$ is $C^2$ and has no periodic points then $f$ has simple dynamics—it is topologically conjugate to a rotation through an irrational angle. This is not the case in the $C^{1+\delta}$ category.

Suppose $\Gamma \subset S^1$ is a Cantor set. If there exists a homeomorphism $f: S^1 \to S^1$ for which $\Gamma$ is minimal, then the pair $(f, \Gamma)$ is a Denjoy Cantor set. Denjoy Cantor sets provide the key ingredient to classifying homeomorphisms of the circle. Poincaré defined the **rotation numbers** and showed that all homeomorphisms of the circle have
them. Furthermore, any homeomorphism of the circle with irrational rotation number \( \alpha \) is either topologically conjugate to a rigid rotation through \( \alpha \) or has a minimal Cantor set. Denjoy proved that these examples can all exist as \( C^1 \) diffeomorphisms but not \( C^2 \) (actually \( C^{1+\epsilon} \) is impossible). We call these examples Denjoy Cantor sets. More generally, a homeomorphism \( g: \Gamma \to \Gamma, \Gamma' \) contained in an \( n \)-manifold \( M \) is also called a Denjoy Cantor set if the pair \( (g, \Gamma) \) is topologically conjugate to a Denjoy Cantor set \( (f, \Gamma) \) in \( S^1 \). That is, there exists an embedding \( h: \Gamma \to M \) such that \( h(\Gamma) = \Gamma' \) and \( h \cdot f = g \cdot h \).

It is not completely understood under what circumstances Denjoy Cantor sets can exist. Hall [21] showed that it is possible to have a Denjoy Cantor set in a \( C^\infty \) annular diffeomorphism. However, it is attracting, and so no orbit is asymptotic to both boundary components. Do there exist \( C^3 \) diffeomorphisms \( f \) of \( A \) with no periodic points, a Denjoy Cantor set \( (f, \Gamma) \) and one orbit asymptotic to both boundary components? We consider some possibilities.

There are two features of \( \Gamma \) for us to study—its structure as a subset of \( A \) and the properties of the first derivative of \( f \) at \( \Gamma \), the distortion of \( f \) at \( \Gamma \). Using the methods of Denjoy, one can rule out any Denjoy Cantor set \( \Gamma \) contained in a smooth Jordan curve as long as \( f \) is \( C^2 \). It is not possible for \( \Gamma \) to have totally arbitrary topological structure since minimality implies homogeneity. A natural question arises—how wild can \( \Gamma \) be?

In Denjoy's theory, the simplest \( C^1 \) examples have first derivative, the identity at the minimal Cantor set. It is quite easy to show there are no \( C^2 \) diffeomorphisms of the circle with this condition at the Cantor set: Let \( L_n \) denote the intervals complementary to \( \Gamma \), indexed so that \( f(L_n) = L_{n+1} \). Let \( a_n = |L_n| \). Since \( f \) is \( C^1 \), we can apply the mean value theorem and continuity of the first derivative to conclude that \( a_n/a_{n+1} \to 1 \) as \( n \to \infty \). If \( f \) were \( C^2 \), we could apply the mean value theorem to \( f' \) and use continuity of the second derivative to conclude that

\[
1 - \frac{a_n}{a_{n+1}} \to 0 \quad \text{as} \quad n \to \infty.
\]

It follows that \( \sum a_n = \infty \), contradicting the finite arc length of \( S^1 \).

The same proof extends to \( \mathbb{R}^2 \) and shows there are no \( C^3 \) annular diffeomorphisms with the first derivative the identity and the second derivative the 0 bilinear transformation at a square rectifiable Denjoy Cantor set.

4. Isometry sets

In this paper we consider a large class of 'simplest' examples. We assume that the first derivative of \( f \) at each point of \( \Gamma \) is an isometry for some Riemannian metric on \( A \). The isometry may vary from point to point. We call \( \Gamma \) an isometry set. Note that this isometry condition on \( \Gamma \), even for the usual metric on \( A \), is weaker than the identity hypothesis. The annulus may be replaced by any Riemannian manifold \( M \).

We need a little more background before we can state the main results. Smooth curves \( Q \) are \( n \)-rectifiable for all \( n \geq 1 \). That is, there exists a constant \( L > 0 \) such that \( \sum |x_i - x_{i+1}|^n < L \) for all partitions \( x_1 < x_2 < \cdots < x_i < x_{i+1} < \cdots \) of \( Q \). A curve is square rectifiable if it is 2-rectifiable. One can similarly define the notion of
n-rectifiable for subsets of curves or for any set on which there is a well-defined order. In particular, we may consider whether Denjoy Cantor sets are n-rectifiable. Topological curves and Cantor sets may be so wild that they fail to be n-rectifiable, for any n. However, additional restrictions guarantee that curves be square-rectifiable.

A curve \( Q \) is called a quasi.curve if there exists \( K > 0 \) such that if \( x, y \in Q \), the arc connecting \( x \) and \( y \) is contained in a disc of radius \( Kd(x, y) \). A quasi-circle is a Jordan curve which is the union of quasi-arcs. We prove

**PROPOSITION**

Quasi-arcs are square rectifiable.

**Theorem.** Let \( f \) be a \( C^3 \) diffeomorphism of a compact Riemannian n-manifold \( M \) and \( \Gamma \subset M \) a minimal isometry set. If \( \Gamma \) is a Denjoy Cantor set then \( \Gamma \) is not square rectifiable.

The Proposition and Theorem imply the following

**COROLLARY**

Let \( f \) be a \( C^3 \) diffeomorphism of the annulus \( A \) and \( Q \subset A \) a quasi-circle. If \( \Gamma \subset Q \) is an infinite, minimal isometry set then \( \Gamma = Q \).

**Proof.** Since \( \Gamma \) is an infinite minimal set, the rotation number of \( f |_\Gamma \) is irrational. Then \( \Gamma \) can only be a Cantor set or all of \( Q \).

These results depend on a general estimate for the asymptotic behaviour of pairs of orbits of isometry minimal sets. This is an example of an estimate of 'non-linear' distortion.

**Theorem.** Let \( E \) be an isometry minimal set of a \( C^3 \) mapping \( f \) of a compact Riemannian 2-manifold \( M \). If \( y \in M \) and \( z \in E \) then

\[
\sum_{n=1}^{\infty} d(y_n, x_n)^2 = \infty.
\]

The proof of Denjoy's result depends on the divergence of the Poincaré series for \( C^2 \) maps \( f \)

\[
\sum_{n=1}^{\infty} |Df^n_x|^1.
\]

(See Sullivan [10], for example). In practice \( Df^n_x \) is sometimes replaced by \( d(x_n, y_n) \) for \( s \) in the minimal set and \( y \) arbitrary in the manifold where \( d \) is the Riemannian metric on \( M \). The exponent is related to the degree of differentiability of \( f \). If \( f \) is \( C^r \), it is natural to estimate the general dynamic sum

\[
\sum_{n=1}^{\infty} d(y_n, X_n)^{r-1}
\]

even on higher dimensional manifolds. In this paper we restrict ourselves to Riemannian two-manifolds and \( r = 3 \), although generalizations to higher dimensions and degrees of differentiability are possible.
5. Isometry and the second derivative

A linear transformation from one normed space to another, say \( T:E_1 \to E_2 \), is an isometry if \( T \) is a bijection and

\[
\| Tx \|_{E_2} = \| x \|_{E_1} \quad \text{for all } x \in E_1.
\]

The quantity

\[
\rho(T) = \max \left[ \sup_{\| x \|_{E_1} = 1} \| Tx \|_{E_2}, \sup_{\| x \|_{E_1} = 1} 1/\| Tx \|_{E_2} \right]
\]

measures how non-isometric is \( T \). Then \( \rho(T) \geq 1 \); \( T \) is an isometry if and only if \( \rho(T) = 1 \) and \( T \) is a bijection.

Lemma 5.1. Let \( f \) be a \( C^2 \) diffeomorphism of a Riemannian \( m \)-manifold \( M \) and \( p \in M \), fixed. Suppose there exists \( p_n \to p \) such that \( T_{p_n}f \) is an isometry respecting the given Riemann structure \( g \) of \( M \). Let \( \bar{f} \) be the lift of \( f \) from \( M \) to \( T_p M \approx \mathbb{R}^m \) at \( p \).

Let \( \bar{p}_n = \exp_p^{-1}(p_n) \), then the sequence of linear maps

\[
(D\bar{f})\bar{p}_n : \quad T_p M \to T_{\bar{f}p} M
\]

is non-isometric only to the extent:

\[
|1 - \rho(D\bar{f}\bar{p}_n)| = O(\| \bar{p}_n \|^2)
\]

Remarks. The norms on \( T_p M \) and \( T_{\bar{f}p} M \) are \( g(p) \) and \( g(\bar{f}p) \) respectively. The length \( \| p_n \| \) is also calculated with respect to \( g(p) \), although we could replace \( \| p_n \| \) with \( d(p_n, p) \) since

\[
\frac{\| p_n \|}{d(p_n, p)} \to 1 \quad \text{as } p_n \to p.
\]

Proof. To calculate \( \rho((D\bar{f})\bar{p}_n) \) one considers the pulled-up Riemann structure on \( T_p M \) and \( T_{\bar{f}p} M \), namely

\[
\bar{g}_p : \bar{g}_p(w_p; u, v) = \langle T_{w_p, \exp_p} u, T_{w_p, \exp_p} v \rangle_{\exp_p(w_p)}
\]

for all \( w_p \in T_p M \) near \( O_p \) and

\[
\text{for all } u, v \in T_{w_p}(T_p M) \approx T_p M
\]

\[
\bar{g}_{\bar{f}p} : \bar{g}_{\bar{f}p}(w_{\bar{f}p}; u, v) = \langle T_{w_{\bar{f}p}, \exp_{\bar{f}p}} u, T_{w_{\bar{f}p}, \exp_{\bar{f}p}} v \rangle_{\exp_{\bar{f}p}(w_{\bar{f}p})}.
\]

The map \( D\bar{f} \) at the point \( \bar{p}_n = \exp_p^{-1}(p_n) \) is an isometry from the tangent space to
$T_pM$ at $p_n$, equipped with the metric $\bar{g}_p(p_n; \ast)$ to the tangent space to $T_{f_p}M$ at $\bar{f(p_n)}$ equipped with the metric $\bar{g}_{f_p}(\bar{p}_n; \ast)$. Let $e^1, \ldots, e^n$ be an orthonormal basis for $T_pM$.

The map $T_pf: T_pM \rightarrow T_{f_p}M$ is an isometry (being the limit of isometries) so $T_pf(e^1), \ldots, T_pf(e^n)$ is an orthonormal basis at $T_{f_p}M$. Both these bases give rise to $\bar{g}_i$-expressions for the metrics $\bar{g}_p$ and $\bar{g}_{f_p}$ on $T_pM$ and $T_{f_p}M$. Besides,

$$\bar{g}_{ij}p(w_p) = \bar{g}(w_p; e^i, e^j) = \delta_{ij} + O(\|w_p\|^2)$$

$$\bar{g}_{ij}f_p(w_{f_p}) = \bar{g}(w_{f_p}; T_pf(e^i), T_pf(e^j)) = \delta_{ij} + O(\|w_{f_p}\|^2).$$

Thus

$$\langle (D\bar{f})\bar{p}_n(u), (D\bar{f})\bar{p}_n(u) \rangle_{g(f_p)} = \sum (i\text{-th component of } v)^2 \text{ where } v = (D\bar{f})\bar{p}_n(u)$$

$$= \sum \delta_{ij}(i\text{-th component of } v)(j\text{-th component of } v)$$

$$= \sum \bar{g}_{ij}f_p(p_n)(i\text{-th component of } v)(j\text{-th component of } v)$$

$$+ \sum (\delta_{ij} - \bar{g}_{ij}(f_p))(\bar{f}_p)(i\text{-th component of } v)(j\text{-th component of } v)$$

$$= \langle (D\bar{f})\bar{p}_n(u), (D\bar{f})\bar{p}_n(u) \rangle_{\bar{g}_p(\bar{p}_n)}$$

$$+ \sum (\delta_{ij} - \bar{g}_{ij}(f_p))(\bar{f}_p)(i\text{-th component of } v)(j\text{-th component of } v)$$

$$= \langle u, u \rangle_{\bar{g}_p(p_n)} + O(\|\bar{f}_p\|^2)\cdot\|D\bar{f}_p\|\|u\|^2$$

$$= \|u\|^2 + O(\|\bar{p}_n\|^2)\text{-constant }\|u\|^2.$$ 

Since $f$ is a diffeomorphism and $T_pf$ is an isometry we have

$$\|\bar{p}_n\|/\|D\bar{p}_n\| \rightarrow 1 \text{ as } n \rightarrow \infty.$$ 

Hence $\|D\bar{f}_p\| = (1 + O(\|\bar{p}_n\|^2)^{1/2} = 1 + O(\|\bar{p}_n\|^2).$ q.e.d.

Let $x_i$ be a sequence of points in $\mathbb{R}^n$. Suppose there exists a finite set of unit vectors $v^1, v^2, \ldots, v^m$ in $\mathbb{R}^n$ which is the limit set of $\{x_i/\|x_i\|\}$. If $\|x_i\| \rightarrow 0$, we say that the sequence $x_i$ converges to 0 from $m$ directions $v^1, v^2, \ldots, v^m$.

Lemma 5.2. Let $g$ be a $C^2$ diffeomorphism of $\mathbb{R}^n$. Suppose there exist points $x \in \mathbb{R}^n$ converging to 0 from $m$ directions $v^1, \ldots, v^m$ such that $|1 - \rho(Dg_x)| = O(\|x\|)$, then $D^2g_0(v, w) - Dg_0(u) = 0$ for $u, v, w \in \text{sp}(v^1, \ldots, v^m)$. If $\text{sp}(v^1, \ldots, v^m) \cong \mathbb{R}^n$ then $D^2g_0(v, w) = 0$.

The author is grateful to M Shub and C Robinson for the following proof.

Proof. Observe that $Dg_0$ is an isometry since $\rho(Dg_0) = 1$.

The set of linear transformations $\{Dg_0^{-1}Dg_x\}$ is tangent to the orthogonal matrices at the identity (where $x = 0$). The antisymmetric metrics form the tangent space to the orthogonal matrices based at the identity. Hence, if $A = Dg_0^{-1}D^2g_0$, then $A(u) = A_v$.
is antisymmetric for \( v \in \text{sp}\{v', \ldots, v^n\} \). Thus \( A_v(w_1) \cdot w_2 = -w_1 \cdot A_v(w_2) \) for all vectors \( w_1 \) and \( w_2 \).

Let \( v, w \in \text{sp}\{v', \ldots, v^n\} \). Then \( A_v(v) \cdot w = -v \cdot A_v(w) = -v \cdot A_w(v) = A_w(v) \cdot v = -w \cdot A_w(v) = 0 \). Write \( A_v(v) = x + y \in \text{sp}\{v', \ldots, v^n\} \times R^l \) where \( R^l \) is the orthogonal complement to \( \text{sp}\{v', \ldots, v^n\} \). Then \( 0 = A_v(v) \cdot x = x \cdot x + y \cdot y \). Since \( y \cdot x = 0 \) we have \( x \cdot x = 0 \). Thus \( x = 0 \) and \( A_v(v) \in R^l \). Since \( A_{v+w}(v + w) = A_v(v) + 2A_w(v) + A_w(w) \in R^l \), then \( A_{v+w}(w) \in R^l \) for all \( u, v, w \in \text{sp}\{v', \ldots, v^n\} \). Note that if \( \text{sp}\{v', \ldots, v^n\} \approx R^n \) then \( A_{v+w}(w) = 0 \).

Hence \( 0 = u \cdot Dg_0^{-1}Dg_0(v, w) = (Dg_0(u) \cdot Dg_0(v, w) \cdot v) = A_v(v) \cdot v = -w \cdot A_v(v) = 0 \).

Write \( A_{v+w}(v) = x + y \in R^n \) where \( R^l \) is the orthogonal complement to \( \text{sp}\{v', \ldots, v^n\} \). Then \( 0 = A_{v+w}(v) \cdot x = x \cdot x + y \cdot y \). Since \( y \cdot x = 0 \) we have \( x \cdot x = 0 \). Thus \( x = 0 \) and \( A_{v+w}(v) \in R^l \). Hence \( u \cdot A_{v+w}(w) = 0 \) for all \( u, v, w \in \text{sp}\{v', \ldots, v^n\} \). Note that if \( \text{sp}\{v', \ldots, v^n\} \approx R^n \) then \( 0 = Dg_0^{-1}Dg_0(v, w) \).

The proof is identical to that of Lemma 5.2.

6. Minimal isometry sets

**DEFINITION**

A set \( E \subset M \) is **minimal** under \( f \) if it is invariant and contains no invariant subsets.

A set \( E \) is an isometry set if \( Df \) is an isometry at each \( x \in E \).

**Theorem 6.1.** Let \( f : M \rightarrow M \) be a \( C^3 \) mapping of a compact Riemannian 2-manifold \( M \). Let \( E \subset M \) be an isometry minimal set. Let \( y, z \in E \) where \( z \) is in a local geodesic coordinate chart about \( y \). Then \( \| Df(y) (z - y) + \frac{1}{2} D^2 f(y) (z - y)^2 \| \leq \| z - y \| + O(\| z - y \|^3 \)."

Here, \( "(z - y)" \) refers to the vector \( u \in R^2 \) such that \( \exp_y(u) = z \); \( "D_f(y)" \) and \( "D^2 f(y)" \) are the first and second derivatives at 0 of the lift \( \tilde{f} : T_y M \rightarrow T_{f(y)} M \).

Notice that Taylor’s theorem alone merely implies that \( \| Df(z - y) + \frac{1}{2} D^2 f(y) (z - y)^2 \| \leq \| z - y \| + O(\| z - y \|^3) \) if \( Df \) is an isometry. The proof of Theorem 2.1 uses the dynamics of \( f \) as well as the isometry condition on \( Df \) to sharpen the estimate.

**Proof.** We may assume that \( E \) is an infinite set, otherwise the result is trivial. It follows that \( E \) is perfect since it is an infinite minimal set.

First, suppose there exists a point \( x \in E \) with sequences \( p \) and \( q \) in \( E \) approaching \( x \) from two directions.

Since \( f \) is \( C^1 \) (in fact \( C^3 \)) and \( E \) is invariant, each point \( f^n(x) = x_n \) has limit points in \( E \) from two directions. Since \( Df \) is an isometry at these points it follows from Lemma 5.1 and Lemma 5.2 that \( D^2 f = 0 \) at \( x_n \). Since \( E \) is minimal, \( \{x_n\} \) is dense in \( E \). Since \( y \in E \), \( D^2 f(y) = 0 \). The estimate follows.

Now suppose there is no point in \( E \) with limit points from two directions. Then for each \( x \in E \) there is a unique unit vector \( v_x \) along which points in \( E \) are converging towards \( x \). Either \( v_x \) varies continuously or it does not. Suppose \( v_x \) is not continuous at \( p \). Then there exist at least two distinct limits \( v' \) and \( v'' \) of \( v_x \) as \( x \rightarrow p \). By
Lemma 5.3 we have $D^2f_{p} = 0$. By the smoothness of $f$ this discontinuity is preserved, so $D^2f_x = 0$ for each orbit point $x = x_n$ and hence for each $x \in E$.

Suppose that $v_x$ is a continuous function of $x$. We know by Lemma 5.2 only that $D^2f_x(v_x, v_x) \cdot Df_x(v_x) = 0$. We need to estimate the dot product $|D^2f_y(w, w) \cdot Df_y(w)|$ where $w = z - y$.

(6.2) Let $g : R^2 \to R^2$ be $C^2$. If $v \cdot D^2g_0(v, v) = 0$ then $|w \cdot D^2g_0(w, w)| \leq O \|w\|^3 \cdot \sin(w, v)$.

Proof of 6.2. Assume $v = (1, 0)$ and $w = (1, y)$. Then $w \cdot D^2g_0(v, v) = (1, 0) \cdot (\partial^2 g_1/\partial x \partial x, \partial^2 g_2/\partial x \partial x) = 0$. Thus $\partial^2 g_1/\partial x \partial x = 0$. Using the Hessian matrix to express $D^2g_0(w, w)$ in coordinates we obtain

\[
|w \cdot D^2g_0(w, w)| = \left| (1, y) \cdot \left( y \frac{\partial^2 g_1}{\partial y \partial x} + y \frac{\partial^2 g_1}{\partial x \partial y} + y \frac{\partial^2 g_1}{\partial y^2}, \frac{\partial^2 g_2}{\partial x \partial y} + y \frac{\partial^2 g_1}{\partial y^2} \right) \right|.
\]

Therefore there exist constants $C > 0$ and $C' > 0$, depending on the second derivative of $g$, such that $|w \cdot D^2g_0(w, w)| \leq C \|w\| \leq C' \sin(w, v)$. The estimate 6.2 follows from the linearity of the dot product and the second derivative. Hence

(6.3) $|D^2f_y(z - y) \cdot Df_y(z - y)| \leq O \|z - y\|^3 \cdot |\sin(v_y, z - y)|$.

We next estimate $|\sin(v_y, z - y)|$ for $y, z \in E$. The function $v_y$ is continuous. It is uniformly continuous by compactness of $M$. Since $v_y$ approximates $(z - y)/\|z - y\|$ it follows that $|\sin(v_y, z - y)| \leq 0 \|z - y\|$. Hence, $|D^2f_y(z - y) \cdot Df_y(z - y)| \leq O \|z - y\|^4$.

(6.4) If $a, b \in R^2$, $\|a\| \leq 1$, $|a \cdot b| \leq O \|a\|^4$ and $\|b\| \leq O \|a\|^2$ then $\|a + b\| \leq \|a\| + O \|a\|^3$.

Proof of 6.4. Let $c$ be the unique vector such that $a \cdot (b - c) = c \cdot (b - c) = 0$. Then $|\cos \theta| = \|c\|/\|b\|$ where $\theta$ is the angle between $a$ and $b$. We have $|a \cdot b| = \|a\| \|b\| \|c\| \leq 0 \|a\|^4$. Hence $\|a\| \|c\| \leq 0 \|a\|^4$ and thus $\|c\| \leq 0 \|a\|^3$. Note $\|b - c\| \leq \|b\| \leq 0 \|a\|^2$. Since $a \cdot (b - c) = 0$ it follows that $\|a + (b - c)\| = \sqrt{(\|a\|^2 + \|b - c\|^2)} \leq \sqrt{(\|a\|^2 + C \|a\|^4)} \leq \|a\| + O \|a\|^3$. Hence $\|a + b\| \leq \|a + (b - c)\| + \|c\| \leq \|a\| + O \|a\|^3$.

By (6.3) and (6.4) we conclude

\[
\|Df_y(z - y) + \frac{1}{2} D^2f_y(z - y)^2\| \leq \|z - y\| + O(\|z - y\|^3).
\]

q.e.d.

COROLLARY 6.5

Let $E$ be an isometry minimal set of a $C^3$ mapping $f$ of a compact Riemannian 2-manifold $M$. If $y \in M$ and $z \in E$ then

\[
\sum_{n=1}^{\infty} (d(f^n(y), f^n(z)))^2 = \infty
\]

Proof. Let $y \in M$ and $z \in E$. Since $E$ is compact, there exists a constant $\mu > 0$ and geodesic coordinate charts $U_n = U_n$ based at $f_n(z) = z_n$ with radius $\mu$. We can assume
that there exists a positive integer \( N \) such that \( y_n \) lies in \( U_n \) for \( n \geq N \). Otherwise the result is immediate. Assume \( n \geq N \).

By Taylor's theorem, since \( f \) is \( C^3 \) and \( E \) is compact, there exists a constant \( A > 0 \) such that in the coordinates of the chart \( U_n \),

\[
\| f(y_n) - (f(z_n) + Df_{z_n}(y_n - z_n) + \frac{1}{2} D^2f_{z_n}(y_n - z_n)^2) \| \leq A.
\]

But

\[
\| Df_{z_n}(y_n - z_n) + \frac{1}{2} D^2f_{z_n}(y_n - z_n)^2 \| - \| y_n - z_n \| \leq A^t.
\]

Hence

\[
\| f(y) - (f(z_n) + (y_n - z_n)) \| \leq A^t.
\]

Since \( f(y_n) = y_{n+1} \) lies in \( U_{n+1} \) we may replace \( f(y_n) \) by \( y_{n+1} \). By the triangle inequality

\[
\| y_{n+1} - z_{n+1} \| - \| y_n - z_n \| \leq \| y_{n+1} - (z_{n+1} - (y_n - z_n)) \| \leq A^t.
\]

6.6 If a sequence of positive numbers \( a_n \to 0 \) satisfies

\[
\frac{a_{n+1} - a_n}{|a_n|^r} < A \text{ and } \frac{a_{n+1}}{a_n} > B > 0
\]

then

\[
\sum_{n=1}^{\infty} a_n^{-1} = \infty.
\]

Proof of (6.6). Since \( a_n \to 0 \) it follows that

\[
\prod_{n=1}^{N} a_n \to \infty \quad \text{as } N \to \infty.
\]

Hence

\[
\sum_{n=1}^{\infty} \left| 1 - \frac{a_n}{a_{n+1}} \right| = \infty.
\]

By hypothesis \( \frac{a_{n+1}}{a_n} > B \) and \( a_n > (|a_{n+1} - a_n|/A) \), so

\[
a_n^{-1} > \frac{1}{A} \left| \frac{a_{n+1}}{a_n} - 1 \right| = \frac{1}{A} \left| \frac{a_{n+1}}{a_n} \right| - \frac{a_n}{a_{n+1}} > \frac{B}{A} \left| 1 - \frac{a_n}{a_{n+1}} \right|.
\]

It follows from the comparison test that \( \sum_{n=1}^{\infty} a_n^{-1} = \infty \).

With (6.6) we may complete the proof of Corollary 6.5. Since \( Df \) is an isometry at \( z_n \), there exists a constant \( B > 0 \) such that \( \| y_{n+1} - z_{n+1} \| / \| y_n - z_n \| > B \). Let \( a_n = \| y_n - z_n \| \) and \( r = 3 \).

**q.e.d.**

**COROLLARY 6.7**

**Corollary** Let \( f \) be a \( C^3 \) diffeomorphism of a compact Riemannian 2-manifold \( M \) and \( Q \subset M \) a minimal isometry set. If \( Q \) is a Denjoy Cantor set then \( Q \) is not square-rectifiable.
Proof. By definition, if $Q$ is a Denjoy Cantor set in $M$ there exists $h: Q \to S^1$ mapping $Q$ onto a Denjoy Cantor set $\Gamma$ in $S^1$. Let $x_0$ and $y_0$ be the inverse image of endpoints of an interval complementary to $\Gamma$ in $S^1$. Since $f$ preserves the order of $Q$ the intervals $(x_n, y_n)$ are each disjoint from $Q$ and each other. By Corollary 6.5, $\sum d(x_n, y_n)^2 = \infty$. Thus $Q$ is not square-rectifiable.

**DEFINITION**

A Jordan curve $Q$ in a Riemannian two-manifold is a **quasi-circle** if there exists a positive constant $K$ such that if $x, y \in Q$ then one of the arcs of $Q$ connecting $x$ and $y$ is contained in a disk of radius $Kd(x, y)$.

**PROPOSITION 6.8**

A quasi-circle $Q$ is square-rectifiable.

**Proof.** Let $x_1 < x_2 < \cdots$ be a partition of $Q$.

Let $a_n = d(x_n, x_{n+1})$ and $B_n$ the disk of radius $a_n/8K$ centered at $x_n$ where $K$ is the quasi-constant for $Q$. The result follows from compactness of $Q$ if the $B_n$ are disjoint. Suppose $B_n \cap B_m \neq \emptyset$. Then $d(x_n, x_m) < (a_n + a_m)/8K \leq a_n/4K$, assuming $a_m \leq a_n$. Since $Q$ is a quasi-circle, one of the arcs connecting $x_n$ and $x_m$ is contained in a disk of radius $a_n/4$. Let $z$ be a point in this arc equidistant from $x_n$ and $x_m$. Then $d(x_n, z) = d(x_m, z) < a_n/2$. Thus $a_n = d(x_n, x_m) \leq d(x_n, z) + d(z, x_m) < a_n$.

**COROLLARY 6.9**

Let $f$ be a $C^3$ diffeomorphism of a compact Riemannian 2-manifold $M$ and $Q \subset M$ a minimal isometry set. If $Q$ is a Denjoy Cantor set then $Q$ is not square-rectifiable.

**References**