DENJOY FRACTALS

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INTRODUCTION

POINCARE defined the rotation number $\rho(f)$ for a homeomorphism $f$ of the circle $S^1$:

$$\rho(f) = \lim_{n \to \infty} \frac{f^n(x) - x}{n}$$

where $f$ is any lift of $f$ to the real line $\mathbb{R}$ and $x \in \mathbb{R}$. The limit $\rho(f)$ is a topological invariant of $f$ and is independent of the lift $f$ and the starting point $x$. Every homeomorphism of a compact manifold has a minimal set. If $\rho(f) = \alpha$ is rational then every minimal set for $f$ is finite and conversely. Henceforth we assume $\alpha$ is irrational. Since every infinite minimal set is perfect and homogeneous, an infinite minimal set of $S^1$ is either a Cantor set or $S^1$ itself.

**DENJOY THEOREM** (1932) [3]. If $f$ is $C^2$ then $f$ is topologically conjugate to a rotation through $\alpha$.

As a complement to this theorem, Denjoy produced uncountably many topologically distinct examples of $C^1$ diffeomorphisms $D$ having Cantor minimal sets. Each $D$ permutes the countable set of intervals $\{I_n\}$ complementary to the minimal set.

In this paper we give examples of $C^{2+\delta}$ diffeomorphisms of the annulus $A$ permuting a countable set of disjoint disks $\{R_n\} \subset A$.

**Theorem A.** For $\delta > 0$ sufficiently small there exists a Jordan curve $Q \subset A$, a family of disjoint disks $\{R_n\} \subset A$ with $R_n \cap Q \neq \emptyset$ and a $C^{2+\delta}$ diffeomorphism $f: A \to A$ such that $Q \cup \{R_n\}$ is $f$-invariant and has no periodic points.

The curve $Q$ has Hausdorff dimension $1 + \delta$.

The derivative of $f$ at the minimal set in $Q$ is an isometry, a feature shared by the canonical Denjoy counterexample $D$. This property is useful in [6] where $C^{2+\delta}$ counterexamples to the Seifert conjecture are found.

There may be periodic points of $f$ in a neighborhood of $Q \cup \{R_n\}$. In [6] $f$ is made to be semi-stable so there is no longer any periodicity.

An overview of this paper and its sequel [6] may be found in [11].

**Problem.** If $f: M \to M$ is a $C^3$ diffeomorphism of a compact surface $M$ with no periodic points and $B \subset M$ is a disk with $\{f^n(B)\}$ disjoint, must the shape of $f^n(B)$ become distorted? That is, must (outer diameter $(f^n(B)))$/inner diameter $(f^n(B))$ be unbounded?

Denjoy proved that if $D$ were $C^2$ then $\sum |I_n| = \infty$, contradicting the finite arc length of $S^1$. The fact that there is bounded distortion in $D$ is important — that is, $\|Df^k\|$ is bounded for a
subsequence $q \to \infty$. There are several examples of embeddings of one-dimensional Denjoy examples in the plane where the distortion of $f$ is unbounded and $f$ is at least $C^2$. In each of these $\Sigma |I_i| < \infty$, but there is no contradiction.

The first was due to R. Knill [16]. He embedded the Denjoy minimal set in the plane as part of a $C^\infty$ diffeomorphism. Mather [18] proved there exist embedded Denjoy minimal sets in some area-preserving twist maps of the annulus. In 1980 G. R. Hall [5] embedded an entire Denjoy counterexample in the plane as part of a $C^\infty$ diffeomorphism. M. Herman [13] produced an area-preserving $C^{3+\varepsilon}$ example.

These examples have unbounded distortion in the derivative and putting limits on the distortion makes such examples impossible. The author proved that if $Df$ is an isometry at the Cantor set and $Q$ is a quasi-circle then $f$ cannot be $C^3$ [7]. Ghys [4] showed that if $x$ satisfies a Diophantine condition then $f$ cannot be complex analytic and restrict to a Denjoy counterexample on $Q$.

The construction of R. Knill preceded and influenced this paper. He embedded the forward Denjoy intervals horizontally and the backward intervals vertically. His diffeomorphism was hyperbolic and had infinitely many periodic orbits in a neighborhood of the Denjoy minimal set. At the time the author was looking for a two-dimensional "Denjoy" example which had intervals of length $1/n^\gamma$, $\gamma < 1$ and disjoint, invariant disks. The longer intervals were needed to satisfy analytic requirements of [8, 9] for a $C^{2+\delta}$ extension and the disks were needed to help rid the example of periodic points. Although Knill's example was two-dimensional, it did not satisfy these two additional properties. The author learned of Knill's work from C. Rourke. In January, 1978 Rourke and she made an unsuccessful attempt to incorporate these properties into a modification of Knill's example.

There are several mathematicians whose help I have appreciated while doing this research. My Seifert education began with M. Handel. G. Levitt and H. Helson suggested the possibility of a relationship between the estimates of my initial work on weighted uniform distribution and the "unweighted" estimates of Ostrowski and Kesten. As a result, I was able to simplify the proof significantly. Yoccoz made the suggestion of using two orbits instead of one in the construction of $h: S^1 \to A$. This makes the embedding easier to describe. Unfortunately, the number theoretic estimates become more difficult.

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§1. GEOMETRY OF CONTINUED FRACTIONS

Let \( a_n \) be a sequence of positive integers and

\[
\alpha = \lim_{n \to \infty} \left[ \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots} \frac{1}{a_n}} \right]
\]

The limit \( \alpha \) exists and is a positive irrational \(< 1\). If there exists a positive integer \( N \) such that \( a_n \leq N \) then \( \alpha \) is of constant type.

**Definition 1.1.** Define sequences of integers \( p_n \) and \( q_n \):\[
\begin{align*}
p_0 &= 0, \quad p_1 = 1 \\
n &= a_n p_{n-1} + p_{n-2} \\
q_0 &= 1, \quad q_1 = a_1 \\
n &= a_n q_{n-1} + q_{n-2}.
\end{align*}
\]

Let \( r_0 = 1, \quad r_1 = \alpha, \quad \text{and} \quad r_{n+1} = r_n - a_n r_{n-1} \).

The fractions \( \frac{p_n}{q_n} \) are rational convergents of \( \alpha \) and \( \lim_{n \to \infty} \frac{p_n}{q_n} = \alpha \). Any rational \( \frac{p}{q} \) satisfying \( (p, q) = 1 \) and \( |x - p/q| < 1/q^2 \) is called a rational approximate of \( \alpha \). Every rational convergent of \( \alpha \) is a rational approximate of \( \alpha \). (see Khintchine [15], for example)

**Lemma 1.2.** For \( n \geq 1 \),

(i) \( q_n r_n + q_{n-1} r_{n+1} = 1 \)

(ii) \( p_n r_{n+1} + p_{n-1} r_n = \alpha \).

**Proof.** For \( n = 1 \) both statements follow from (1.1). Assume (i) is true for fixed \( n \geq 1 \). Then

\[
q_{n+1} r_{n+1} + q_n r_{n+2} = (a_{n+1} q_n + q_{n-1}) r_{n+1} + q_n r_{n+2}
= (a_{n+1} r_{n+1} + r_{n+2}) q_n + q_{n-1} r_{n+1}
= r_n q_n + q_{n-1} r_{n+1} \quad \text{by (1.1)}
\]

Now assume (ii) is true for \( n \geq 1 \). Then

\[
p_n+1 r_{n+1} + p_n r_{n+2} = (a_{n+1} p_n + p_{n-1}) r_{n+1} + p_n (r_n - a_n r_{n+1})
= p_n r_{n+1} - a_n r_{n+1} - \alpha.
\]

Q.E.D.

Let \( S^1 = \mathbb{R}^1 / \mathbb{Z} \). If \( x \in \mathbb{R}^1 \) let \( \langle x \rangle = x \pmod{1} \) be the rotation of the circle through \( x \). If \( A \subset S^1 \) let \( A + x = R_x(A) \). If \( n \in \mathbb{Z} \), we call \( \langle nx \rangle \) the orbit point of \( R_x \) with index \( n \).

Give \( S^1 \) the orientation inherited from \([0, 1)\). Any two points \( x \neq y \subset S^1 \) bound a unique oriented arc \((x, y)\). If its length is \( \leq 1/2 \) define \( x < y \).

**Lemma 1.3.** For \( n \geq 0 \), \( p_n - q_n \alpha = (-1)^{n+1} r_{n+1} \). Hence \( \langle q_n \alpha \rangle \) alternates on either side of \( \langle 0 \alpha \rangle \).

**Proof.** If \( n = 0 \) this follows from (1.1). Assume \( p_{n-1} - q_{n-1} \alpha = (-1)^n r_n \). Multiplying by \(-r_{n+1}/r_n\) gives

\[
(-r_{n+1}/r_n)(q_n - \alpha r_{n+1}/r_n) = (-1)^{n+1} r_{n+1}.
\]
On the other hand, the L.H.S. is precisely \( p_n - q_n \alpha \) by (1.2).

**Lemma 1.4.** If \( a_n(\alpha) = N \) for \( n \geq 1 \) then \( r_k = \alpha^k \) for \( k \geq 0 \).

**Proof.** Definition (1.1) implies the result for \( k = 0 \) or 1. Assume \( r_k = \alpha^k \) and \( r_{k-1} = \alpha^{k-1} \). Then by (1.1) \( r_{k+1} = \alpha^k - N\alpha^k = \alpha^k(\alpha^{-1} - N) \). But \( \alpha^{-1} = N + \alpha \). Thus \( r_{k+1} = \alpha^{k+1} \).

\[ \text{Q.E.D.} \]

Let \( n \in \mathbb{Z}^+ \). We define \( W_n \) to be the collection of intervals \( I \) in \( S^1 \setminus \{ \langle 0 \alpha \rangle, \langle 1 \alpha \rangle, \ldots, \langle (q_n + q_{n-1} - 1) \alpha \rangle \} \).

\( I \) is a \( W_n \)-interval and \( \langle j \alpha \rangle \), \( 0 \leq j < q_n + q_{n-1} \), is a \( W_n \)-point. Let \( I_0(n) \) be the \( W_n \)-interval with endpoints \( \langle 0 \alpha \rangle \) and \( \langle q_n \alpha \rangle \) and \( J_0(n) \) the interval bounded by \( \langle 0 \alpha \rangle \) and \( \langle q_{n-1} \alpha \rangle \). By (1.3), \( I_0(n) \) and \( J_0(n) \) are on opposite sides of \( \langle 0 \alpha \rangle \) in \( S^1 \).

**Lemma 1.5.** The intervals of \( W_n \) consist of the first \( q_n \) iterates of \( I_0(n) \) and the first \( q_{n-1} \) iterates of \( J_0(n) \) under rotation by \( R_\alpha \). In particular, all \( W_n \)-intervals have length \( r_n \) or \( r_{n+1} \).

**Proof.** Without loss of generality, assume \( I_0(n) = \langle 0 \alpha \rangle, \langle q_{n-1} \alpha \rangle \) and \( J_0(n) = \langle q_n \alpha \rangle, \langle 0 \alpha \rangle \). Denote by \( V \) the collection of \( q_n \)-iterates of \( I_0(n) \):

\( \langle 0 \alpha \rangle, \langle q_{n-1} \alpha \rangle, \langle \alpha \rangle, \langle q_{n-1} + 1 \alpha \rangle, \ldots, \langle (q_n - 1) \alpha \rangle, \langle (q_n + q_{n-1} - 1) \alpha \rangle \)

together with the \( q_{n-1} \)-iterates of \( J_0(n) \):

\( \langle q_n \alpha \rangle, \langle 0 \alpha \rangle, \langle (q_n + 1) \alpha \rangle, \langle \alpha \rangle, \ldots, \langle (q_n + q_{n-1} - 1) \alpha \rangle, \langle (q_{n-1} - 1) \alpha \rangle \).

Then \( V \) consists precisely of the \( W_n \)-intervals: The collection of endpoints of \( V \)-intervals and \( W_n \)-intervals is the same. By (1.2) the total length of \( V \)-intervals is one. If two \( V \)-intervals had intersection then there would exist some interval of the circle not covered by \( V \) with its left endpoint, say, a \( W_n \)-point. But each \( W_n \)-point appears as a left endpoint of one of the intervals of \( V \).

\[ \text{Q.E.D.} \]

We now remove the orbit points \( \langle nx \rangle \) for integers \( n < 0 \) as well.

1.6. Define \( t_n = \left[ \frac{(q_n + q_{n-1})}{2} \right] \). Then

\( (q_n + q_{n-1} - 1)/2 \leq t_n \leq (q_n + q_{n-1})/2 \).

**Lemma 1.7.**

(i) \( 1/2 < q_n r_n < 1 \) for all \( n \geq 1 \);

(ii) If \( a_n(\alpha) \leq N \) for all \( n \geq 1 \) then \( t_n < q_n < 2(N^2 + 1)t_{n-2} \).

**Proof:** (i) By (1.2), \( q_n r_n < 1 \). By (1.1) \( r_{n-1} > 2r_{n+1} \). Then

\[ r_n q_n = 1 - q_{n-1} r_{n+1} > 1 - r_{n+1}/r_{n-1} > \frac{1}{2} \]

(ii) By (1.1) and (1.6)

\[ t_n < q_n \leq (N^2 + 1)q_{n-2} + Nq_{n-3} < (N^2 + 1)(q_{n-2} + q_{n-3} - 1) \leq 2(N^2 + 1)t_{n-2} \]

\[ \text{Q.E.D.} \]

An interval \( I \) of \( S^1 \) is in the collection \( W_n \) if it is the image of a \( W_n \)-interval under rotation by \( -t_n \alpha \); \( I \) is called a \( W_n \)-interval and its endpoints are \( W_n \)-points.
LEMMA 1.8. \( W_n \) is isometric to \( W_n' \) and consists of the first \( q_n \) iterates of \( I_0(n) \) and the first \( q_{n-1} \) iterates of \( J_0(n) \) under rotation by \( R_n \). The intervals in the former set have length \( r_n \) and those in the latter have length \( r_{n+1} \). Each \( J_0(n) \)-iterate in \( W_n \) is also an \( I_0(n+1) \)-iterate in \( W_{n+1} \).

Proof. Let \( I \) be a \( J_0(n) \)-iterate in \( W_n \). Assume that

\[
J_0(n) = (\langle (q_n - t_n)\alpha \rangle, \langle -t_n\alpha \rangle) \quad \text{and} \quad I_0(n+1) = (\langle (q_n - t_{n+1})\alpha \rangle, \langle -t_{n+1}\alpha \rangle).
\]

Then the index \( p \) of the right endpoint of \( I \) satisfies \( -t_n \leq p < -t_n + q_n \). By (1.6) \(-t_{n+1} < p < -t_{n+1} + q_{n+1}\); these bounds are also the bounds for the indices of the right endpoints of the \( I_0(n+1) \)-iterates.

Q.E.D.

Remarks 1.9. The \( W_n \)-intervals \( I \) are open. Sometimes we need \( I \) to be half closed or closed. Even so, we refer to \( I \) as a \( W_n \)-interval and specify its type.

The largest positive index \( t'_n \) of a \( W_n \)-point will equal \( t_n \) only if \( q_n + q_{n-1} \) is odd. This will be the case when \( q_n(x) \) is even. We assume in our proofs that \( t'_n = t_n = (q_n + q_{n-1} - 1)/2 \) for our notation to be tractable. Otherwise \( t'_n \) differs from \( t_n \) by 1. (But all theorems are valid as stated.)

§2. NUMBER THEORETIC ESTIMATES

In this section we discuss the nature of a weighted distribution of the orbit points \( \langle na \rangle \) in \( S^1 \).

\[
\left| \sum_{i=s}^{t-1} \left( \chi_U \langle ix \rangle - |U| \right)i^{-\gamma} \right|
\]

The estimates will depend on the degree of "irrationality" of \( \alpha \), the given "weight" \( i^\gamma \) and on the type of interval \( U \) over which we are testing the distribution.

The classical Denjoy-Koksma theorem (2.1) gives an upper bound of two for this series with \( \gamma = 0, s = 0 \) and \( t = q \), a denominator of a rational approximate of \( \alpha \). If \( U = [x, x + \langle q\alpha \rangle) \) and \( \gamma = 0 \) then, according to Kesten, two is again an upper bound for arbitrary integers \( s \) and \( t \).

Theorem 2.1. Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \). Let \( p/q \) be a rational approximate of \( \alpha \). Let \( f \) be a homeomorphism of the circle with rotation number \( \rho(f) = \alpha \). Let \( \theta : S^1 \to \mathbb{R}^1 \) be a function with bounded variation and \( \mu \) an invariant probability measure of \( f \). Then, for every \( x \in S^1 \),

\[
\left| \sum_{i=s}^{t-1} \left( \theta - f'(x) - \int_{S^1} \theta \, d\mu \right) \right| \leq \text{Var}(\theta).
\]

Proof. See [17].

Theorem 2.2.

(i) (Kesten [14]) Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) and \( p/q \) a rational approximate of \( \alpha \). If \( I = [x, \langle x + qa \rangle) \subseteq S^1 \), for \( x \in S^1 \), then for every \( s < t \in \mathbb{Z} \)

\[
\left| \sum_{i=s}^{t-1} \left( \chi_I \langle ix \rangle - |I| \right) \right| < 2.
\]
(ii) (Hecke [12] and Ostrowski [19]) Let $\alpha$ be of constant type. There exists a constant $C > 0$ such that if $U$ is an interval of $S^1$ and $s \leq t \in \mathbb{Z}$ then
\[ \left| \sum_{t}^{s-1} \left( \chi_U\left( \langle t \alpha \rangle - |U| \right) \right) \right| < C \log(t-s). \]

Proof of (i). Let $I_0 = [0, q\alpha)$. Note that for any real numbers $a$ and $b$,
\[ \langle b - a \rangle = \chi_{(0, q\alpha)}(b) + \langle b \rangle - \langle a \rangle. \]
Then
\[
\left| \sum_{t=s}^{s-1} \chi_U(\langle t \alpha \rangle - |U|) \right| = \left| \sum_{t=0}^{s-1} \chi_U(\langle (i+s)\alpha - x \rangle - |I_0|) \right|
\leq \sum_{t=0}^{s-1} \left| \langle (i+s-q)\alpha - x \rangle - \langle (i+s)\alpha - x \rangle \right|
\leq \sum_{t=0}^{s-1} \left| \langle (i+q+t)\alpha - x \rangle - \langle (q+t)\alpha - x \rangle \right|
\leq \sum_{t=0}^{s-1} \chi_{(0, q\alpha)}(\langle (i+s-q)\alpha - x \rangle - \langle (s-t)\alpha \rangle) < 2 \quad \text{(by (2.1))}
\]

Proof of (ii). See [10, 15, 17].

The next proposition involves “summation by parts”.

**Proposition 2.3.** Let $C > 0$ and $f_\alpha$ a monotone increasing sequence of positive real numbers. Let $b_i$ be a sequence of real numbers satisfying
\[ \sum_{j=s}^{j=N} b_j < C f_N \quad \text{for all } j, N \geq 0. \]
Let $d_j > 0$ be a monotone decreasing sequence and $0 \leq k \leq m$. Then

(i) \[ \left| \sum_{k=m-k}^{m-1} b_k d_k \right| \leq C f_m d_k. \]
(ii) Define $n$ and $r$ by $m-1 = 2^n + l$, $0 \leq l < 2^n$ and $k = 2^r + p$, $0 \leq p < 2^r$. Then
\[ \left| \sum_{k=1}^{m-1} b_k d_k \right| \leq C \left[ \sum_{l=0}^{r} f_{2^l} d_{2^l} \right]. \]

Proof. By Abel’s partial summation formula,
\[
\left| \sum_{k=s}^{s-1} b_k d_k \right| = \left| (b_s + \cdots + b_{s-1})d_{s-1} + b_s(d_s - d_{s+1}) + (b_{s+1} + \cdots + b_{s+2}) \right|
\leq \left| b_s + \cdots + b_{s-1} \right| + \left| b_s \right| \left| d_s - d_{s+1} \right|
\leq C f_{s-1} d_{s-1} + C f_s (d_s - d_{s+1}) + \cdots + C f_{s-1} (d_{s-2} - d_{s-1})
= C f_{s-1} d_s.
\]
Thus
\[ \left| \sum_{i=k}^{n-1} b_i d_i \right| = \left| \sum_{i=2}^{\frac{1}{2} \log_2 k} \sum_{i=2}^{\frac{1}{2} \log_2 k} b_i d_i \right| + \left| \sum_{i=2}^{\frac{1}{2} \log_2 k} b_i d_i \right| \leq C \left( \sum_{i=2}^{\frac{1}{2} \log_2 k} f_i d_i \right). \]

Q.E.D.

We conclude with the estimates underlying Lemmas 3.2 and 3.7.

**Theorem 2.4.**

(i) Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \). If \( I = [x, \langle x + q_n \alpha \rangle) \), \( g: \mathbb{Z} \to \mathbb{Z}^+ \) is monotone decreasing and \( 0 < s < t \) then
\[ \left| \sum_{i=2}^{\frac{1}{2} \log_2 k} \left( \chi_{i} \langle i \alpha \rangle - |I| \right) g(i) \right| < 2g(s). \]

(ii) Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) be of constant type. There exists a constant \( C > 0 \) such that if \( \frac{1}{2} < \gamma < 1 \), \( U \) is an interval of \( S^1 \) and \( s < t \), then
\[ \left| \sum_{i=2}^{\frac{1}{2} \log_2 k} \left( \chi_{U} \langle i \alpha \rangle - |U| \right) \right| \times \left( \frac{2^\gamma}{2^{\gamma+1}} \right) \leq C \log s/s^\gamma. \]

**Proof of (i).** The result follows immediately from \((2.2)(ii)) and \((2.3)(i))

**Proof of (ii).** Let \( C_1 \) be the constant depending on \( \alpha \) obtained from \((2.2)(ii))\. Apply \((2.3)(ii)) to \( b_i = \chi_U \langle i \alpha \rangle - \left| U \right|, d_i = 1/i^\gamma \) and \( f_i = \log(i) \). We have
\[ \left| \sum_{i=2}^{\frac{1}{2} \log_2 k} b_i d_i \right| < C_1 \left( \sum_{i=2}^{\frac{1}{2} \log_2 k} \log(2^\gamma/2^{\gamma+1}) \right) \]

where \( s = 2^\gamma + p, 0 \leq p < 2^\gamma \). The series is bounded by a geometric series since the ratio of successive terms is \( (\lambda + 1)/2^{\gamma+1} \) which is less than 1 for \( \lambda > 1 \). Thus the series is bounded by \((\log(2^\gamma/2^{\gamma+1}) \times (\Sigma (\lambda + 1)/2^{\gamma+1}) \). The latter series is bounded independently of \( \frac{1}{2} < \gamma < 1 \). The result follows.

Q.E.D.

**§3. The Embedding**

Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) and \( D_\alpha: S^1 \to S^1 \) a Denjoy counterexample with rotation number \( \alpha \). Let \( \rho: S^1 \to S^1 \) be a monotonic, continuous mapping semi-conjugating \( D_\alpha \) to the rotation \( R_\alpha \). Assume that \( \rho^{-1} \langle n \alpha \rangle \) is an interval with interior denoted by \( \Delta'_n = (y'_n, z'_n) \) and that the left endpoint of \( \Delta'_n \) is 0. Define \( \pi_1(x, y) = x \) and \( \pi_2(x, y) = y \) for \( (x, y) \in S^1 \times R^1 \).

We construct a mapping \( h: S^1 \to S^1 \times R^1 \) as a limit of mappings \( h_n: S^1 \to S^1 \times R^1 \). The image of \( h \) resembles the Denjoy circle except that each Denjoy interval \( \Delta'_n \) is replaced by a diagonal \( \Delta_n \), with slope \( (-1)^\gamma \). The total length of the \( \Delta_n \) is unbounded.

More precisely, let \( \frac{1}{2} < \gamma < 1 \) and \( a > 0 \) be real numbers; define \( g(i) = a/i \gamma \) for integers \( i \neq 0 \). (The constants \( \gamma \) and \( a \) will be specified later.) Recall the sequence \( t_n \) (see 1.6), and define
\[ g(0) = \sum_{i=1}^{t_n} (-1)^{i+1} g(i). \]

The lengths of the diagonals will be \( \sqrt{2} \cdot g(i) \). The choice of \( g(0) \) will insure \( \pi_2 h_n \langle 0 \alpha \rangle \) is well defined. Note that \( g(0) \) is positive and depends on \( n \).
Define
\[ m_n = 1 + \sum_{|i|=0}^{t_n} y(i). \]
This constant will insure that \( \pi_1 h_n(0, x) \) is well-defined.

Use the isomorphism \( S^1 \cong \mathbb{R}^1 \setminus \mathbb{Z} \) to carry the Euclidean metric \( d(\cdot, \cdot) \) and norm \( ||\cdot|| \), locally, to \( S^1 \times \mathbb{R}^1 \). Then the meaning of line and slope in \( \mathbb{R}^2 \) naturally carry over to \( S^1 \times \mathbb{R}^1 \).

Call a line segment in \( S^1 \times \mathbb{R}^1 \) \emph{horizontal} if its slope is 0.

Now fix \( n \in \mathbb{Z}^+ \). Order the endpoints of the intervals \( \Delta_i : 0 \leq |i| \leq t_n \), beginning with 0, from left to right:
\[ 0 = p_0 < q_0 < p_1 < q_1 < \cdots < p_m = 1 = 0. \]
Let
\[ \langle i, x \rangle = \rho(p_i) = \rho(q_i) \]
for
\[ 0 \leq i < m \] and \( \langle i, x \rangle = 1 = \rho(p_m) \).

Define a piecewise linear function \( h_n : S^1 \rightarrow S^1 \times \mathbb{R}^1 \) inductively: Define \( h_n(p_0) = (0,0) \) and \( h_n[p_0, q_0] \) to be the line segment with slope \( +1 \), length \( \sqrt{2} g(0) \) and decreasing first coordinate. Suppose that \( h_n[p_0, q_{k-1}] \) has been defined for \( 1 \leq k \leq m \). Suppose that its endpoints are \( (0,0) \) and \( (x, y) \). Map \( h_n[d_{k-1}, p_k] \) linearly to the horizontal line segment with one endpoint \( (x, y) \), increasing first component and length \( m_n \langle i, x \rangle - \langle h_{k-1}, x \rangle \). Suppose that the image of \([p_0, p_k] \) under \( h_n \) has been defined for \( 1 \leq k \leq m \). If \( k = m \) then \( h_n \) has been defined on all of \( S^1 \). Otherwise, suppose that its endpoints are \( (0,0) \) and \( (x, y) \). Map \( h_n[p_k, q_k] \) linearly to the diagonal line segment \( \Delta \) with length \( \sqrt{2} g(i_k) \), slope \( (-1)^k \) so that \( h_n(p_k) = (x, y) \) and \( h_n[p_k, q_k] \) has decreasing first coordinate. This uniquely specifies a backward sloping diagonal \( \Delta \).

From this description we derive an explicit formula for \( h_n \). Fix \( x \in S^1 \) and let \( W = [0, \rho(x)] \). The diagonal associated to \( \langle jx \rangle \in W, |j| \leq t_m \), contributes \( -g(j) \) to the first component \( \pi_1 h_n(x) \), no matter what its slope. It contributes \( (-1)^{j+1} g(j) \) to the second coordinate \( \pi_2 h_n(x) \). Each \( W_x \)-interval \( I, I \cap W \neq \emptyset \), contributes its normalized length \( m_n |I \cap W| \) to \( \pi_1 h_n(x) \). The total of the normalized lengths is \( m_n |W| \). Finally, if \( x \) lies in \( \text{cl}(\Delta_x) \) for \( |p| \leq t_m \) its relative position is preserved on the diagonal \( h_n(\text{cl}(\Delta_x)) \) and a correction term \( \pm d_x \) is added to each of the coordinates. For such \( x \) define
\[ d_x = (-1)^p g(p)|x - y'|/|z_p' - y_p'|. \]
Otherwise, let \( d_x = 0 \). Then
\[ \pi_1 h_n(x) = m_n |W| - \sum_{|i|=0}^{t_n} \chi_W \langle i, x \rangle g(i) - |d_x|. \quad (3.1) \]
\[ \pi_2 h_n(x) - \sum_{|2i+1| \leq t_n} \chi_W \langle 2i+1, x \rangle g(2i+1) - \sum_{|2i| \leq t_n} \chi_W \langle 2i, x \rangle g(2i) - d_x. \]

Let \( x = 1 \), the second copy of 0. Then \( W = S^1 \) and \( d_x = 0 \). Hence \( h_n(1) = (1,0) = (0,0) = h_n(0) \) and \( h_n \) is well-defined. See Fig. 2.

Define \( h = \lim h_n \) as \( n \to \infty \).

It is not at all clear \emph{a priori} that \( h \) exists or is continuous, much less that it is an embedding for any choice of \( \alpha, \gamma \) and \( a \). We first prove that if \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) is of constant type, \( a > 0 \) and \( \frac{1}{2} < \gamma < 1 \), then \( h \) exists and is continuous. (It can be shown that, indeed, \( h \) exists and is continuous for any \( \alpha \) satisfying a Diophantine condition [10].) Choosing \( a, \gamma = 2 \) and placing further restrictions on \( \gamma \) and \( a \) enables us to prove that \( h \) is an embedding.
Estimates on horizontal and vertical components of the mapping $h$

Let $U$ be an interval in $S^1$ with endpoints $p < q$. Define $H_n(U) = \pi_1 h_n(q) - \pi_1 h_n(p)$ and $V_n(U) = \pi_2 h_n(q) - \pi_2 h_n(p)$. If $H_n$ and $V_n$ converge uniformly over intervals of $S^1$ then $h_n$ converges uniformly to a continuous function $h$.

**Lemma 3.2.** If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is of constant type, $\frac{1}{2} < \gamma < 1$ and $\alpha > 0$ then $H_n = H_n(\alpha, \gamma, \alpha)$ and $V_n = V_n(\alpha, \gamma, \alpha)$ converge uniformly over intervals of $S^1$.

**Proof.** Let $C$ be the constant depending on $\alpha$ from (2.4) (ii). Recall $A_i = (y_i, z_i) = \rho^{-1}(i\alpha)$. We verify the uniform convergence of $H_n$ and $V_n$ over two types of intervals $U$ with endpoints $p < q$.

1. $p \notin [y_i, z_i]$ and $q \notin (y_i, z_i]$;
2. $U \subset [y_i, z_i]$ for some $i \in \mathbb{Z}$.

Uniform convergence over all intervals of $S^1$ follows.

Given $\delta > 0$ choose $N$ such that if $n \geq N$ then

$$0 < m_n r_n + 2C \log (t_n) g(t_n) + 2g(t_n) < \delta.$$  

(Recall $r_n$ (1.1); estimate $m_n$ by an integral and apply (1.7) to see that $m_n r_n < (1 + 4a t_n^{-\gamma} / (1 - \gamma)) r_n < r_n + 4a t_n / (1 - \gamma) \to 0$ as $n \to \infty$. By the definition of $g$, $\log (t_n) g(t_n) \to 0$, so it is possible to find such an $N$.)

Fig. 2.
Proof of the uniform convergence of $H_n$. Let $U$ be an interval of type (1) with endpoints $p < q$. Then in (3.1) $d_q = 0$. Letting $\omega = \text{int}(\rho(U))$ and $N < m < n$ it follows that

$$H_n(U) = \left(1 + \sum_{|i| = 0}^{\infty} g(i) \right) |W| - \sum_{|i| = j}^{\infty} \chi_W \langle ix \rangle g(i).$$  \hspace{1cm} (3.3)$$
where $j = \min \{|i|; \langle ix \rangle \in \text{int}(W)\}$. (Whenever $j > t_n$, the last sum is zero.)

Suppose $j \leq t_m < t_n$. Apply (2.4) (ii). Then

$$|H_n(U) - H_m(U)| = \left| \sum_{|i| = t_m + 1}^{\infty} \left( |W| - \chi_W \langle ix \rangle \right) g(i) \right|$$

$$< C \log(t_m) g(t_m) < \delta.$$

Suppose $t_m < t_n < j$. Then $W$ is disjoint from the $W_i$-points. Therefore $W$ is contained in a $W_i$-interval and $|W| \leq r_i$ by (1.8). Since the last sum in (3.3) is 0 for both $H_n(U)$ and $H_m(U)$ we have

$$|H_n(U) - H_m(U)| = |m_n - m_m||W| < m_n r_i < \delta.$$

If $t_m < j \leq t_n$ let $i$ satisfy $t_m \leq i < j \leq t_m + 1 \leq t_n$. Then $W$ is disjoint from the $W_i$-points and $|W| \leq r_i$ as in the preceding paragraph. The last sum in (3.3) for $H_m(U)$ is zero so

$$|H_n(U) - H_m(U)| = |W| \left| \sum_{|i| = t_m + 1}^{\infty} g(i) - \sum_{|i| = t_m + 1}^{\infty} \chi_W \langle ix \rangle g(i) \right|$$

$$= |W| \left| \sum_{|i| = t_m + 1}^{\infty} g(i) + \sum_{|i| = t_m + 1}^{\infty} \left( |W| - \chi_W \langle ix \rangle \right) g(i) \right|$$

$$< m_n r_i + C \log(t_i) g(t_i) \text{ (by (2.4) (ii))}$$

$$< \delta.$$

Thus $H_n$ converges uniformly over the intervals of type (1).

Now suppose $U \subset \Delta_r$. If $|i| \leq t_n$ then $|H_n(U)| = q(i)$. If $|i| > t_n$ then $\rho(U) = \langle ix \rangle \subset I_n$, an open $W_n$-interval. Hence

$$|H_n(U)| = m_n |I_n| |U|/|\rho^{-1}(I_n)| \leq m_n |I_n| \leq m_n r_n.$$

Therefore, if $t_m < t_n < |i|$,

$$|H_n(U) - H_m(U)| < \max \{|H_n(U)|, |H_m(U)|\}$$

$$\leq \max \{|m_n r_n, m_n r_m|\}$$

$$< \delta.$$

If $|i| \leq t_m < t_n$ then $|H_n(U) - H_m(U)| = 0$. Finally, if $t_m < |i| \leq t_n$

$$|H_n(U) - H_m(U)| < \max \{|H_n(U)|, |H_m(U)|\}$$

$$\leq \max \{|g(i), m_n r_m|\}$$

$$\leq \max \{|g(t_m), m_n r_m|\}$$

$$< \delta.$$

Proof of uniform convergence of $V_n$. Let $U \subset S^1$ be an interval of type (1) and $W = \text{int}(\rho(U))$, as before. Then

$$V_n(U) = \sum_{|2i+1| \leq t_n} \chi_W \langle (2i+1)x \rangle g(2i+1) - \sum_{|2i| \leq t_n} \chi_W \langle 2ix \rangle g(2i).$$  \hspace{1cm} (3.4)$$
Let \( n > m > N \). Then

\[
|V_n(U) - V_m(U)| = \left| \sum_{t_m < |2i + 1| \leq t_n} \chi_w \langle 2i + 1 \rangle g(2i + 1) - \sum_{t_m < |2i| \leq t_n} \chi_w \langle 2i \rangle g(2i) \right| \\
= \left| \sum_{t_m < |2i + 1| \leq t_n} \left( \chi_w \langle 2i + 1 \rangle - \chi_w \langle 2i \rangle \right) g(2i + 1) + \sum_{t_m < |2i| \leq t_n} \left( |W| - \chi_w \langle 2i \rangle \right) g(2i) \right| \\
+ |W| \sum_{t_m < |2i + 1| \leq t_n} \left( \langle 2i + 1 \rangle - \langle 2i \rangle \right)
\]

Apply the triangle inequality and (2.4 (ii)) to estimate the first two sums. (The third sum is bounded by \( 2g(t_m) \) since \(|W| < 1\)). Thus

\[
|V_n(U) - V_m(U)| < 2C \log(t_m) g(t_m) + 2g(t_m) < \delta.
\]

Suppose \( U \subset \Delta_i \). Then \( V_j(U) \) is zero for \( t_j < |i| \). If \(|i| \leq t_j\) then \( V_j(U) \) is a constant depending only on \( U \). It is bounded by \( g(i) \).

Therefore, if \(|i| \leq t_m < t_n\) then \( |V_n(U) - V_m(U)| = 0 \). If \( t_m < |i| \leq t_n\) then \( |V_n(U) - V_m(U)| = |V_n(U)| < g(i) < g(t_m) < \delta \).

Hence \( V_n \) converges uniformly.

Q.E.D.

**Corollary 3.5.** If \( x \in \mathbb{R} \setminus \mathbb{Q} \) is of constant type, \( \gamma > 0 \), and \( a > 0 \) then \( h = h(x, \gamma, a) \) exists and is continuous.

Q.E.D.

It remains to find constants \( a, \gamma \) and \( a \) such that \( h = h(x, \gamma, a) \) is an embedding. The next lemma gives us some restrictions. It provides the main estimate for the proof of the embedding.

If \( U \) is an interval of \( S^1 \) define

\[
H(U) = \lim H_d(U) \quad \text{and} \quad V(U) = \lim V_d(U) \quad \text{as} \quad n \to \infty.
\]

**Definition 3.6.** Define \( \tau: \mathbb{Z} \to \mathbb{Z}^+ \) by \( \tau(n) = k \) where \( t_{k-1} < |n| \leq t_k \). Then \( \langle \tau x \rangle \) is an endpoint of a \( W_k \)-interval \( I \) iff \( \tau(j) < k \). If \(|I| = r_k\) then at least one endpoint \( \langle \tau x \rangle \) has \( k - 1 \leq \tau(j) \leq k \).

Otherwise both are endpoints of \( W_{k-1} \)-intervals implying \(|I| \geq r_{k-1} \).

**Lemma 3.7.** Suppose \( \alpha = \sqrt{2} - 1 \). There exist constants \( C_1, C_2, C_3 \) such that if

\[
0 < (1 - \gamma) < 1/1000, \quad a = (1 - \gamma)/4 \quad \text{and} \quad I \text{ is a half closed } W_k \text{-interval with } \tau(n) \geq k - 1 \quad \text{for any } \langle \tau x \rangle \in I
\]

(i) \( C_1 |I|^\gamma < H(U) < C_2 |I|^\gamma \),

(ii) \( |V(U)| < C_3 (1 - \gamma) |I|^\gamma \)

where \( U = \rho^{-1}(I) \).

For the embedding, we need \( |V(U)| < H(U) \). This follows if \( 1 - \gamma \) is sufficiently small. However, the smaller \( 1 - \gamma \), the closer the differentiability of the eventual Denjoy example is to two. There is a sharp, but more difficult, version of this lemma which leads to an embedded curve for any \( 0 < 1 - \gamma < \frac{1}{3} \). One then obtains \( C^{3 - \varepsilon} \) Denjoy examples for any \( \varepsilon > 0 \). Here, we sacrifice sharpness for simplicity, time and again.

In what follows, keep in mind that the actual value of the constants \( C_1, C_2, C_3 \) is not important. That they exist is a consequence, in part, of (1.1) and (1.7): \( q_k, t_k, q_{k-1}, 1/r_k \) are all-
proportional. (We use the fact that \(a_4(a) = 2\) since \(a = \sqrt{2} - 1\).) The constant \(a\) is only used to keep large diagonals disjoint, \(\gamma\) controls the rest of the diagonals.

**Proof.** Let \(\langle jx \rangle\) be the included endpoint. Since \(\tau(n) \geq k - 1\) for \(\langle nx \rangle \in I, k - 1 \leq \tau(j) \leq k\).

Assume \(j > 0\). Then \(t_{k-2} < j < t_k\). By (1.7)

\[ q_k/10 < j < q_k. \]

**Proof of (i).** Since \(j\) is the minimum absolute index of orbit points in \(I\), by (3.1), (cf. (3.3)), we have

\[ |H_a(U) - \left( |I| + |I| \sum_{|i| = 0}^{j-1} g(i) \right) | = \left| \sum_{|i| = j}^{t_{k-1}} \left( |I| - \chi_i \langle i \alpha \rangle \right) g(i) \right| \]

It follows from the uniform convergence of \(H_a\) (3.2) and (2.4(i)) that

\[ \left| H(U) - \left( |I| + |I| \sum_{|i| = 0}^{j-1} g(i) \right) \right| < 4g(j). \]

Therefore

\[ H(U) > 2a|I| \int_{1}^{j-1} x^{-\gamma}dx + |I| - 4g(j) \]

\[ = 2a|I|(j-1)^{1-\gamma}/(1-\gamma) - 4aq^{-\gamma} + (|I| - 2a|I|/(1-\gamma)) \]

\[ > \frac{1}{2}j^{1-\gamma}(1-2(1-\gamma)) \quad \text{since } a = (1-\gamma)/4 \]

\[ > \frac{1}{2}|I|(1/20 - 2(1-\gamma)) \quad \text{by the bounds on } j \text{ and (1.7)} \]

\[ > |I|^\gamma/44 \quad \text{by the bounds on } \gamma. \]

Let \(C_1 = 1/44\). On the other hand,

\[ H(U) < 2a|I| \int_{0}^{j} x^{-\gamma}dx + |I|g(0) + |I| + 4g(j) \]

\[ = 2a|I|j^{-\gamma}/(1-\gamma) + |I|g(0) + |I| + 4g(j) \]

\[ < 2a/q_k(1-\gamma) + 2a/q_k + 1/q_k + 40a/q_k^2 \quad \text{by the bounds on } j, \]

\[ \text{since } q_k|I| < 1, \text{ and } g(0) < 2g(1) = 2a \]

\[ < 2/q_k \quad \text{since } a = (1-\gamma)/4 \]

\[ < 4|I|^\gamma \quad \text{since } q_k|I| > 1/2. \]

Let \(C_2 = 4\).

**Proof of (ii).** By (3.1) (cf. (3.4))

\[ |V(U)| = \left| \sum_{|i| \leq 2k+1} x_i \langle (2i+1)x \rangle g(2i+1) - \sum_{|i| \leq 2i} x_i \langle 2ix \rangle g(2i) \right| \]

\[ < \left| \sum_{|i| \leq 2k+1} \left( x_i \langle (2i+1)x \rangle - |I| \right) g(2i+1) \right| + \left| \sum_{|i| \leq 2i} \left( |I| - x_i \langle 2ix \rangle \right) g(2i) \right| + 2|I|g(j). \]

We would like to apply (2.4(i)) at this point. But the rotation number is 2\(x\), not \(x\), and (2.4(i)) limits application to intervals with length \(r_k(2x)\), not \(r_k(x)\) (Indeed, Kesten proved that the unweighted sum of (2.2(ii)) is unbounded for intervals with length \(\neq r_k(2a)\).) However, since \(a_4(a) = 2\), it follows easily from (1.1), that \(r_k(2x) = r_k(x)\) if \(k\) is even and \(r_k(2x) = 2r_k(x)\) if \(k\) is
odd. Thus for even \( k \), we may apply (2.4(i)) and obtain
\[
|V(U)| \leq (8 + 2|I|) \alpha j < 10 \alpha j^2
\]
where \( j \) is the index of the included endpoint of \( I \). Hence, for even \( k \),
\[
|V(U)| \leq 100 \alpha q \| < 200 \alpha |I|\|^2.
\]

Suppose \( I = [\langle j \alpha \rangle, \langle (j - q_{k-1}) \alpha \rangle] \) where \( k \) is odd. (The proof is similar for \( I = [\langle j \alpha \rangle, \langle (j - q_{k-2}) \alpha \rangle] \) and \( d(j - q_{k-1}) > k - 1 \).) Then \( |I| = r_n \). We decompose \( I \) into segments of length \( r_n \) with \( m \) even:

Let \( p \) be the midpoint of \( I \) and \( I_0 = [\langle j \alpha \rangle, p] \). Let \( J_0 = [\langle j \alpha \rangle, \langle (j + q_k) \alpha \rangle] \) and for \( n \geq 1 \),
\[
J_n = \langle (j + q_k + q_{k+2} + \ldots + q_{k+2n-2}) \alpha \rangle, \langle (j + q_k + q_{k+2} + \ldots + q_{k+2n}) \alpha \rangle.
\]
Then \( |J_n| = r_{k+2n+1} = \alpha^{k+2n+1} \) by (1.4). Since \( a_d(z) = 2, 1/\alpha = 2 + \alpha \). Therefore \( \alpha^2 + 2\alpha - 1 = 0 \) and hence
\[
\alpha^2/2 = \alpha \alpha^{2n-1}, n \geq 0. \text{ Thus } |I_0| = \Sigma |J_n|. \text{ Since the } J_n \text{ are consecutive, } I_0 = \cup J_n.
\]
Let \( U_0 = \rho^{-1}(I_0) \). By (3.8)
\[
|V(U_0)| \leq \sum_{n=0}^{\infty} V(p^{-1}(J_n)) \leq 100(1/j^2 + 1/(j + q_k)^2 + 1/(j + q_k + q_{k+2})^2 + \ldots ) < 120 \alpha q \|
\]
< 240 \alpha |I|^2 \text{ since } j > \alpha/10 \text{ and } a_{n-2} > 5 a_n \text{ for all } n
\]
< 60(1 - y)|I|^2.

The estimate may be doubled for |V(U)|.
Let \( C_\gamma = 120 \). (This estimate is particularly coarse!)

Q.E.D.

Geometry of the embedding \( h:S^1 \to S^1 \times R^1 \). Define \( h:S^1 \to S^1 \times R^1 \) as in (3.1), so that the hypothesis of (3.7) are satisfied. It depends only on \( \gamma \).

Let \( Q = h(S^1) \). Since \( Q \) is compact, there exist real numbers \( \varepsilon_1 < \varepsilon_2 \) such that \( Q \subset A = \{(x, t) \in S^1 \times R^1: \varepsilon_1 < t < \varepsilon_2 \} \). Denote the diagonal \( h(\Delta_{r}) \subset A \) by \( \Delta_{r} \). Let \( y_n \) and \( z_n \) be the images of the endpoints \( y_n < z_n \) of \( \Delta_{r} \).

For \( \beta > 0 \) and \( x \in R^2 \), define \( C_\beta(x) = \{w \in R^2: |\text{slope}(w - x)| \leq \beta \} \), the cone of slope \( \pm \beta \) based at \( x \). For \( x, y \in R^2 \), define \( T_\beta(x, y) \) to be the compact component of \( C_\beta(x) \cap C_\beta(y) \). If \( \|x - y\| < 1 \) then \( T_\beta(x, y) \) projects to a “parallelogram” in the annulus \( A \), also denoted by \( T_\beta \).

If \( U = \langle \langle m \alpha \rangle, \langle n \alpha \rangle \rangle \subset S^1 \) define \( T(U) = T_\beta(U) = T_\beta(z_m, y_n) \). See Fig. 3.

Form a rectangle \( R_n \subset A \) with one edge containing \( y_n \) and slope \( (-1)^n/2 \). Let the endpoints \( p \) and \( q \) of this edge satisfy \( \pi_1(p) - \pi_1(y_n) = \pi_1(z_n) - \pi_1(q) = 2 \delta(n) \). The opposing parallel edge passes through \( z_n \). This determines \( R_n \). We prove the \( R_n \) are disjoint. They are
permuted under a $C^2$ diffeomorphism $f$ of the annulus and have a Cantor limit set. The disks $R_n$ are analogous to the complementary Denjoy intervals in the circle.

Now identify $S'$ with $S' \times \{0\}$ in $S' \times R^1$. Let $R'_n$ be the square centered at the midpoint of $\Delta'_n$; let its sides have length that of $\Delta'_n$. Then the $R'_n$ are disjoint. Extend $h$ continuously to $S' \cup \{R'_n\}$ so that $h|_{R'_n}$ is a homeomorphism onto $R_n$. See Fig. 4. Let $\pi$ be the normal projection of $S' \cup \{R'_n\}$ to $S'$.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{fig4.png}
\caption{Fig. 4.}
\end{figure}

The next lemma describes the underlying geometric pattern used to show $h$ is an embedding and $Q$ is a quasi-circle.

Let $U \subset S'$ and $n \in \mathbb{Z}$. Then $\tau(n)$ is minimal over $U$ if $\langle px \rangle \in U$ implies $\tau(n) \leq \tau(p)$. (See (3.6)). For example, if $U = I$ is a half-closed $W_j$-interval with endpoint $\langle m, x \rangle$ then $\tau(m)$ is minimal over $\text{int}(I)$.

**Lemma 3.9.** Let $0 < \beta < 1$ and $(1 - \gamma) < \beta C_1/2C_3$. If $U = \langle mx \rangle, \langle nx \rangle \subset S_1$, and $\tau(m)$ and $\tau(n)$ are minimal over $U$, then $R_p \subset T_\beta(U)$ for every $\langle px \rangle \in U$.

**Proof.** Suppose $\tau(m)$ is minimal over $U$. Let $\langle px \rangle \in U$. The idea is to write $\langle mx \rangle, \langle px \rangle$ as a union of right closed $W_j$-intervals and show the vertical displacement $V$ of each $W_j$-interval is small compared to its horizontal displacement $H$ (by (3.7)). By “zig-zagging” from $z_m$ to $z_p$ and taking into account the diameter of $R_p$ we conclude that $R_p$ is to the “right” of $z_m$ and $\sigma = |\text{slope}(x - z_m)| < \beta$ for all $x \in R_p$. The analysis is similar for the interval $\langle px \rangle, \langle nx \rangle$.

(See Fig. 3).

More precisely, given $\beta > 0$ let $(1 - \gamma) < \beta C_1/2C_3$.

Let $U' = \langle mx \rangle, \langle px \rangle \subset U'$ with $\tau(m_1)$ minimal over $U'$. Let $j = \tau(m_1)$. Then $t_{j-1} < m_1 \leq t_j$. By the minimality of $\tau(m_1)$ we have $|m_1|, |m_1| \leq t_j$. Then $U_1 = \langle m_0, x \rangle, \langle m_1, x \rangle \rangle$ is a union of right closed $W_j$-intervals. Let $J$ be one of these and $\langle qx \rangle$ its right endpoint. Then $|J| = r_j$ or $r_{j+1}$ and $\tau(q) = j$. Therefore the estimates (i) and (ii) of (3.7) are valid for this decomposition of $U_1$.

Since $\tau(m_1)$ is minimal over $U \setminus U_1$ and $\langle m_1, x \rangle$ is an endpoint of $U \setminus U_1$, we may repeat the process: Choose a point $\langle m_2, x \rangle$ in $U \setminus U_1$ with $\tau(m_2)$ minimal and define $U_2 = \langle m_0, x \rangle, \langle m_2, x \rangle \rangle$.

Break $U_2$ into standard intervals over which we may apply (3.7) and compare $V$ with $H$. Inductively choose $\langle m_k, x \rangle$ in $U \setminus U_1 \setminus U_2 \setminus \ldots \setminus U_{k-1}$ until eventually $\langle px \rangle$ is chosen. Then $U'$ is a union of $W_j$-intervals $J$ satisfying $|V(\rho^{-1}(J))| < C_3(1 - \gamma)|J| < \beta C_1|J|/2 < \beta H(\rho^{-1}(J))/2$.

Let $J_p$ be the $W_j$-interval in this decomposition containing $\langle px \rangle$ and with $|J_p| = r_n$. Then $\tau(p) \geq n - 1$ and $\rho > q_n/10$. It follows that $g(p) < \beta H(\rho^{-1}(J_p))/2$. The rectangle $R_p$ contributes an extra term of $g(p)$. Hence $\sigma < \beta$. Q.E.D.
Recall the constants $C_1, C_3$ of (3.7).

**Corollary 3.10.** If $(1-\gamma)<C_1/6C_3$ then $h:S^1\cup\{R_{x}\}\rightarrow S^1\times R^1$ is an embedding onto $Q\cup\{R_x\}$.

**Proof.** Let $\beta=1/3$. Then $1-\gamma<\beta C_1/2C_3$ as required in (3.9).
Let $w\neq z\in S^1\cup\{R_{x}\}$ and $x=\rho w$, $y=\rho z$.

If $x=y$ then $w, z\in R_{x}$ for some $j\in Z$. Since $h$ embeds $R_{x}$, $h(w)\neq h(z)$.

Suppose $x \neq y$. Let $U$ be an interval of $S^1$ with endpoints $x$ and $y$ and $|U|\leq 1/2$. Choose $\langle px \rangle \in U$ with $t(p)=k$ minimal over $U$. Let $\langle mx \rangle \leq x$ be the $W_k$-point closest to $x$ and $y \leq \langle nx \rangle$ be the $W_k$-point closest to $y$. Let $J_1=\langle mx \rangle, \langle px \rangle$ and $J_2=\langle px \rangle, \langle nx \rangle$ and $J=J_1\cup\langle px \rangle\cup J_2$. Then $\tau(p)$ is minimal over $J$. This implies $\tau(m)$ is minimal over $J$: otherwise there exists $\langle qx \rangle \in J$ such that $\tau(m) > \tau(q) \geq \tau(p) = k$. But $\tau(m) \leq k$ since $\langle mx \rangle$ is a $W_k$-point. Similarly $\tau(n)$ is minimal over $J$. By restriction, the endpoints of $J_1$, $J_2$ and $J$ are minimal over their respective intervals.

It follows from (3.9) and the continuity of $h$ that $h(w)\in T_{1/3}(J_1)\cup R_m$ and $h(z)\in T_{1/3}(J_2)\cup R_m$. It is only necessary to show these two sets are disjoint.

By (3.9) $T_{1/3}(J_1)\cup R_p \cup T_{1/3}(J_2) \subset T_{1/3}(J)$. The diagonal $\Delta_p$ connects $T_{1/3}(J_1)$ and $T_{1/3}(J_2)$ while $\Delta_m$ and $\Delta_n$ are attached to the vertices $z_m$ of $T_{1/3}(J_1)$ and $y_n$ of $T_{1/3}(J_2)$, respectively. See Fig. 5.

3.11. Either $\tau(m)$ or $\tau(n) \geq k-2$. If $\tau(m), \tau(n) \leq k-3$ then $J$ is a union of $W_{k-3}$ intervals. It must contain at least two since $\langle px \rangle \in J$ is a $W_k$-point. Hence $J$ contains an interval of length $r_k$. Any such interval is the union of 7 $W_k$-intervals.

$J$ is also a union of $W_k$-intervals since its endpoints are $W_k$-points. If $J$ contained a half closed $W_{k-1}$ interval with its endpoint $\langle jx \rangle$ then $\tau(j) \leq k-1$, contradicting the minimality of $\tau(p)$. Since $a(x)=2$, any 5 consecutive $W_k$-intervals must contain a $W_{k-1}$-interval. Therefore $J$ can contain no more than 4 half closed consecutive $W_k$-intervals. Therefore $J$ itself is the union of no more than 6 $W_k$-intervals.
3.12. \( R_m, R_n, R_p, T(J_1) \) and \( T(J_2) \) are pairwise disjoint. Since \( \partial T_{1/3}(J) \) has slope \( 1/3 \) and \( \partial R_p \) has slope \( 1/2 \), it follows that \( R_p, T_{1/3}(J_1), T_{1/3}(J_2) \) are disjoint sets contained in \( T_{1/3}(J) \); \( R_m \) and \( R_n \) each are disjoint from \( T_{1/3}(J) \). It remains to see that \( R_m \) and \( R_n \) are disjoint. By (3.11) we may assume \( n(n) \geq k - 2 \). Since \( \langle nx \rangle \) is a \( W_k \)-point, \( t_k - 3 < n \leq t_k \). Let \( K \subset J \) be the open \( W_k \)-interval with right endpoint \( \langle nx \rangle \). By (1.7) \( |K| \geq r_{k+1} > 1/2q_{k+1} \) and \( q_{k+1} < 10t_{k-1} < 60t_{k-3} \). Then \( |K| > 1/120|n| \). Hence

\[ |T_{1/3}(J)| > |T_{1/3}(K)| > H(\rho^{-1}(K)) > C_1|K|^s > C_1/(120|n|)^s > 3|R_n|. \]

It follows from simple geometry that \( R_m \) and \( R_n \) are disjoint no matter what the length of \( \Delta_m \).

Hence \( h(w) \neq h(z) \) if \( w \neq z \). Thus \( h \) is an embedding.

Q.E.D.

**Definition.** A Jordan curve \( J \) contained in a metric space with metric \( d \) is called a quasi-circle (in the sense of Ahlfors [2]) if there exists a constant \( K > 0 \) such that for \( x \neq y \in J \) then one of the arcs of \( J \) connecting \( x \) and \( y \) is contained in a disk of diameter \( Kd(x, y) \).

We prove next that \( Q \) is a quasi-circle and slightly more, that \( Q \cup \{R_n\} \) has a quasi-structure. Since \( h \) is an embedding we may work with \( h^{-1} \).

**Theorem 3.13.** There exists a constant \( K > 0 \) such that if \( x, y \in Q \cup \{R_n\} \) then one of the arcs connecting \( h(h^{-1}(x)) \) and \( h(h^{-1}(y)) \) together with the line segments \( (x, h(h^{-1}(x))) \) and \( (y, h(h^{-1}(y))) \) are contained in a disk of radius \( Kd(x, y) \). In particular, \( Q \) is a quasi-circle.

**Proof.** Let \( x \neq y \in Q \cup \{R_n\} \). As in (3.10) we have \( x \in R_m, y \in R_n \). An arc connecting \( h(h^{-1}(x)) \) and \( h(h^{-1}(y)) \) consists of a portion in \( T(J_1) \), a portion in \( T(J_2) \), the entire diagonal \( \Delta_p \subset R_p \) and possible segments of \( \Delta_m \subset R_m \) and \( \Delta_n \subset R_n \) (if \( x \in R_m \) or \( y \in R_n \), respectively). By (3.12) these sets are disjoint. The theorem follows by simple plane geometry.

Q.E.D.

**Proposition 3.14.** \( Q \) has winding number 1.

**Proof.** This follows from (3.9) since \( Q \subset T(y_0, z_0) \cup cl(\Delta_0) \).

Q.E.D.

**Definition.** Given a metric space \( X \), for each non-negative real number \( s \) there is a corresponding \( s \)-dimensional Hausdorff measure \( \mu_s \) defined as follows. Let \( B \subset X \) be an arbitrary set. The zero-dimensional measure \( \mu_0 \) is the number of points in \( B \). For \( s > 0 \), let

\[
\mu_{s,x}B = \inf \sum_i [\text{diam}(B_i)]^s,
\]

where the infimum is taken over all covers \( \{B_i\} \) of \( B \) such that \( \text{diam}(B_i) < \alpha \) for each \( i \). Then

\[
\mu_sB = \lim_{\alpha \to 0^+} \mu_{s,x}B.
\]

A set \( B \) has Hausdorff dimension \( s \), denoted \( HD(B) = s \), iff \( \mu_sB = 0 \) for all \( r > s \) and \( \mu_sB = \infty \) for all \( r < s \). If \( HD(B) = s \), then \( \mu_sB \) is the Hausdorff measure of \( B \) within its dimension.

**Theorem 3.15.** The Hausdorff dimension of \( \Gamma \) is \( 1/\gamma \). The Hausdorff measure of \( \Gamma \) within its dimension is a positive, real number.

**Proof.** Let \( S = S^1 \setminus \{\langle nx \rangle\} \). Define \( g:S \to \mathbb{R}^2 \) by \( g = h \circ \rho^{-1} \).
Claim. There exist constants $A_1, A_2, A_3$ and $K_1$ such that, (1) If $I$ is a $W_k$-interval, $k \geq 0$, then
\[ A_2 |I| < |g(I)| < A_1 |I| \cdot \]
(2) If $J$ is an arbitrary interval of $S^1$ then,
\[ |g(J)| > A_3 |J| \cdot \]
(3) If $B \subset \mathbb{R}^2$ is a disk with $B \cap g(S) \neq \emptyset$ then
\[ cl(g^{-1}(K_1 B)) \supset \text{interval} \supset g^{-1}(B) \cdot \]

Proof of claim. (1) This follows from (3.7). (2) The ratios of lengths of $W_k$-intervals and $W_k$-intervals is bounded from 0 and $\infty$. For each $k \geq 0$ the collection of $W_k$-intervals forms a cover of $S$. Hence there exists a constant $C$ (uniform over intervals $J$) and a $W_k$-interval $I \subset J$ and $|I| \geq C |J|$. Then $|g(J)| \geq |g(I)| > A_2 |I| > A_3 |J|$. (3) This follows since $g(S) \subset \Gamma \subset Q$ and $Q$ is a quasi-circle (3.13).

By (1) the cover $\bigcup g(I)$, $I \in W_k$ satisfies $\sum |g(I)|^{1/\gamma} \leq A_1^{1/\gamma} \sum |I| \leq A_1^{1/\gamma}$. Hence $\mu_{1/\gamma}(g(S)) < \infty$ implying
\[ HD(g(S)) \leq 1/\gamma. \]

Suppose that $HD(g(S)) < 1/\gamma$. Then $\mu_{1/\gamma}(g(S)) = 0$. Then there is a cover of $g(S)$ by disks $B_i$ such that $|B_i|^{1/\gamma} < \delta$ for any $\delta$. Notice that $|K_1 B_i|^{1/\gamma} < K_1^{1/\gamma} \delta$. Consider the cover $\{K_1 B_i\}$. Using (3) for each $i$ we pick $I_i$ such that
\[ g^{-1}(B_i) \subset \Gamma \subset cl(g^{-1}(K_1 B_i)). \]

Therefore by (2) we have
\[ A_2 |I_i| < |g(I_i)| \leq K_1 |B_i|. \]

Hence $|I_i| < (K_1 |B_i| / A_3)^{1/\gamma}$. Since $\bigcup I_i$ covers $S$ we have
\[ 1 \leq \sum |I_i| \leq (K_1 / A_3)^{1/\gamma} \sum |B_i|^{1/\gamma}. \]

This contradicts the assumption that $\mu_{1/\gamma} = 0$. From the preceding paragraph we have $0 < \mu_{1/\gamma}(g(S)) < \infty$. The theorem follows since $cI(g(S)) = \Gamma$.

Q.E.D.

The author thanks Curt McMullen for his assistance in the preceding proof.

§4. A $C^{1+\epsilon}$ DIFFEOMORPHISM $F: A \to A$

Recall $\Gamma$, the Cantor set $Q \setminus \cup \{\Delta_n\}$. Define a homeomorphism $f_1: \Gamma \to \Gamma$ by $f_1(y_\ast) = y_{\ast+2}$, $f_1(z_\ast) = z_{\ast+2}$ and if $x \notin cl(\{\Delta_n\})$
\[ f_1(x) = h_0^{-1} R_{2\xi} h_0^{-1}(x). \]

Lemma 4.1. There exists $C > 0$ such that if (i) $x$ and $y$ are the endpoints of a diagonal $\Delta_n \subset Q$ or (ii) $x$ and $y$ are the endpoints of $hp^{-1}(I)$ where $I$ is a half closed $W_k$-interval with $\tau(f) \geq k - 1$ for any $\langle \xi \rangle \in I$ then
\[ ||f_1(x) - f_1(y) - (x - y)|| < C \|x - y\|^{1 + 1/\gamma}. \]
Proof. Note that \(1/n^\gamma - 1/(n+2)^\gamma < 2/n^{1+\gamma}\) for all \(n \geq 1\). (i) If \(x\) and \(y\) are the endpoints of \(\Delta_n\), then,
\[
\|f_r(x) - f_r(y) - (x - y)\| = \sqrt{2(g(n) - g(n+2))} = \sqrt{2a(1/n^\gamma - 1/(n+2)^\gamma)} < 2\sqrt{2a/n^{1+\gamma}} - A_1\|x - y\|^{1+\gamma} \quad \text{for some } A_1 > 0.
\]

(ii) Suppose \(x\) and \(y\) are the endpoints of \(h_\rho^{-1}(I)\) and \(I = [\langle px \rangle, \langle qx \rangle]\) is a \(W_k\)-interval of length \(r_k\), \(\tau(p) \geq k - 1\). Let \(U = \rho^{-1}(I), \quad U' = \rho^{-1}(R_2(I))\). Then
\[
\|\pi_1 f_r(x) - \pi_1 f_r(y) - \pi_1 (x - y)\| = |H(U) - H(U')|
\]
\[
= \sum_{n=1}^{\infty} \chi_t(\langle nx \rangle)(g(n) - g(n+2)) < 2a \sum_{n=1}^{\infty} \chi_t(\langle nx \rangle)/n^{1+\gamma}
\]
\[
< 2a |l| \sum_{n=1}^{\infty} 1/n^{1+\gamma} + 8a/p^{1+\gamma} \quad \text{(by } 2.4)(i)) \text{).}
\]

Since \(k - 1 \leq \tau(p) \leq k\) apply (1.7) to get \(q_k/10 < p < q_k\); also \(1/2 < q_k |l| < 1\). It follows that there exists \(A_2 > 0\) with
\[
\|\pi_1 f_r(x) - \pi_1 f_r(y) - \pi_1 (x - y)\| < A_2 |l|^{1+\gamma}.
\]

On the other hand,
\[
\|\pi_2 f_r(x) - \pi_2 f_r(y) - \pi_2 (x - y)\| = |V(U) - V(U')|
\]
\[
= \sum_{n=[p/2]}^{\infty} \chi_t(\langle 2nx \rangle)(g(2n) - g(2n+2)) - \chi_t(\langle (2n+1)x \rangle)(g(2n+1) - g(2n+3))
\]
\[
< 2a \sum_{n=[p/2]}^{\infty} \chi_t(\langle 2nx \rangle)/(2n)^{1+\gamma} + \chi_t(\langle (2n+1)x \rangle)/(2n+1)^{1+\gamma}
\]
\[
< 4a |l| \sum_{n=[p/2]}^{\infty} 1/(2n)^{1+\gamma} + 16a/p^{1+\gamma}
\]
\[
< A_3 |l|^{1+\gamma} \text{ as above.}
\]

Hence
\[
\|f_r(x) - f_r(y) - (x - y)\| < (A_2 + A_3)|l|^{1+\gamma}. \quad (4.2)
\]

By (3.7)(i) \(C_1 |l|^\gamma < H(\rho^{-1}(I)) \leq \|x - y\|\). Let \(C = (A_1 + A_2 + A_3)/C_1^{1+\gamma}\). If \(x\) and \(y\) satisfy either (i) or (ii),
\[
\|f_r(x) - f_r(y) - (x - y)\| < C \|x - y\|^{1+\gamma}.
\]

Q.E.D.

**Lemma 4.3.** There exists \(C_r > 0\) such that if \(x, y \in \Gamma\) then
\[
\|f_r(x) - f_r(y) - (x - y)\| < C_r \|x - y\|^{1+\gamma}.
\]

**Proof.** Let \(K\) be the quasi-circle constant for \(Q\). Suppose \(x, y \in \Gamma\). At least one of the arcs \(A(x, y)\) connecting \(x\) and \(y\) is contained in a disk of diameter \(K \|x - y\|\). Assume \(x < y\) in \(A(x, y)\). If \(x \in \Delta_m\) define \(x' = z_m\), otherwise let \(x' = x\). If \(y \in \Delta_m\) define \(y' = y_m\), otherwise let \(y' = y\). Then
\[
\|f_r(x) - f_r(y) - (x - y)\| \leq \|f_r(x) - f_r(x') - (x - x')\| + \|f_r(y) - f_r(y') - (y - y')\|
\]
\[
+ \|f_r(x') - f_r(y') - (x' - y')\|.
\]

The first two terms are bounded above by \(C(\|x - x'\|^{1+\gamma} + \|y - y'\|^{1+\gamma})\) by (4.1)(i).
We use (4.2) to estimate the third term for \( x' \neq y' \). Let \( U \) be the open interval \( \rho^{-1}(A(x', y')) \). Let \( I_j \subset U \) be a half closed \( W'_i \)-interval with length \( r_j \) and \( l \) minimal. \( I_j \) must satisfy \( t(j) \geq 1 - 1 \) for any \( \langle jx \rangle \epsilon I_i \). Beginning with \( I_j \) we may inductively write \( U \) as an infinite union of no more than 8 half closed \( W'_k \)-intervals, \( k \geq l \), satisfying the hypothesis of (3.7). Then by (4.2)

\[
\|f(x') - f(y') - (x' - y')\| \leq 8(A_2 + A_3)\Sigma r_k^{1+\gamma}, \quad k \geq l
\]

\[
\leq 16(A_2 + A_3)r_k^{1+\gamma} \quad \text{since } r_k = 2^k (1.4)
\]

\[
= 16(A_2 + A_3)\|I\|^{1+\gamma}
\]

\[
< 16CH(\rho^{-1}(I_j))^{1+1/\gamma} \quad \text{by (3.7)(i) and the definition of } C \quad (4.1)
\]

Since \( h(\rho^{-1}(I_j)) \subset A(x, y) \) and \( x', y' \in A(x, y) \) we have

\[
\|x - x'\|, \quad \|y - y'\|, \quad H(\rho^{-1}(I_j)) < 2K\|x - y\|.
\]

Hence there exists \( C_\tau \) such that

\[
\|f(x') - f(y') - (x' - y')\| \leq C_\tau\|x - y\|^{1+1/\gamma}.
\]

Q.E.D.

**Definition of \( f_\tau^* \):** \( R \to R + 2 \). For \( n \in Z \) define \( G_n(x) = (g(n + 2)/g(n))(x - z_n) + z_{n+2} \) for \( x \in R^2 \). (Recall \( g(n) = \|y_n - z_n\|/\sqrt{2} \).) Define

\[
H_n(x) = \begin{cases} 
  x - z_n + z_{n+2} & \text{if } x \in D(z_n, g(n)/4) \\
  y_n + y_{n+2} & \text{if } x \in D(y_n, g(n)/4).
\end{cases}
\]

Extend \( H_n \) to \( R^2 \) arbitrarily.

Let \( \phi \) be a smooth bump function on \( R^2 \) which is identically one on \( D(0, 1/5) \) and vanishes off \( D(0, 1/4) \). Let

\[
\phi_n(x) = \begin{cases} 
  \phi((x - z_n)/g(n)) & \text{if } x \in D(z_n, g(n)/4) \\
  \phi((x - y_n)/g(n)) & \text{if } x \in D(y_n, g(n)/4).
\end{cases}
\]

Let \( \phi_n(x) = 0 \) elsewhere.

Define

\[
f_n(x) = \phi_n(x)H_n(x) + (1 - \phi_n(x))G_n(x).
\]

Recall the constant \( C_\tau \) of (4.3).

**Lemma 4.4.** There exists \( C' > 0 \) such that if \( \|y_n - z_n\| < (3C'C_\tau)^{-\gamma} \) then,

(i) \( Df_n(p_n) = 1d \) and \( D^2f_n(p_n) = 0 \) for \( p = y \) or \( z \);

(ii) \( \|D^rf_n\| < C'(g(n) - g(n+2))/g(n) \) for \( 1 \leq r \leq 3 \);

(iii) \( f_n \) maps \( \Delta_n \subset R_n \) diffeomorphically onto \( \Delta_n + 2 \subset R_n + 2 \).

**Proof:** (i) Simply observe \( f_n - H_n \) in neighborhoods of \( y_n \) and \( z_n \). (ii) This is a straightforward calculation from the definitions. (iii) \( G_n \) preserves boundary components since it is affine and \( R_n + 2 \) is a homothetic replica of \( R_n \). Inside \( D(p_n, g(n)/4) \), \( p = y \) or \( z \), \( H_n \) preserves boundary components since it is a translation. Thus any convex combination of the two preserves boundary components.

Let \( x \in R_n \). By (i) and (ii) and since \( R_n \) is convex,

\[
\|Df_n(x) - 1d\| = \|Df_n(x) - Df_n(z_n)\| < C'\|x - z_n\| \|D^2f_n\|
\]

\[
< 3C'(g(n) - g(n+2))/g(n) < 3C'C_\tau\|y_n - z_n\|^{1/\gamma} \quad \text{by (4.3)}.
\]

If \( \|y_n - z_n\| < (3C'C_\tau)^{-\gamma} \) then \( \|Df_n(x) - 1d\| < 1 \).
The open unit ball of linear functions centered at the identity are all invertible. Thus \(Df_n(x)\) is invertible for \(\|y_n-z_n\| < (3C'C_r)^{-\gamma}\).

Hence \(f_n\) sends \(\Delta_n\) and the boundary of \(R_n\) diffeomorphically onto \(\Delta_{n+2}\) and the boundary of \(R_{n+2}\). Any smooth mapping which sends the boundary of a disk \(D_1\) diffeomorphically onto the boundary of another disk \(D_2\) and which is locally invertible, is a diffeomorphism from \(D_1\) to \(D_2\).

Q.E.D.

For \(\|y_n-z_n\| \geq (3C'C_r)^{-\gamma}\) redefine \(f_n\) to satisfy (i) and (iii) above. They will satisfy (ii) automatically for some constant \(C > C'\). Hence

**Lemma 4.5.** There exists a constant \(C > 0\) such that, if \(n \in \mathbb{Z}\),

(i) \(Df_n(p_n) = I_d\) and \(D^2f_n(p_n) = 0\) for \(p = r, z\);

(ii) \(\|Df_n\| < C(g(n) - g(n+2))/g(n)^{r}\) for \(1 \leq r \leq 3\);

(iii) \(f_n\) maps \(\Delta_n \subset R_n\) diffeomorphically onto \(\Delta_{n+2} \subset R_{n+2}\).

Q.E.D.

If \(x \in \Gamma \cup \{R_n\}\), define

\[
\theta_0(x) = \begin{cases} 
1d, & x \in \Gamma \\
Df_n(x), & x \in R_n 
\end{cases}
\]

\[
\theta_1(x) = \begin{cases} 
1d, & x \in \Gamma \\
Df_n(x), & x \in R_n 
\end{cases}
\]

**Theorem 4.6.** Let \(\delta = 1/\gamma - 1\). There exists \(B > 0\) such that for \(x, y \in Q \cup \{R_n\}\)

(i) \(\|\theta_2(x) - \theta_2(y)\| < B\|x - y\|^\delta\),

(ii) \(\frac{\|\theta_1(x) - \theta_1(y) - \theta_2(y)(x - y)\|}{\|x - y\|} < B\|x - y\|^\delta\),

(iii) \(\frac{\|\theta_0(x) - \theta_0(y) - \theta_1(y)(x - y) - \theta_2(y)(x - y)^2/2\|}{\|x - y\|^2} < B\|x - y\|^\delta\).

**Proof:** Suppose \(x, y \in R_n\). Since \(R_n\) is convex, (i)–(iii) are all estimated by \(\|x - y\|\|D^3f_n\|\) by Taylor’s theorem. By (4.5)(ii) this is bounded by \(\|x - y\|C(g(n) - g(n+2))/g(n)^{3} < C \|x - y\|^\delta\) for some \(C > 0\) since \(\gamma < 1\).

Suppose \(x, y \not\in \Gamma\). By (4.3)

\[
\|f_1(x) - f_1(y)\| < C_1 \|x - y\|^{1 + 1/\gamma} - C_1 \|x - y\|^{2 + \delta},
\]

Since \(\theta_1|\Gamma = 1d\) and \(\theta_2|\Gamma = 0\) (i)–(iii) are valid for \(B > C_1\).

The general case is proved with the triangle inequality and the fact that \(Q\) is a quasi-circle.

Let \(x, y \in Q \cup R_n\) and not in the same disk \(R_n\) and not both in \(\Gamma\). Let \(K\) be the quasi-constant for \(Q \cup \{R_n\}\). There exists an arc \(A(x, y) \subset Q \cup \{R_n\}\) connecting \(x\) and \(y\) and contained in a disk of diameter \(= K\|x - y\|\). If \(x \in R_n\), let \(x'\) be a boundary point of \(R_n \cap A(x, y)\). Otherwise let \(x = x'\). Similarly define \(y'\). Then \(x', y' \in \Gamma\).

Since \(x, y, x', y'\) are all in the arc \(A(x, y)\),

\[
\|x' - y'\|, \|x - x'\|, \|y - y'\| < K\|x - y\|.
\]
(i) \[ \|\theta_2(x) - \theta_2(y)\| \leq \|\theta_2(x) - \theta_2(x')\| + \|\theta_2(x') - \theta_2(y')\| + \|\theta_2(y') - \theta_2(y)\| \]
\[ \leq C_k(\|x - x'\|^\delta + \|y - y'\|^\delta + C_T\|x' - y'\|^\delta) \]
\[ \leq (2C_k + C_1)k^\delta \|x - y\|^\delta \]
\[ = B_1\|x - y\|^\delta. \]

(ii) \[ \frac{\|\theta_1(x) - \theta_1(y) - \theta_2(x - y)\|}{\|x - y\|} \leq \frac{\|\theta_1(x) - \theta_1(x')\| + \|\theta_1(x') - \theta_1(y')\| + \|\theta_1(y') - \theta_1(y)\|}{\|x - x'\|} + \frac{\|\theta_2(y) - \theta_2(y')\|}{\|x - y\|} \]
\[ \leq K\frac{\|\theta_1(x) - \theta_1(x')\|}{\|x - x'\|} + K\frac{\|\theta_1(y') - \theta_1(y'')\|}{\|y' - y\|} + \frac{\|\theta_2(y) - \theta_2(y')\|}{\|x - y\|} \]
\[ \leq KC_k\|x - x'\|^\delta + KC_R\|y' - y\|^\delta + C_R\|y - y'\|^\delta \]
\[ \leq 3K^{1+\delta}C_R\|x - y\|^\delta - B_2\|x - y\|^\delta. \]

(iii) \[ \frac{\|\theta_0(x) - \theta_0(y) - \theta_1(x - y) - \theta_2(x - y)^2/2\|}{\|x - y\|^2} \]
\[ \leq \frac{\|\theta_0(x) - \theta_0(x') - (x - x')\|}{\|x - x'\|^2} + \frac{\|\theta_0(x') - \theta_0(y') - (x' - y')\|}{\|x' - y'\|^2} + \frac{\|\theta_0(y') - \theta_0(y) - (y' - y)\|}{\|y' - y\|^2} \]
\[ \leq 3KC_R\|x - y\|^\delta + C_R\|y' - y\|^\delta + C_R\|y - y'\|^\delta \leq B_1\|x - y\|^\delta \text{ for some } B_3 \geq 0. \]

The first term of \(*\) is bounded by \[ K^2\left(\frac{\|\theta_0(x) - \theta_0(x') - (x - x')\|}{\|x - x'\|^2} + \frac{\|\theta_0(x') - \theta_0(y') - (x' - y')\|}{\|x' - y'\|^2} + \frac{\|\theta_0(y') - \theta_0(y) - (y' - y)\|}{\|y' - y\|^2}\right) \]
\[ \leq K^2(C_R\|x - x'\|^\delta + C_T\|x' - x\|^\delta + C_R\|y' - y\|^\delta) \leq B_4\|x - y\|^\delta. \]

The last two terms of \(*\) are non-zero only if \(y \in R_\alpha\). In this case \[ \frac{\|\theta_1(y) - y\|}{\|x - y\|} \leq K\frac{\|\theta_1(y) - \theta_1(y')\|}{\|y' - y\|} \leq KC_R\|y' - y\|^\delta \leq K^{1+\delta}C_R\|x - y\|^\delta = B_4\|x - y\|^\delta. \]

And \[ \|\theta_2(y)\| = \|\theta_2(y) - \theta_2(y')\| \leq C_R\|y - y'\|^\delta \leq C_RK^{1+\delta}\|x - y\|^\delta = B_4\|x - y\|^\delta. \]

Let \[ B = B_1 + B_2 + B_3 + B_4 + B_5. \]

The first conditions of Whitney extension theorem have been met. (See [1] or [20].)

**Corollary 4.7.** There exists a \(C^{2+\delta}\) mapping \(F:S^1 \times R^1 \rightarrow S^1 \times R^1\), \(\delta = 1/\gamma - 1\), such that if \(x \in Q \cup \{R_\alpha\}\) then \(F(x) = \theta_0(x)\), \(DF_x = \theta_1(x)\) and \(D^2F_x = \theta_2(x)\).

Q.E.D.

In summary we have proved

**Proposition 4.8.** Suppose \(f: \Gamma \rightarrow \Gamma\) is a homeomorphism and \[ \|f(x) - f(y) - (x - y)\| < C\|x - y\|^{2+\delta} \]
for all \(x, y \in \Gamma\) where \(x\) and \(y\) are endpoints of \(\Delta\) or \(x\) and \(y\) are the endpoints of \(h^{-1}(I)\) where \(I\) is a half closed \(W_\kappa\)-interval with \(\tau(j) \geq k + 1\) for any \(j \alpha \in \ell\). There exists a \(C^{2+\delta}\) embedding \(F\) of a
neighborhood of $Q \cup \{R_{\alpha}\}$ into $U$ such that,

(i) $F|\Gamma = f$, $DF|\Gamma = \text{Id}$, $D^2 F|\Gamma = 0$;
(ii) $F$ maps each $\Delta_{n} \subset R_{\alpha}$ diffeomorphically onto $\Delta_{n+2} \subset R_{n+2}$.

Q.E.D.

Since $F$ is a diffeomorphism of $Q \cup \{R_{\alpha}\}$, there exists a smooth neighborhood $N$ of $Q \cup \{R_{\alpha}\}$ on which $F$ is a $C^{2+\theta}$ embedding into $S^1 \times R^1$.

Since $Q \cup \{R_{\alpha}\}$ is $F$-invariant and $Q$ has winding number 1, (see (3.14)) we may assume that $N$ is a smooth deformation retract of the annulus $A$ and that its image $F(N)$ is contained in $A$. $F(N)$ is therefore a smooth deformation retract of $A$. Denote these two retractions by $d_N: A \to N$ and $d_{F(N)}: A \to F(N)$. Then define $f: A \to A$ by $f(x) = d_N^{-1}(F(d_{F(N)}(x)))$. This is a $C^{2+\theta}$ diffeomorphism of the annulus extending $f: Q \cup \{R_{\alpha}\} \to Q \cup \{R_{\alpha}\}$. This establishes Theorem A.

REFERENCES


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