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C^2 COUNTEREXAMPLES TO THE SEIFERT CONJECTURE

J. HARRISON

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INTRODUCTION

IN 1974 [11] Paul Schweitzer published his celebrated C^1 counterexample to the Seifert conjecture which asserted;

Every vector field on the three sphere S^3 has either a zero or a closed integral curve.

His construction has two main steps. In the first he embeds the C^1 Denjoy vector field [2] in a smooth, thickened, punctured torus T_0 in S^3. The way T_0 is embedded in S^3 is called a clerical collar. (See Fig. 1.) The vector field on T_0 is vertical at ∂T_0 and lets it be used as a periodic orbit annihilator or flow plug in the manner of Fuller [3] and Wilson [15]. This is Schweitzer's second step.

Denjoy's vector field on the two-torus cannot be made C^2, so Schweitzer's example left open the possibility that the Seifert Conjecture is true for C^r vector fields on S^3 with $r \geq 2$. Estimates relating dimension and differentiability in [4] led the author to the simple idea of smoothing the Denjoy vector field by sacrificing the smoothness of the torus. Indeed, the more smooth the vector field, the worse the toral structure. The Hausdorff dimension of the torus equals the degree of differentiability of the vector field; so $r = 3$ is a natural limit to these methods. cf [5] and [8].

Let $S^n = \{x \in \mathbf{R}^{n+1} : \|x\| = 1\}$, the n-sphere.

The construction begins with a Denjoy diffeomorphism

$$D: S^1 \to S^1.$$

Dynamically, D has no periodic points and has an exceptional orbit. (An exceptional orbit is one that is dense in a Cantor set.) We embed D in a diffeomorphism of S^2

$$f: S^2 \to S^2.$$

The north pole N is repelling, the south pole S is attracting. One orbit is asymptotic to both N and S. Apart from that, the dynamics of f is like that of D: there are no periodic points in $S^2 \setminus (\mathrm{N} \cup \mathrm{S})$ and there is an exceptional orbit.

The last step is to "Schweitzer-suspend" f to obtain a tangent vector field on S^3

$$X: S^3 \to \mathbf{R}^4$$

which has no zeros and no closed integral curves. It is as smooth as the diffeomorphism f.

Such vector fields X exist of class $C^{3-\varepsilon}$ for arbitrary $\varepsilon > 0$. In this paper we construct X with $1 - \varepsilon$ small, sacrificing sharpness for simplicity of the estimates.

THEOREM A. *Suppose there exists a C^r orientation preserving diffeomorphism $f: S^2 \to S^2$ with no other periodic points than the fixed repelling north pole N and the fixed attracting south pole S. Suppose there exist one orbit asymptotic to both N and S and one orbit that is not. Then there exists a C^r vector field on S^3 which has no zeros and no closed integral curves.*

Theorem A describes a dynamic component $f: S^2 \to S^2$ more fundamental than a Seifert counterexample. Since it is a diffeomorphism in dimension two rather than a vector field in dimension three, it is more amenable for study.

Identify S^1 with $\mathbf{R}^1 \backslash \mathbf{Z}^1 \cong [0, 1)$. Denote the annulus $S^1 \times [-1, +1]$ by $\mathbf{A}$ and its boundary components $S^1 \times \{+1\}$ by $\partial^+ A$ and $S^1 \times \{-1\}$ by $\partial^- A$.

Let **Id** be the identity transformation, $Id(x) = x$.

Proof of Theorem A. The existence of $f: S^2 \to S^2$ of Theorem A is equivalent to the existence of a C^r orientation preserving diffeomorphism $f_1: A \to A$ such that;

(i) $f_1 | \text{int}(A)$ has no periodic points;

(ii) There exists $\delta > 0$ such that if $B^+ = \{(x, t): 1 - \delta < t < 1\}$ and $B^- = \{(x, t): -1 < t < -1 + \delta\}$ and $(x, t) \in B^+ \cup f_1^{-1}(B^-)$ then $f_1(x, t) = (x, t')$ where $t' < t$; $f_1 | \partial A = Id$; and $B^+ \cap B^- = \varnothing$.

(iii) There exists a point $q \in A$ whose orbit is bounded away from ∂A.

(iv) There exists a point $p \in f_1(B^+)$ such that $f_1(p) \in f_1^{-1}(B^-)$.

Since f_1 is orientation preserving, it is isotopic to $f_0 = Id$ by a C^r isotopy f_s. (See [13].) By (ii) we may assume that the isotopy decreases t-levels in $B^+ \cup f^{-1}(B^-)$, i.e., $f_s(x, t) = (x, t')$ where $t' \leq t$ and $f_s | \partial A = Id$. For s near 0, let $f_s = Id$ and for s near 1, let $f_s = f$.

Let $A \times S^1$ have coordinates (x, t, s) where $(x, t) \in A$ and $s \in [0, 1)$.

The isotopy defines a suspension flow $F_u: A \times S^1 \to A \times S^1$ by

$$F_u(x, t, s) = (f_{s+u} \circ f_s^{-1}(x, t), s + u).$$

The flow conditions $F_{u+v} = F_u \circ F_v$ and $F_0 = Id$ are easily verified. By the chain rule, F_u is clearly C^r away from the slice $A \times \{0\}$. Since the isotopy is constant near $s = 0$ or $s = 1$, the flow is trivial in a neighborhood of $A \times \{0\}$: $F_u(x, t, s) = (x, t, s + u)$. Thus F_u is C^r on $A \times S^1$.

If $K \subset A$, define the *suspension* of K to be $\{F_u(K \times \{0\}): 0 \leq u \leq 1\}$.

Let B denote the suspension of $B^+ \cup f^{-1}(B^-)$. Then the suspension η of the entire orbit of q is disjoint from B. Otherwise the orbit of q meets ∂A, contradicting (iii). The suspension ζ of p is a C^r graph – it passes through each s-slice only once. It is disjoint from B since $p \notin B^+ \cup f_1^{-1}(B^-)$.

Let F' denote the C^{r-1} tangent vector field of the flow F_u. It follows from (i) that F' has no closed integral curves on int$(A \times S^1)$. It has no zeros since it is a suspension.

Let T be the thickened torus $S^1 \times [-2, 2] \times S^1$ with coordinates (x, t, s). It contains $A \times S^1$ in its interior. Let N be the vector field $\partial/\partial t$ defined on T.

Choose a smooth, real-valued function ψ which is 1 on $(A \times S^1)\backslash B$, 0 on $T\backslash(A \times S^1)$ and $0 < \psi < 1$ on B. Let

$$Y = \psi F' + (1 - \psi) N.$$

Y has no zeros and has no closed integral curves: On $T\backslash(A \times S^1)$, where $\psi = 0$, we have $Y = N = \partial/\partial t$. On B, we have $0 < \psi < 1$ and both F' and N are t-level reducing. Since N strictly reduces t-levels, Y is strictly t-level reducing on $(T\backslash A \times S^1) \cup B$. The dynamics on $(A \times S^1)\backslash B$ is identical to that of F'. Therefore there are no zeros and no closed integral curves.

Since $\psi | (A \times S^1)\backslash B = 1$ and η and ζ are disjoint from B, they are contained in maximal integral curves η' and ζ' of Y. But $\eta = \eta'$ since η is already maximal. The curve ζ' enters on the outer boundary of T at a point p' and exits on the inner boundary of T at q'. Let us verify that ζ' is unknotted: The "ends" of ζ', that is, the two components of $\zeta'\backslash\zeta$, may be continuously isotoped to become vertical without disturbing ζ. Since ζ is a graph, the new curve may be isotoped to become a graph disjoint from the $s=0$ slice of T. These isotopies may be realized by an ambient isotopy of T. Therefore ζ' is unknotted.

There exists a small disk $D\subset\partial T$ containing p' such that $U=\{$integral curves of Y meeting $D\}$ is a C^r tubular neighborhood of ζ'. Then Y is tangent to ∂U. Let $T_0=T\backslash U$. Since ζ' is unknotted there exists a C^r diffeomorphism h of R^3 such that $h(T_0)$ is the Schweitzer clerical collar (see Fig. 1).

The non-zero vector field

$$Z=hYh^{-1},$$

on $h(T_0)$ satisfies the necessary properties to make a flow plug. (See [9]). That is, in R^3 coordinates (x,y,z), $Z=-\partial/\partial z$ in a neighborhood of $\partial h(T_0)$, there are no closed integral curves in $h(T_0)$, and there is one integral curve $h(\eta)$ contained entirely in $h(T_0)$. Extend with Z' which has the "mirror image property" with respect to Z. (See [15], [11] or Fig. 1.) Assume that the domain of $Z\cup Z'$ is contained in the unit cube C in R^3. Use $-\partial/\partial z$ to extend $Z\cup Z'$ to a non-zero vector field P defined on all of C. Then at least one integral curve of P enters the top face of C and never exits. Otherwise, the entering integral curves would completely foliate C, contradicting the existence of $h(\eta)$. By the mirror image property, if any integral curve enters C also exits, it does so directly below where it entered. Therefore P has no closed integral curves. Choose a C^∞ non-zero vector field on S^3 with only finitely many closed integral curves. For each of these curves choose a flow box meeting it. They may be chosen to be disjoint. Replace the vector field in each flow box by a copy of P so that the previously closed integral curve enters it and never exits. No new closed integral curves are introduced. The resulting vector field V on S^3 has no zeros and no closed integral curves. Its flow G_t is of class C^r.

According to Hart [10] there exists a C^r diffeomorphism of S^3 conjugating G_t to a flow which is generated by a C^r vector field X. Since X is conjugate to V it also has no zeros and no closed integral curves.

Q.E.D.

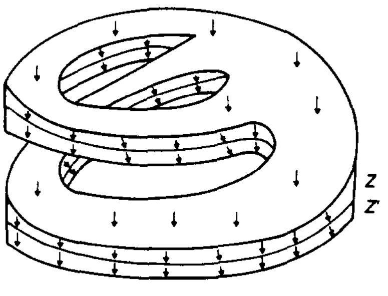

Fig. 1.

The object of this paper, then, is to prove

THEOREM B. *For* $(1-\varepsilon)$ *sufficiently small there exists a* $C^{3-\varepsilon}$ *diffeomorphism* $f{:}A\to A$ *satisfying (i)–(iv).*

Hence

COROLLARY. *For $(1-\varepsilon)$ sufficiently small there exists a $C^{3-\varepsilon}$ counterexample to the Seifert conjecture.*

The Denjoy fractal diffeomorphism g of [5] satisfies (iii) – it has a minimal Cantor set in int(A). It can be easily modified to satisfy all but (i) – lack of periodicity.

In §1–5, we describe a method for perturbing g to destroy periodicity. *It is not necessary to know the exact definition of g nor the estimates in* [5].

We define a general class ϕ of diffeomorphisms of the annulus which satisfy simple properties sufficient to allow a non-periodic perturbation. All we need know from [5] is that this class ϕ is non-empty.

A diffeomorphism from ϕ has an invariant Jordan curve Q which itself lies in a class γ of Jordan curves described in §1. A Lipschitz graph is in γ; any curve in γ is a quasi-circle. We use the geometric properties of γ-circles to perturb away periodic points in A.

Conjecture. Suppose $f: S^2 \to S^2$ a C^3 diffeomorphism which repels N, attracts S and has no other periodic points. If one orbit is asymptotic to both N and S then f is equivalent to the standard "north pole, south pole" diffeomorphism.

If there is a counterexample $f: S^2 \to S^2$ then f satisfies the hypotheses of Theorem A and thus the Seifert conjecture is false.

It is dual to the Birkhoff conjecture:
Suppose $f: S^2 \to S^2$ is an area-preserving diffeomorphism and has no other periodic points than the fixed poles. Then f is equivalent to a rigid rotation.

NOTATION. *The projections*

$$\pi_i: \mathbf{R}^2 \to \mathbf{R}'$$

for $i = 1, 2$ and defined by $\pi_i(x_1, x_2) = x_i$. The usual metric on $\mathbf{R}^2$ is denoted by $d(x, y) = \|x - y\|$ and for $x_0 \in \mathbf{R}^2$ and $K \subset \mathbf{R}^2$ define

$$d(x_0, K) = \inf\{d(x_0, y): y \in K\}$$

and

$$|K| = sup\{d(x, y): x, y \in K\}.$$

For $x \in \mathbf{R}^2$ and $\varepsilon > 0$

$$D(x, \varepsilon) = \{y \in \mathbf{R}^2: d(x, y) \leq \varepsilon\}.$$

For $v \in (\mathbf{R}' \backslash \{0\}) \times \mathbf{R}'$ the slope of v is defined by

$$\sigma(v) = \pi_2(v)/\pi_1(v).$$

Define the infinite strip U by

$$U = \mathbf{R}' \times (0, 1)$$

and the components of the boundary:

$$\partial^+ U = \mathbf{R}' \times \{1\}, \partial^- U = \mathbf{R}' \times \{0\}, \partial U = \partial^+ U \cup \partial^- U.$$

Let $\mathbf{Z}$ act on U by translation:

$$U \times \mathbf{Z} \to U: ((x_1, x_2), n) \to (x_1 + n, x_2)$$

so that the quotient space $U \backslash \mathbf{Z}$ is an annulus.

We shall not distinguish between U and $U \backslash \mathbf{Z}$ nor between any subset of U and its projection in $U \backslash \mathbf{Z}$. All functions on U are assumed to be $\mathbf{Z}$-invariant and are identified with the corresponding functions defined on $U \setminus \mathbf{Z}$ and all maps from U to U are assumed to be $\mathbf{Z}$-equivariant and are identified with the corresponding maps from $U \setminus \mathbf{Z}$ to $U \setminus \mathbf{Z}$.

If Q is a Jordan curve with winding number one in the open annulus, then $U^+ = U^+(Q)$ (resp. $U^- = U^-(Q)$) is the component of the complement of Q in U containing $\partial^+ U$ (resp. $\partial^- U$) in its closure. Thus

$$U = U^+ \cup Q \cup U^-.$$

If $K \subset U$ let $K^+ = K \cap U^+$ and $K^- = K \cap U^-$.

§1. DIAMOND CIRCLES

A *diagonal* is an open line segment $\Delta \subset U$ of slope ± 1:

$$\Delta(p, q) = \{(1-s)p + sq \colon 0 < s < 1\}$$

where $|\sigma(p-q)| = 1$ and $p, q \in U$; the points p and q are called the *endpoints* of Δ.

Fix $\beta \in (0, 1)$. A *β-diamond* is a closed subset of U with non-empty interior of the form

$$T_\beta(x, y) = \left\{ z \in U \colon \begin{array}{c} \pi_1(x) \leq \pi_1(z) \leq \pi_1(y) \\ -\beta \leq \sigma(z-x) \leq \beta \\ -\beta \leq \sigma(z-y) \leq \beta \end{array} \right\}$$

where x and y are distinct. The points x and y are called the *endpoints* of the diamond. The condition that $T_\beta(x, y)$ have non-empty interior is equivalent to the inequalities

$$\pi_1(x) < \pi_1(y)$$

and

$$|\sigma(y-x)| < \beta.$$

Note that (since $0 < \beta < 1$) the diameter of a diamond is the distance between its endpoints:

$$|T_\beta(x, y)| = d(x, y).$$

A *β-diamond chain* is a set $N = N_\beta \subset U$ of the form

$$N = T_\beta(x_0, y_0) \cup \Delta(y_0, x_1) \cup T_\beta(x_1, y_1) \cup \ldots \cup \Delta(y_{m-1}, x_m) \cup T_\beta(x_m, y_m)$$

where each $T_\beta(x_j, y_j)$ (for $j = 0, 1, \ldots, m$) is a β-diamond (and so has non-empty interior) and each $\Delta(y_j, x_{j+1})$ is a diagonal sloping backwards:

$$\pi_1(x_{j+1}) > \pi_1(y_j)$$

(for $j = 0, \ldots, m-1$). The points $x_0, y_0, x_1, y_1, \ldots, y_{m-1}, x_m, y_m$ are called the *endpoints* of the β-diamond chain N.

We denote by

$$\Xi(N) = \{T_\beta(x_0, y_0), T_\beta(x_1, y_1), \ldots, T_\beta(x_m, y_m)\}$$
$$\Gamma(N) = \{\Delta(y_0, x_1), \Delta(y_1, x_2), \ldots, \Delta(y_{m-1}, x_m)\}.$$

A *constituent* of a diamond chain N is either a diamond or a diagonal of N; thus $\Xi(N) \cup \Gamma(N)$ is the set of constituents of N. We call N *simple* iff no two constituents of N intersect.

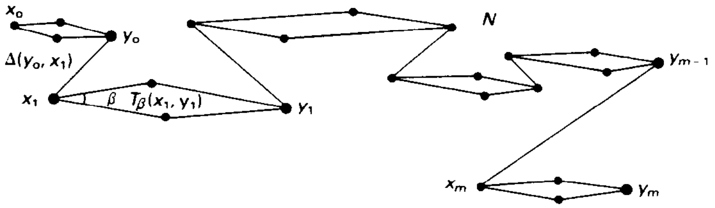

Fig. 2.

LEMMA 1.1. *Assume N is a diamond chain satisfying*

$$|\Delta| < |T|/2$$

for $\Delta \in \Gamma(N)$ and $T \in \Xi(N)$. Then N is simple.

Proof. The inequalities imply

$$|\pi_1(y_{k-1} - x_k)|, \quad |\pi_1(y_k - x_{k+1})| \leq \pi_1(y_k - x_k)/2$$

for $k = 1, \ldots, m$ which imply $\pi_1(y_{k-1}) < \pi_1(x_{k+1})$. This says that the maximum value of π_1 on $\Delta(y_{k-1}, x_k)$ is less than the minimum value of π_1 on $\Delta(y_k, x_{k+1})$ so that no two constituents of the chain can intersect if they are separated by a diamond. Since it is clear that no two adjacent constituents and no two diamonds which are separated by a single diagonal can intersect, it follows that N is simple.

Q.E.D.

We call N *closed* iff

$$T_\beta(x_m, y_m) = T_\beta(x_0, y_0) + (1, 0)$$

i.e. iff $x_m = x_0 + (1, 0)$ and $y_m = y_0 + (1, 0)$.

A β-diamond chain N' is a *refinement* of a β-diamond chain N iff $N' \subset N, \Gamma(N) \subset \Gamma(N')$ and

$$|\Delta| \leq \beta |T|/2$$

for$\Delta \in \Gamma(N') \backslash \Gamma(N)$ and $T \in \Xi(N')$. It follows from the preceding remarks that if N is simple, and N' refines N, then N' is simple.

Given a constant $\kappa > 1$ and diamond chains N and N', say that N' is a refinement of N with *uniformity* κ iff N' is refinement of N and

$$|T| < \kappa |T'|$$

for all $T \in \Xi(N)$ and all $T' \in \Xi(N')$.

A *β-diamond structure with uniformity* κ is a sequence $\{N^k : k = 1, 2, \ldots\}$ of simple closed β-diamond chains such that for each k, N^{k+1} refines N^k with uniformity κ and

$$\limsup \{|T| : T \in \Xi(N^k)\} = 0$$

A *β-diamond circle with uniformity* κ is a set Q of the form

$$Q = \bigcap_k N^k$$

where $\{N^k\}_k$ is a β-diamond structure with uniformity κ. The set $\Gamma \subset Q$ defined by

$$\Gamma = \bigcap_k \cup \{T : T \in \Xi(N^k)\}$$

is called the *Cantor set* of Q. By a *diagonal* of Q we mean a diagonal of one of the approximating diamond chains N^k; note that $Q\backslash\Gamma$ is precisely the disjoint union of the diagonals of Q. We denote the set of diagonals of Q by $\Gamma(Q)$.

NOTATION. *Given $T_\beta = T_\beta(x, y)$ and $\beta < \gamma$ we may write T_γ for $T_\gamma(x, y)$ (without mentioning that T_β and T_γ have the same endpoints). If N_β^k is a fixed β-diamond chain then N_γ^k is a γ-diamond chain where $\Xi(N_\gamma^k) = \{T_\gamma: T_\beta \in \Xi(N_\beta^k)\}$ and $\gamma(N_\gamma^k) = \gamma(N_\beta^k)$.*

The next lemma follows easily from the definitions and induction. It says that every β-diamond circle is a γ-diamond circle for $0 < \beta < \gamma < 1$.

LEMMA 1.2. *Let $Q = \cap N_\beta^k$ where $\{N_\beta^k\}_k$ is a β-diamond structure. If $0 < \beta < \gamma < 1$, then $Q = \cap N_\gamma^k$ and $\{N_\gamma^k\}_k$ is a γ-diamond structure.*

Given a diagonal $\Delta = \Delta(p, q)$ define sets R_Δ and S_Δ by

$$R_\Delta = cl\left\{ x + v: x \in \Delta,\ v \in \mathbf{R}^2,\ \begin{array}{l} \|\gamma\| < 2|\Delta|, \\ 1/9 < |\sigma(v)| < 1/7, \\ \sigma(v)\sigma(p-q) > 0 \end{array} \right\}$$

$$S_\Delta = cl\left\{ x + v: x \in \Delta,\ v \in \mathbf{R}^2,\ \begin{array}{l} \|\gamma\| < 3|\Delta|/2, \\ |\sigma(v)| = 1/8, \\ \sigma(v)\sigma(p-q) > 0 \end{array} \right\}$$

Thus S_Δ is a parallelogram which is cut by Δ and R_Δ is a "bow tie" containing S_Δ and "pulled tight" by Δ. Finally, for each diamond circle Q we define $S = \cup S_\Delta$ and $R = \cup R_\Delta$ where the unions are over all diagonals Δ of Q.

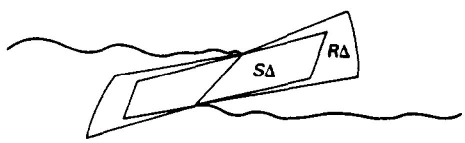

Fig. 3.

We prove that the R_Δ are pairwise disjoint. This will be a consequence of the next lemma which gives us a simple approximation of Q in a neighborhood of Δ.

Let $\eta > 0$. For $x \in \mathbf{R}^2$, let

$$C_\eta(x) = \{v \in \mathbf{R}^2: |\sigma(v - x)| \leq \eta\},$$

the cone of slope $\pm\eta$ based at x. Define $C_\eta^\Delta(x)$ to be the lower half of the cone $[C_\eta(x)]^c$ and $C_\eta^\nabla(x)$ to be the upper half.

If $\Delta \in \Gamma(Q)$, let D_Δ denote a disk centered at the midpoint of Δ.

LEMMA 1.3. *Let $\kappa > 1$. There exists $0 < \beta_0 < 1$ with the following property: Suppose $Q = \cap N^k$ is a β-diamond circle with uniformity κ and $0 < \beta < \beta_0$. Let $T_\beta \in \Xi(N^k)$. Then there are disks $\{D_\Delta: \Delta \in \Gamma(N^{k+1}), \Delta \subset T_\beta\}$ such that;*

(i) $R_\Delta \subset D_\Delta \cap T_{\sqrt{\beta}}$;
(ii) *The* $\{D_\Delta\}$ *are disjoint*;
(iii) $D_\Delta \subset T_{1/10}$;
(iv) $D_\Delta \cap Q \subset \Delta \cup C_\beta(p) \cup C_\beta(q)$, *where* p *and* q *are the endpoints of* Δ;
(v) $T_{\sqrt{\beta}} \subset \cup\{T'_{1/10}\colon T'_\beta \in \Xi(N^{k+1}), T'_\beta \subset T_\beta\} \cup \{D_\Delta\}$.

Proof. The chain N^{k+1} restricts to a subchain in T_β. Its constituents alternate between diamonds $(T'_i)_\beta (i=0, \ldots, n)$ and diagonals $\Delta_i (i=1, \ldots, n-1)$. By uniformity, $|(T'_i)_\beta| > |T_\beta|/\kappa (i=0, \ldots, n)$. It follows that for β sufficiently small, $(T'_i)_{1/10}$ meets both sides of $T_{\sqrt{\beta}}$. (It helps to think of β as extremely small so that an N^k_β-diamond T is virtually a straight line.)

Then $T_{\sqrt{\beta}}$ is an alternating union of sets contained in $(T'_i)_{1/10}$ and sets B_i containing Δ_i as depicted in Fig. 4.

In order to prove (v) we need $B_i \subset D_{\Delta_i} (i = 1, \ldots, n-1)$. This is achieved by letting L_i be the line segment in B_i in Fig. 4 and setting $|D_{\Delta_i}| = 2|L_i|$.

To prove (i) first note that $|R_\Delta| < 5|\Delta|$, for all Δ. For $R_{\Delta_i} \subset D_{\Delta_i}$ to hold we need $|D_{\Delta_i}| > 5|\Delta_i|$ or $|L_i| > 5|\Delta_i|/2$. This is clearly true for β sufficiently small since $\Delta_i \subset T_\beta$ and L_i meets both Δ_i and the boundary of $T_{\sqrt{\beta}}$. That $R_\Delta \subset T_{\sqrt{\beta}}$ is easy since $|R_\Delta| < 5|\Delta|$ and $\Delta \subset T_\beta$.

We estimate $L = \max\{|L_i|\}$ in terms of β and show that for β sufficiently small, L is so small that (ii), (iii) and (iv) are valid. It suffices to prove that $2L < \min\{|(T'_i)_\beta|\colon 0 \le i \le n\}/10$. But $L < 2\sqrt{\beta}\,|T_\beta|$; so if $\sqrt{\beta} < 1/40\kappa$ then $2L < |(T'_i)_\beta|/10$.

Q.E.D.

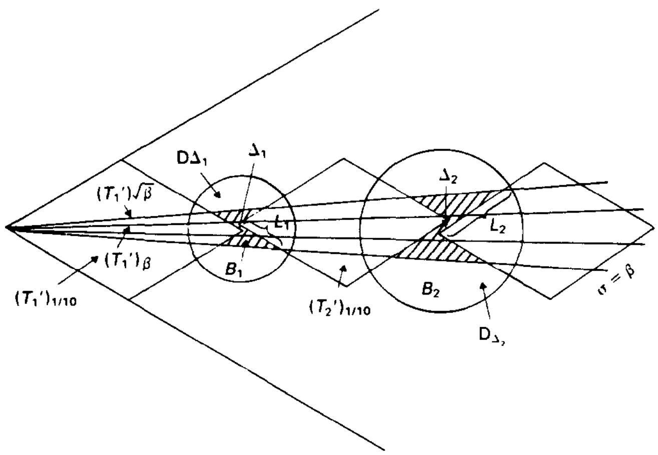

Fig. 4.

COROLLARY 1.4. *Let* $\kappa > 1$. *There exists* $0 < \beta_0 < 1/10$ *such that if* Q *is a* β*-diamond circle with uniformity* κ *and* $0 < \beta < \beta_0$, *then the sets* $R_\Delta(\Delta \in \Gamma(Q))$ *are pairwise disjoint.*

Proof. Let $\beta_0 < 1/10$ be as in the conclusion of (1.3). Let $T_\beta \in \Xi(N^k)$ and $\Delta \in \Gamma(N^{k+1}), \Delta \subset T_\beta$. By (1.3) (i) $R_\Delta \subset D_\Delta$. Then by (1.3) (ii), the $\{R_\Delta\colon \Delta \in \Gamma(N^{k+1}), \Delta \subset T_\beta\}$ are disjoint. By (1.3) (iii) it remains to show that $(T_1)_{1/10}$ and $(T_2)_{1/10}$ are disjoint for $(T_1)_\beta, (T_2)_\beta \in \Xi(N^k)$. But this follows by (1.2): each chain $N^k_{1/10}$ is simple.

Q.E.D.

A Jordan curve Q is a *quasi-circle* if there exists a constant $K > 1$ such that if $x, y \in Q$ then one of the arcs connecting x with y is contained in a disk of radius $Kd(x, y)$. In general, a path connected set $B \subset \mathbf{R}^2$ has a *quasi-structure* if there exist a constant $K > 1$ such that if $x, y \in B$ then there exists a path in B connecting x and y which is contained in a disk of radius $Kd(x, y)$.

LEMMA 1.5. *Let Q be a β-diamond circle with uniformity κ, $0 < \beta < 1/100$. Then $Q \cup R$ has a quasi-structure depending on κ.*

Proof. Set $K = 40\kappa$. Let $w \in Q \cup R$. If $w \in Q$, define $\gamma_w = w$. Otherwise $w \in R_\Delta$ and define γ_w to be the shortest line segment connecting w and Δ.

Let $x, y \in (Q \cup R) \cap \{[0,1) \times (0,1)\}$. Then γ_x and γ_y can be extended to a path γ in Q with endpoints x and y.

Let k be maximal with $\gamma \subset T \in \Xi(N^k)$. (Let $\Xi(N^0) = [0,1) \times (0,1)$.) Hence x and y are not both in any single element of $\Xi(N^{k+1})$. Suppose x and y are separated by $T' \in \Xi(N^{k+1})$. Since $T \cap N^{k+1}$ refines T it follows that $|T'| < 2d(x,y)$. By uniformity,

$$|T| < \kappa|T'| < 2\kappa d(x,y) < Kd(x,y).$$

Assume x and y are not separated by any element of $\Xi(N^{k+1})$. The result is immediate if they are both in some R_Δ. Otherwise x, say, must lie in an element T' of $\Xi(N^{k+1})$ and γ meets a diagonal Δ which is attached to T' at p. Let j be maximal so that $p, x \in T_x \in \Xi(N^j)$. Then an element of $\Xi(N^{j+1})$ separates x and p. The preceding estimate applies, so $|T_x| < 2\kappa d(x,p)$.

Now either $y \in R_\Delta$ or y is contained in the element T'' of $\Xi(N^{k+1})$ which contains the other endpoint q of Δ. In either case $d(x,p) < 20d(x,y)$. If $y \in R_\Delta$ then $d(y,p) < 20d(x,y)$. Note that $\gamma \subset T_x \cup R_\Delta$ and

$$|\gamma \cap T_x| + |\gamma \cap R_\Delta| < 2\kappa(d(x,p) + d(y,p)) < Kd(x,y)$$

If $y \in T''$, let j be maximal so that $q, y \in T_y \in \Xi(N^j)$. Then $|T_y| < 2\kappa d(y,q)$. But $d(y,q) < 2d(y,p) < 4d(x,y)$. Also $|\Delta| < 2d(x,y)$. Thus $|T_x \cup \Delta \cup T_y| < 2\kappa(d(x,p) + |\Delta| + d(y,q)) < Kd(x,y)$. The result follows since $\gamma \subset T_x \cup \Delta \cup T_y$.

Q.E.D.

Graph-like properties of β-diamond circles

Diamond circles are certainly not graphs, but we will see in Proposition 1.9 some properties they have in common with graphs. It uses the next three lemmas.

LEMMA 1.6. *Let $Q = \cap N^k$ be a β-diamond circle, $0 < \beta < 1$, and $T \in \Xi(N^k)$. If l is an unbounded line in $\mathbf{R}^2$ with absolute slope > 1 which meets T then the points in $Q \cap T \cap l$ with maximal and minimal π_2 coordinates lie in Γ.*

Proof. Let $x \in Q \cap T \cap l$ have maximal π_2 coordinate. Suppose that $x \notin \Gamma$. Then $x \in \Delta \in \Gamma(Q)$. Let j be maximal with $\Delta \subset T' \in \Xi(N^j)$. Let $T'' \in \Xi(N^{j+1})$ be attached to the endpoint of Δ with largest π_2-coordinate. Since $T' \cap N^{j+1}$ refines T', $|\Delta| \leq \beta|T''|/2$. It follows from geometry that l meets $\mathrm{int}(T'')$. Since Q contains the endpoints of T'', $Q \cap T'' \cap l \neq \phi$. Any point in $T'' \cap l$ must have larger π_2 coordinate than x. Hence x must lie in Γ.

Q.E.D.

LEMMA 1.7. *Let $Q = \cap N^k$ be a β-diamond circle, $0 < \beta < 1/10$, and $T_\beta \in \Xi(N^k)$. If $x \in T_{1/10}$ and w is a closest point to x in Q then $w \in T_\beta$.*

Proof. By (1.2) $\{N^k_{1-\varepsilon}\}_k$ is a $(1-\varepsilon)$-diamond structure for $\beta < 1-\varepsilon$. Hence $N^k_{1-\varepsilon}$ is simple for all $0 < \varepsilon < 1$ and

$$Q \cap \mathrm{int}(T_1) \subset T_\beta.$$

Let l be the vertical line crossing $\mathrm{int}(T_\beta)$ with one endpoint x and one endpoint in ∂T_β. It follows easily from the geometry that $D(x, |l|) \subset T_1$ for $\beta < 1/10$. Q contains the endpoints of T_β, so l meets Q. Hence any closest point w of Q to x must lie in $D(x, |l|)$. Then $w \in T_\beta$.

Q.E.D.

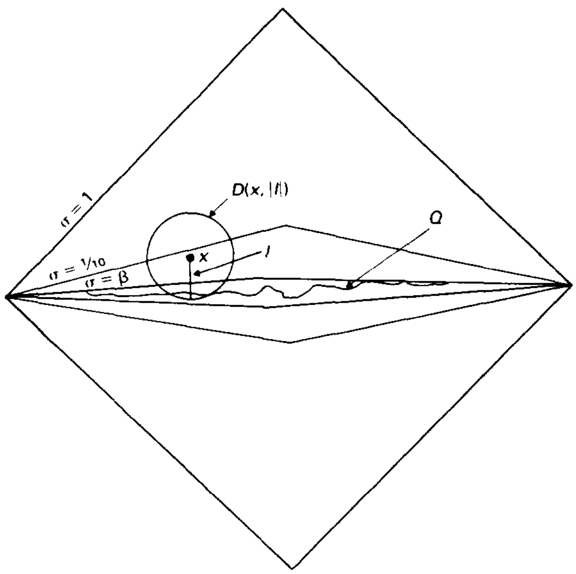

Fig. 5.

Let Q be a Jordan curve in U with winding number 1. Let $x, y \in Q$ and $\alpha(x, y)$ be an arc of Q with endpoints x and y, $\pi_1 x < \pi_1 y$. If $\alpha(x, y) \subset T_\beta(x, y)$ then α is a *β-diamond arc.* Note that all arcs are β-diamond arcs iff Q is a Lipschitz graph with Lipschitz constant β.

We next remark that Q is graph-like over its diamond arcs.

LEMMA 1.8. *Given $0 < \eta < \infty$ there exists $0 < \beta_0 < 1$ such that if $0 < \beta < \beta_0$, $w, x, y \in Q \subset \mathbf{R}^2$, and $z \in \mathbf{R}^2$ satisfy;*

(i) $w \in \alpha(x, y) \subset T_\beta(x, y)$;
(ii) $\pi_1 x \le \pi_1 z \le \pi_1 y$ *and* $z \notin T_{\sqrt{\beta}}(x, y)$;
(iii) $d(z, w) = d(z, \alpha(x, y))$;

then $|\sigma(z - w)| > \eta$.

This geometric property of Lipschitz graphs carries over to diamond circles as long as $z \in \mathbf{R}^2$ is not close to a diagonal Δ.

Henceforth assume that if $\Delta(p, q) \in \Gamma(Q)$ then $\pi_2(p) > \pi_2(q)$.

PROPOSITION 1.9. Graph-like property of Q. *Let $\kappa > 1$. There exists $0 < \beta_0 < 1/100$ with the following property: Suppose Q is a β-diamond circle, $0 < \beta < \beta_0$, with uniformity κ. Let $\Delta \in \Gamma(Q)$, $x \in U \setminus S$ and w a point of Q closest to x. Then $w \in \Gamma$ and*

(i) *If $x \in U^+$ then $w \in C^{\Delta}_{1/8}(x)$; if $x \in U^- \setminus S$ then $w \in C^{\nabla}_{1/8}(x)$.*
(ii) *If $x \in R^+_\Delta$ and $w \neq p$, then $w \in C_\beta(q) \cap C^{\Delta}_{20}(x)$ and $\|x - p\| > |\Delta|/2$;*
if $x \in R^-_\Delta$ and $w \neq q$, then $w \in C_\beta(p) \cap C^{\nabla}_{20}(x)$ and $\|x - q\| > |\Delta|/2$.
(iii) *If $x \in R^+_\Delta$ and $w = p$ then $\|x - p\| < |\Delta|$;*
if $x \in R^-_\Delta$ and $w = q$ then $\|x - q\| < |\Delta|$.

Proof. Let $\eta = 20$ in (1.8). Fix β_0 to be the minimum of 1/100 and the constraints of (1.3), (1.4) and (1.8).

Assume $Q = \cap N^k_\beta$ where $\beta < \beta_0$. Since $\beta < 1/10$ apply (1.2): $Q = \cap N^k_{1/10}(k \geq 1)$. Set $N^0_{1/10} = [0, 1) \times (0, 1)$. Thus if $x \in N^0_{1/10} \backslash Q$ then $x \in (N^k)_{1/10} \backslash (N^{k+1})_{1/10}$ for some $k \geq 0$. In particular, $x \in T_{1/10} \in \Xi(N^k_{1/10})$. We study $Q \cap T_\beta \subset T_\beta \subset T_{\sqrt{\beta}} \subset T_{1/10} \subset T_1$.

By (1.7) $x \in T_{1/10}$ implies $w \in T_\beta$.

Assume $x \notin T_{\sqrt{\beta}}$. By (1.8) we have $w \in C^{\Delta}_{20}(x)$ if $x \in U^+$ and $w \in C^{\nabla}_{20}(x)$ if $x \in U^-$. By (1.6), $w \in \Gamma$. By (1.3) (i), $x \notin R_\Delta$; so (ii) and (iii) are not possible. Thus the result is valid if $x \notin T_{\sqrt{\beta}}$.

Now assume $x \in T_{\sqrt{\beta}}$. By (1.3) (v), as in Fig. 4, $x \in D_\Delta$ for some $\Delta \in \Gamma(N^{k+1})$ in T_β. (Since k is maximal, x is not in any $N^{k+1}_{1/10}$-diamond.) By (1.3) (i) and (iv) we know $R_\Delta \subset D_\Delta$ and $D_\Delta \cap Q \subset \Delta \cup C_\beta(p) \cup C_\beta(q)$, where p and q are the endpoints of Δ. Figure 6 illustrates this. Since x can lie only in $D_\Delta \backslash S_\Delta$, the proposition follows for $\beta < \beta_0$ by (1.6), (1.8) and the definitions of S_Δ.

Q.E.D.

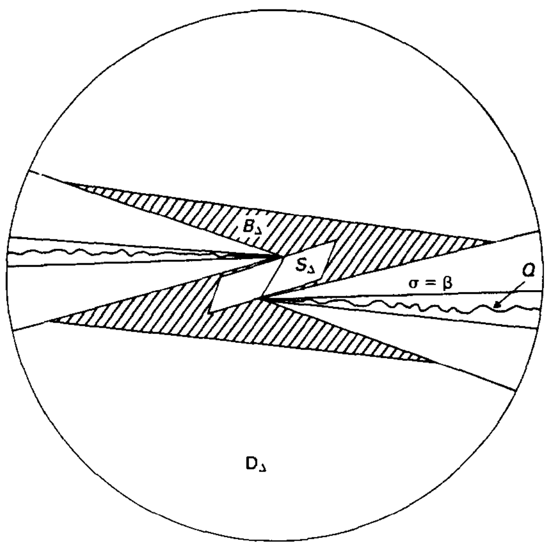

Fig. 6.

§2. MAPPINGS OF DIAMOND CIRCLES

If $U \subset \mathbf{R}^n$ is open and $f: U \to \mathbf{R}^n$ is of class $C^{r+\alpha}$, define

$$\|f\|_k = \sup\{\|D^k f(x)\| : x \in U\}, \quad \text{(We also write } \|f\| \text{ for } \|f\|_0.)$$
$$\|f\|_{k,\alpha} = \sup \|D^k f(x) - D^k f(y)\| / \|x - y\|^\alpha, \quad 0 \leq k \leq r, \ 0 < \alpha < 1.$$

Definition 2.0. Let θ_k: $B \to L^k_s(\mathbf{R}^n, \mathbf{R}^n)$ for $B \subset \mathbf{R}^n$, $k = 0, 1, \ldots, r$. (Notation: θ_0: $B \to L^0_s(\mathbf{R}^n, \mathbf{R}^n)$ is a continuous mapping from B to $\mathbf{R}^n$.) For $k = 0, 1, \ldots, r$, define R_k: $B \times B \to L^k_s(\mathbf{R}^k, \mathbf{R}^k)$ by

$$R_k(x, y) = \theta_k(y) - \sum_{i \leq r-k} \frac{\theta_{k+i}(x)}{i!}(y - x)^i \text{ and } x, y \in B.$$

If there exists a constant $C > 0$ such that for $x, y \in B$ and $0 \leq k \leq r$,

$$\|R_k(x, y)\| < C\|x - y\|^{r+\alpha-k}$$

then C is called a $C^{r+\alpha}$ *bound* for $\{\theta_k : 0 \leq k \leq r\}$.

If $U \subset \mathbf{R}^n$ is open, a function $f: U \to \mathbf{R}^n$ is of *class* $C^{r+\alpha}$ if there exist $\theta_k: U \to L_s^k(\mathbf{R}^n, \mathbf{R}^n)$ $k = 0, \ldots, r$ and a constant $C > 0$ such that $\theta_0 = f$ and C is a $C^{r+\alpha}$ bound for $\{\theta_k | U \cap D(x, 1), x \in U\}$. We write $D^k f = \theta_k$.

THEOREM 2.1. *If $U \subset \mathbf{R}^m$ is convex and open, $f: U \to \mathbf{R}^n$ is of class $C^{r+\alpha}$ and*

$$\| D^2 f(x) - D^2 f(y) \| < C \| x - y \|^\alpha$$

then C is a $C^{r+\alpha}$ bound for $\{f, Df, D^2 f\}$.

We henceforth refer to C as a $C^{r+\alpha}$ bound for f.

Proof. We apply the general formula for C^1 maps g and $x, y \in U$ which is convex.

$$* \qquad g(x) - g(y) = (x - y) \int_0^1 (Dg)(tx + (1-t)y)\, dt$$

Then $$\frac{\|Df(x) - Df(y) - D^2 f(y)(x-y)\|}{\|x-y\|} = \int_0^1 \|D^2 f(tx + (1-t)y) - D^2 f(y)\|\, dt$$
$$\leq C\|x-y\|^\alpha.$$

To estimate $\|R_0(x,y)\|/\|x-y\|^2$ apply * twice and get

$$f(x) - f(y) - Df(y)(x-y) = \int_0^1 \int_0^1 (D^2 f)(s(tx + (1-t)y) + (1-s)y) ds (t(x-y)^2) dt.$$

Therefore $$\frac{\| f(x) - f(y) - Df(y)(x-y) - \frac{1}{2} D^2 f(y)(x-y)^2 \|}{\|x-y\|^2}$$
$$= \int_0^1 \int_0^1 \| D^2 f(s(tx + (1-t)y) + (1-s)y) - D^2 f(y) \|\, ds\, dt$$
$$\leq \tfrac{1}{2} C \|x-y\|^\alpha.$$

Q.E.D

2.2 *The Inverse Function Theorem with $C^{2+\alpha}$ estimates*

Let $U \subset \mathbf{R}^n$ be an open set and $V \subset U$ an open, convex set. Let $f: U \to \mathbf{R}^n$ be $C^{2+\alpha}$ and $\sup \| Df(x) - Id \| = \delta < 1$. Then f is an embedding of V and

(i) $\| f^{-1} \|_1 \leq \left(\frac{1}{1-\delta} \right)$

(ii) $\| f^{-1} \|_2 \leq \left(\frac{1}{1-\delta} \right)^3 \| f \|_2$

(iii) $\| f^{-1} \|_{2,\alpha} = 3 \left(\frac{1}{1-\delta} \right)^5 \| f \|_2^2 + \left(\frac{1}{1-\delta} \right)^{3+\alpha} \| f \|_{2,\alpha}$.

Proof. $f^{-1} f = Id$. Therefore $[(Df^{-1})(fx)]\,[(Df)(x)] \equiv Id$; and

$$(D^2 f^{-1})(fx)(u,v) = -[(Df^{-1})(fx)]\,[(D^2 f)(x)]\,[(Df)(x)^{-1}(u), (Df)(x)^{-1}(v)]$$

Also $\|fx - fy\| = \|(Df)(tx + (1-t)y)\,dt\,(x-y)\| \geq \min\{\| Df(z) \|\}\, \|x-y\| \geq (1-\delta)\, \|x-y\|$

and $\|fx - fy\| \leq \|f\|_1\, \|x-y\| = (1+\delta)\, \|x-y\|$.

Let $\Omega = 1/(1-\delta)$

(i) $(Df^{-1})(fx)=(Df)(x)^{-1}$

$$(Id-(Id-(Df)(x)))^{-1}=Id+(Id-(Df)(x))+\dots+(Id-(Df)(x))^n+\dots.$$

So $\|f^{-1}\|_1\le 1+\delta+\delta^2+\dots=\Omega.$

(ii) $\|f^{-1}\|_2\le\|f^{-1}\|_1\,\|f\|_2\,\|f^{-1}\|_1^2\le\Omega^3\,\|f\|_2.$

(iii) $(D^2f^{-1})(fx)-(D^2f^{-1})(fy)=$

$$=\begin{cases}[(Df^{-1})(fx)-(Df^{-1})(fy)]\,[(D^2f)(x)]\,[(Df)(x)^{-1},(Df)(x)^{-1}]\\ +[(Df^{-1})(fy)]\,[(D^2f)(x)-(D^2f)(y)]\,[(Df)(x)^{-1},(Df)(x)^{-1}]\\ +[(Df^{-1})(fy)]\,[(D^2f)(y)]\,[(Df)(x)^{-1}-(Df)(y)^{-1},(Df)(x)^{-1}]\\ +[(Df^{-1})(fy)]\,[(D^2f)(y)]\,[(Df)(y)^{-1},(Df)(x)^{-1}-(Df)(y)^{-1}].\end{cases}$$

Hence

$$\begin{aligned}\|(D^2f^{-1})(fx)-(D^2f^{-1})(fy)\|&\le\|f^{-1}\|_2\,\|fx-fy\|\,\|f\|_2\,\|f^{-1}\|_1^2\\&\quad+\|f^{-1}\|_1\,\|f\|_{2,\alpha}\,\|x-y\|^{\alpha}\|f^{-1}\|_1^2\\&\quad+2\|f^{-1}\|_1\,\|f\|_2\,\|f^{-1}\|_2\,\|fx-fy\|\,\|f^{-1}\|_1\\&\le\Omega^3\,\|f\|_2^2\Omega^2\|fx-fy\|+\Omega\,\|f\|_{2,\alpha}\Omega^{\alpha}\|fx-fy\|^{\alpha}\Omega^2\\&\quad+2\Omega\,\|f\|_2\Omega^3\,\|f\|_2\,\|fx-fy\|\,\Omega.\\&\leqslant 3\Omega^5\,\|f\|_2^2+\Omega^{3+\alpha}\|f\|_{2,\alpha}.\end{aligned}$$

It remains to show that f is an embedding. It is an immersion. Assume there exist $a, b\in U$ such that $f(a)=f(b)$. Apply $*$ of the proof of (2.1). Then

$$0=\|f(b)-f(a)\|=\|(b-a)\int_0^1 Df(tb+(1-t)a)\,dt\,\|\ge\|b-a\|\min\{\|Df(tb+(1-t)a)\|$$

Since $\|Df\|$ is bounded away from 0, we conclude $\|b-a\|=0$.

Q.E.D.

Special Whitney extension for quasi-structure

We now investigate homeomorphisms of diamond circles $f\colon Q\to Q$ which are extendable to $C^{3-\varepsilon}$ diffeomorphisms.

The first theorem applies to subsets of $\mathbf{R}^2$ with a quasi-structure.

PROPOSITION 2.3. *Let $B\subset U=\mathbf{R}^1\times(0,1)$ be a closed, path connected set with a K quasi-structure. Let $\{R_i\colon i\in\mathbf{Z}\}\subset B$ be a collection of disjoint, closed sets and $\Lambda=Cl(B\setminus\cup\{R_i\})$. Suppose there exist $\theta_n\colon B\to L_s^n(\mathbf{R}^2,\mathbf{R}^2)$, $0\le n\le 2$, and constants $C_\Lambda\ge 0$ and $C_R\ge 0$ satisfying:*

(i) $\theta_1|\Lambda=Id$; $\theta_2|\Lambda=0$;
(ii) *C_Λ is a $C^{3-\varepsilon}$ bound for $\{\theta_n|\Lambda\colon 0\le n\le 2\}$;*
(iii) *C_R is a $C^{3-\varepsilon}$ bound for $\{\theta_n|R_i\colon i\in Z,\ 0\le n\le 2\}$.*

Then there exists a $C^{3-\varepsilon}$ mapping $f\colon U\to\mathbf{R}^2$ such that $D^nf|B=\theta_n$, $n=0,1,2$.

Proof. Let us briefly review the outline Whitney's construction [14]. The complement of B in U has a "Whitney covering" of closed squares whose vertices are on the $\mathbf{Z}/2^n$ lattice (see (3.1) for more details). The size of the sets is on the order of their distance to B. They are each expanded slightly to become open sets U_i which intersect at most four at a time. An "anchor" point a_i is chosen in B for each U_i whose distance to U_i is on the order of $|U_i|$. The candidate derivatives $\{\theta_n\}$ at a_i are used to construct a local polynomial extension on U_i. A partition of unity produces a global extension f. The work comes in showing that f is $C^{r-\varepsilon}$. This construction may easily be **Z**-invariant since the $\{\theta_n\}$ are **Z**-invariant.

Let $C = 9K^3 \max\{C_\Lambda, C_R\}$. We prove, for all $x, y \in B$

(a) $\|\theta_2(x) - \theta_2(y)\| \leq C\|x-y\|^{1-\varepsilon}$;

(b) $\dfrac{\|\theta_1(x) - \theta_1(y) - \theta_2(y)(x-y)\|}{\|x-y\|} \leq C\|x-y\|^{1-\varepsilon}$;

(c) $\dfrac{\|\theta_0(x) - \theta_0(y) - \theta_1(y)(x-y) - \frac{1}{2}\theta_2(y)(x-y)^2\|}{\|x-y\|^2} \leq C\|x-y\|^{1-\varepsilon}$

By assumption, (a)–(c) are satisfied for pairs of points x, y either in Λ or in some R_i, $i \in \mathbf{Z}$. Let $x, y \in B$ and not in the same R_i and not both in Λ. Since K is a quasi-constant for B, there exists an arc γ connecting x and y and contained in a disk of diameter $< K\|x-y\|$. If $x \in R_i$, let x' be a boundary point of $R_i \cap \gamma$. Since R_i is closed, $x' \in R_i$. Otherwise let $x = x'$. Note that $x' \in \Lambda$. Similarly define y'. Since x, y, x', y' are all in the arc γ, it follows that

$$K\|x-y\| > max\{\|x'-y'\|, \|x-x'\|, \|y-y'\|\} \qquad 2.4.$$

Then (a)–(c) follow from (i)–(iii), the triangle inequality and (2.4):

(a) $$\begin{aligned}\|\theta_2(x) - \theta_2(y)\| &\leq \|\theta_2(x) - \theta_2(x')\| + \|\theta_2(x') - \theta_2(y')\| + \|\theta_2(y') - \theta_2(y)\| \\ &\leq C_R(\|x-x'\|^{1-\varepsilon} + \|y-y'\|^{1-\varepsilon}) + C_\Lambda\|x'-y'\|^{1-\varepsilon} \\ &\leq (2C_R + C_\Lambda)K\|x-y\|^{1-\varepsilon} \leq C\|x-y\|^{1-\varepsilon}.\end{aligned}$$

(b) $$\begin{aligned}&\frac{\|\theta_1(x) = \theta_1(y) - \theta_2(y)(x-y)\|}{\|x-y\|} \leq \frac{\|\theta_1(x) - \theta_1(y)\|}{\|x-y\|} + \|\theta_2(y)\| \\ &\leq \frac{\|\theta_1(x) - \theta_1(x')\| + \|\theta_1(x') - \theta_1(y')\| + \|\theta_1(y') - \theta_1(y)\|}{\|x-y\|} + \|\theta_2(y) - \theta_2(y')\| \\ &\leq K\frac{\|\theta_1(x) - \theta_1(x')\|}{\|x-x'\|} + K\frac{\|\theta_1(y') - \theta_1(y)\|}{\|y'-y\|} + \|\theta_2(y) - \theta_2(y')\| \text{ by (2.4)} \\ &\leq KC_R\|x-x'\|^{1-\varepsilon} + KC_R\|y'-y\|^{1-\varepsilon} + C_R\|y-y'\|^{1-\varepsilon} \\ &\leq (2K^2C_R + KC_\Lambda)\|x-y\|^{1-\varepsilon} \leq C\|x-y\|^{1-\varepsilon}.\end{aligned}$$

(c) $$\begin{aligned}&\frac{\|\theta_0(x) - \theta_0(y) - \theta_1(y)(x-y) - \frac{1}{2}\theta_2(y)(x-y)^2\|}{\|x-y\|^2} \\ &\leq \frac{\|\theta_0(x) - \theta_0(y) - (x-y)\|}{\|x-y\|^2} + \frac{\|\theta_1(y) - Id\|}{\|x-y\|} + \tfrac{1}{2}\|\theta_2(y)\| = *.\end{aligned}$$

The first term of $*$ is bounded by

$$K^2\left(\frac{\|\theta_0(x) - \theta_0(x') - (x-x')\|}{\|x-x'\|^2} + \frac{\|\theta_0(x') - \theta_0(y') - (x'-y')\|}{\|x'-y'\|^2} + \frac{\|\theta_0(y') - \theta_0(y) - (y'-y)\|}{\|y'-y\|^2}\right)$$

$$\leq K^2(C_R\|x-x'\|^{1-\varepsilon} + C_\Lambda\|x'-y'\|^{1-\varepsilon} + C_R\|y'-y\|^{1-\varepsilon}) \leq C\|x-y\|^{1-\varepsilon}/3.$$

The last two terms of $*$ are non-zero only if $y \in R_i$. In this case

$$\frac{\|\theta_1(y) - Id\|}{\|x-y\|} \leq K\frac{\|\theta_1(y) - \theta_1(y')\|}{\|y-y'\|} \leq KC_R\|y-y'\|^{1-\varepsilon} \leq K^2C_R\|x-y\|^{1-\varepsilon} \leq C\|x-y\|^{1-\varepsilon}/3$$

and $\quad \|\theta_2(y)\| = \|\theta_2(y) - \theta_2(y')\| \leq C_R \|y - y'\|^{1-\varepsilon} \leq C_R K \|x - y\|^{1-\varepsilon} \leq C \|x - y\|^{1-\varepsilon}/3$

Therefore, there exists a $C^{3-\varepsilon}$ mapping $f: U \to \mathbf{R}^2$ such that $D^n f | B = \theta_n$, $n = 0, 1, 2$.

Q.E.D.

If $B = \Gamma \cup R_\Delta$ where Γ is a diamond circle, we need to know *much* less about the mapping $\theta_0: B \to \mathbf{R}^2$ than (2.3) (i), (ii), (iii) in order to find an extension.

PROPOSITION 2.5. *Let $Q = \cap N^k$ be a β-diamond circle, $0 < \beta < 1$. Suppose $f: \Gamma \to \Gamma$ is a homeomorphism and $C > 0$ such that*

$$* \qquad \|f(x) - f(y) - (x - y)\| < C \|x - y\|^{3-\varepsilon}$$

for all $x, y \in \Gamma$ where $T(x, y) \in \Xi(N^k)$ or $\Delta(x, y) \in \Gamma(N^k)$, $k \geq 1$. There exists a $C^{3-\varepsilon}$ embedding F of a neighborhood of $\Gamma \cup \{R_\Delta\}$ into U such that

(i) $F|\Gamma = f$, $DF|\Gamma = Id$, $D^2F|\Gamma = 0$;
(ii) *F maps each $\Delta \subset S_\Delta \subset R_\Delta$ diffeomorphically onto $f_\Delta \subset S_{f\Delta} \subset R_{f\Delta}$.*

The proof for $S_\Delta = R_\Delta$ is given in [5, (4.8)].

§3. $C^{3-\varepsilon}$ DOMINATION

Let B be a closed subset of an open set $U \subset \mathbf{R}^n$. A function $h: U \to \mathbf{R}$ *vanishes to the rth order at B* if $h|B = 0$ and $h(x)/d(x, B)^r \to 0$ as $x \to p$, where $p \in B$ and $x \in U \setminus B$.

If $h_1 < h_2$ are positive, real-valued functions defined on U which vanish to the rth order at B we wish to find a $C^{r-\varepsilon}$ function h with $h_1 < h < h_2$. This is always possible if the values of h_1 and h_2 are sufficiently far apart. In general, the function $d(x, B)$ is only Lipschitz so some work is necessary. Our next "sandwich" lemma uses some ingredients of the proof of the Whitney extension theorem ([14]). We state the lemma in dimension 2 for simplicity, but it is valid in arbitrary dimension $\mathbf{R}^n$.

LEMMA 3.1. *Let B be a closed subset of an open set $U \subset \mathbf{R}^2$ and $C > 0$. There exists a $C^{r-\varepsilon}$ function $h: U \to \mathbf{R}^+$ such that*

(i) $Cd(x, B)^{r-\varepsilon} \leq h(x) \leq C(15d(x, B))^{r-\varepsilon}$;
(ii) *h vanishes to the $(r-1)$st order at B.*

Proof. Estimates on the Whitney cover of $U \setminus B$.

Define $d(x) = d(x, B)$. Begin with the cover of U by closed unit squares the vertices of which are on the integer lattice of $\mathbf{R}^2$. If a unit square L satisfies $d(L, B) \geq 1/2$ then retain it. Otherwise divide L into 4 equal squares with vertices on the 1/2-integer lattice. Retain each smaller square L with $d(L, B) \geq 1/4$. Subdivide each of the others into 4 equal squares. Repeat indefinitely to provide a closed cover $\{L_j : j \geq 1\}$ of $U \setminus B$. The interiors of the L_j are disjoint. Let e_j be the length of an edge of L_j.

Let K_j denote the open square with the same center as L_j and edge length $(1 + \frac{1}{4})e_j$. Then $(4 - \sqrt{2})e_j < 8d(x) < (8 + 17\sqrt{2})e_j$ for $x \in K_j$, $j \geq 1$. These bounds are unnecessarily sharp so we can replace them: There exist universal, positive constants A_1, A_2 such that,

$$A_1 < d(x)/e_j < A_2 < 15A_1 \text{ for all } x \in K_j, j \geq 1. \qquad 3.2.$$

If $x, y \in K_j$ then $\|x - y\| < \sqrt{2}(1 + \frac{1}{4})e_j$. Thus there exists a universal, positive constant

A_3 such that,

$$\|x-y\| < A_3 d(x) \text{ if } x, y \in K_j \text{ for some } j \geq 1. \qquad 3.3.$$

If x and y are not in the same set K_j for any j, then they cannot be too close together. That is, there exists a universal, positive constant A_4 so that

$$d(x) < A_4 \|x-y\| \text{ if } x, y \in U \setminus B, \text{ and } x, y \text{ are not in the same } K_j \text{ for any } j \geq 1. \qquad 3.4.$$

Furthermore

Each point of $U \setminus B$ has a neighborhood intersecting at most 4 of the K_j. 3.5.

Definition of h. There exists a partition of unity β_j subordinate to the cover $\{K_j\}$ and a constant $A_5 > 0$ such that,

$$|D^n \beta_j(y)| < A_5 d(y)^{-n} \text{ for all } y \in U \setminus B, \quad 0 \leq n \leq r. \qquad 3.6.$$

Let $P_j = C \sup\{d(y)^{r-\varepsilon}: y \in K_j\}$

Define
$$h(x) = \begin{cases} 0, & x \in B \\ \sum_{x \in K_j} P_j \beta_j(x), & x \in U \setminus B \end{cases}$$

It follows that for fixed $x \in U$, $Cd(x)^{r-\varepsilon} \leq h(x) \leq \sup\{P_j: x \in K_j\}$ since $h(x)$ is a convex combination of values $P_j \geq Cd(x)^{r-\varepsilon}$. By (3.2) $P_j = C \sup\{d(y)^{r-\varepsilon}: y \in K_j\} \leq C(A_2/A_1)^{r-\varepsilon} d(x)^{r-\varepsilon}$ for $x \in K_j$. By (3.2) we have

$$P_j \leq C(15d(x))^{r-\varepsilon} \text{ if } x \in K_j. \qquad 3.7.$$

Therefore if $x \in U \setminus B$ then $h(x) \leq C(15d(x))^{r-\varepsilon}$. Thus (i) is verified.

Differentiability class of h. We next prove h is of class $C^{r-\varepsilon}$. By the converse of Taylor's theorem, since U is open, we only have to find a $C^{r-\varepsilon}$ bound for h and its following candidate derivatives θ_n. See (2.0).

$$\text{For } n = 0, \ldots, r-1 \text{ define } \theta_n(x) = \begin{cases} 0, & x \in B \\ \sum_{x \in K_j} P_j D^n \beta_j(x), & x \in U \setminus B \end{cases}$$

If $x, y \in B$, $x \neq y$, then $R_n(x, y) = 0$ for $0 \leq n \leq r-1$.

Assume $x \in B$ and $y \in U \setminus B$. Then $d(y) \leq \|x-y\|$. By (2.0), (3.5), (3.6) and (3.7), for $0 \leq n \leq r-1$.

$$\begin{aligned} \frac{\|R_n(x,y)\|}{\|x-y\|^{r-1-n}} &= \frac{\sum_j P_j D^n \beta_j(y)}{\|x-y\|^{r-1-n}} \\ &\leq \frac{4[C(15d(y))^{r-\varepsilon}][A_5 d(y)^{-n}]}{d(y)^{r-1-n}} \\ &\leq C_1 d(y)^{1-\varepsilon} \leq C_1 \|x-y\|^{1-\varepsilon}. \end{aligned}$$

for some $C_1 > 0$.

The Hölder condition for θ_{r-1} is similar:

$$\|\theta_{r-1}(x)-\theta_{r-1}(y)\| = \|\sum_{y\in K_j} P_j D^{r-1}\beta_j(y)\| \le 4[C(15d(y))^{r-\varepsilon}][A_5 d(y)^{-(r-1)}]$$

$$\le C_1 C\|x-y\|^{1-\varepsilon}.$$

Assume $y\in B$ and $x\in U\backslash B$. Then $d(x)\le \|x-y\|$. We make a preliminary estimate: By (3.6), for $0\le k\le r-1-n$,

$$\frac{\|D^{n+k}\beta_j(x)(y-x)^k\|}{k!\,\|x-y\|^{r-1-n}} \le \frac{A_5 d(x)^{-(n+k)}}{k!\,\|x-y\|^{r-1-n-k}} \le A_5 d(x)^{-r+1}.$$

Thus by (2.0), (3.5) and (3.7), for $0\le n\le r-1$,

$$\frac{\|R_n(x,y)\|}{\|x-y\|^{r-1-n}} = \frac{\left\|\sum_{i\le r-1-n}\frac{\theta_{n+i}(x)}{i!}(y-x)^i\right\|}{\|x-y\|^{r-1-n}}$$

$$= \frac{\|\sum P_j[D^n\beta_j(x)+D^{n+1}\beta_j(x)(y-x)+\ldots+D^{r-1}\beta_j(x)(y-x)^{r-1-n}/(r-1-n)!]\|}{\|x-y\|^{r-1-n}}$$

$$\le 4[C(15d(x))^{r-\varepsilon}](r-n)[A_5 d(x)^{-r+1}]$$

$$\le C_2 d(x)^{1-\varepsilon} \le C_2\|x-y\|^{1-\varepsilon}$$

for some $C_2>0$.

The Hölder condition is the same as in the previous case.

Finally, suppose $x, y\in U\backslash B$. We break the numerator $\|R_n(x,y)\|$ into two sums depending on j. The first sum involves only j where both x and y are in K_j. In this case, by (3.3) $\|x-y\|<A_3 d(z)$ for $z=x$ or y. The last terms are summed over j where x and y are not both in any one K_j. If $x\in K_j$ then $d(x)<A_4\|x-y\|$ by (3.4). So

$$\frac{\|R_n(y,x)\|}{\|x-y\|^{r-1-n}} \le$$

$$\frac{\left\|\sum_{\{j:\,x,\,y\in K_j\}} P_j[D^n\beta_j(x)-D^n\beta_j(y)-D^{n+1}\beta_j(y)(x-y)\ldots-D^{r-1}\beta_j(y)(x-y)^{r-1-n}/(r-1-n)!]\right\|}{\|x-y\|^{r-1-n}}$$

$$+\frac{\left\|\sum_{\{j:\,x\in K_j,\,y\notin K_j\}} P_j D^n\beta_j(x)\right\|}{\|x-y\|^{r-1-n}}$$

$$+\frac{\left\|\sum_{\{j:\,y\in K_j,\,x\notin K_j\}} P_j[D^n\beta_j(y)+\ldots+D^{r-1}\beta_j(y)(x-y)^{r-1-n}/(r-1-n)!]\right\|}{\|x-y\|^{r-1-n}} = *.$$

By (3.6) a C^r bound for β_j is $A_5 d(y)^{-r}$. Thus by (3.5), (3.7) and Taylor's theorem, the first term of $*$ is bounded by $4[C(15d(y))^{r-\varepsilon}]\,\|x-y\|\,[A_5 d(y)^{-r}]$. It follows from (3.3) that this term is bounded by $C_3\|x-y\|^{1-\varepsilon}$ where $C_3>0$.

Use the triangle inequality to separate the last two terms of $*$ into at most $4(1+r-n)$ terms. The numerator of each is of the form $\|P_j D^{n+i}\beta_j(y)(x-y)^i/i!\|$, $0\le i\le r-1-n$. This is bounded above by $[C(15d(y))^{r-\varepsilon}]\,[A_5 d(y)^{-(n+i)}\,\|x-y\|^i]$. It follows from (3.4) that the last two terms of $*$ are bounded by $C_4\|x-y\|^{1-\varepsilon}$ for some $C_4>0$.

Last of all we verify the Hölder condition for θ_{r-1} and for $x, y\in U\setminus B$:

$$\|\theta_{r-1}(x)-\theta_{r-1}(y)\| = \left\|\sum_{x\in K_j} P_j D^{r-1}\beta_j(x) - \sum_{y\in K_j} P_j D^{r-1}\beta_j(y)\right\|$$

$$\leq \left\|\sum_{x,y\in K_j} P_j D^{r-1}(\beta_j(x)-\beta_j(y))\right\|$$

$$+\left\|\sum_{x\in K_j,\, y\notin K_j} P_j D^{r-1}\beta_j(x)\right\| + \left\|\sum_{y\in K_j,\, x\notin K_j} P_j D^{r-1}\beta_j(y)\right\|$$

$$\leq 4[C(15d(x))^{r-\varepsilon}]\,\|x-y\|\,[A_5 d(x)^{-r}] + 4[C(15d(x))^{r-\varepsilon}][A_5 d(x)^{-r+1}]$$

$$+ 4[C(15d(y))^{r-\varepsilon}][A_5 d(y)^{-r+1}] \text{ by (3.3) and (3.4)}$$

$$< C_5\|x-y\|^{1-\varepsilon} \text{ for some } C_5>0.$$

The converse to Taylor's theorem implies h is $C^{r-\varepsilon}$ and vanishes to the $(r-1)$st order at B.
Q.E.D.

Remark. The constants A_1–A_5 and C_1–C_5 are no longer used in the proof.

§4. PRELIMINARY GEOMETRIC ESTIMATES

This section contains estimates which will allow us to bring together the graph-like geometry of Q of section 1 and the extension theorems of sections 2 and 3 to perturb away periodic orbits.

The first estimates (4.1) are purely geometrical. They will apply to all of $U\backslash S$.

For $x, w, u\in U$, $g\colon U \to \mathbf{R}^1$ continuous and $\lambda \geq 0$, define

$$I = I(x, u, w, \lambda, g) = xu \cap D(w, \|x-w\| - \lambda g(x)).$$

Let $u\leq l\leq r\leq x$ be the endpoints of I if it is not empty.

LEMMA 4.1. *Let $0<\phi<\pi/2, A>1$ and $g\colon U\to\mathbf{R}^1$ be continuous. Suppose that $x, w, u\in\mathbf{R}^2$ satisfy $\|x-w\| = \|u-w\|$, $2\cos\phi g(x)\leq \|u-x\|$ and if θ is the angle uxw then $0<\theta\leq\phi$. Then for $0<\lambda\leq \min\{\cos\phi(1-\sin\phi), (\cos\phi)^2/(1+A)\}$*

(i) $I = I(x,u,w,\lambda,g)\neq\varnothing$;

(ii) $\|r-x\| < \dfrac{2\lambda g(x)}{\cos\phi}$;

(iii) $\|l-x\| > \dfrac{A2\lambda g(x)}{\cos\phi}$.

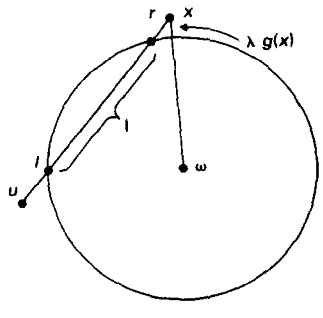

Fig. 7.

(iv) *If* $x' \in I$ *and* $\|w - w'\| < \lambda g(x)$ *then*

$$\|x' - w'\| < \|x - w\|.$$

Proof (i) It suffices to show that $\|x-w\| \sin\theta < \|x-w\| - \lambda g(x)$ as the first quantity is the length of the normal from w to xu and the second is the radius of the disk defining I. Note that $\|u-x\| \le 2\|x-w\|$. Then, by assumption, $2\cos\phi g(x) \le \|u-x\| < 2\|x-w\|$. Since $\lambda \le \cos\phi(1-\sin\phi)$ then $\lambda g(x) < \|x-w\|(1-\sin\theta)$.

(ii) Simple geometry implies $\|r-x\| < 2\lambda g(x)/\cos\theta \le 2\lambda g(x)/\cos\phi$ if $I \ne \varnothing$.

(iii) By symmetry $\|l-u\| = \|r-x\|$. Applying (ii), we have

$$\|l-x\| = \|u-x\| - \|l-u\| > 2\cos\phi\, g(x) - 2\lambda g(x)/\cos\phi \ge 2A\lambda g(x)/\cos\phi$$

since

$$\lambda \le (\cos\phi)^2/(1+A).$$

(iv) $\|x'-w'\| \le \|x'-w\| + \|w-w'\| < [\|x-w\| - \lambda g(x)] + \lambda g(x) = \|x-w\|$. (See Fig 7.)

Q.E.D.

We apply (4.1) in three situations.

Suppose $\Delta = \Delta(p,q) \in \Gamma(Q)$. There are two types of points x in $R_\Delta^+ \backslash S$. There are those of **Type** 1 for which a closest point w of Γ to x lies in $C_\beta(q) \cap C_{20}^\Delta(x)$ and those of **Type** 2 for which p is a closest point. If x is of Type 1 then $\|x-p\| > |\Delta|/2$; if it is of Type 2 then $\|x-p\| < |\Delta|$. (See (1.9). If $x \in U \backslash R$ we say x is of **Type** 3 in which case $w \in C_{1/8}^\Delta(x)$.

Define

$$d = d(x) = d(x, \Gamma).$$

Type 1. Suppose $x \in R_\Delta^+ \backslash S$ and w is a closest point of $C_{20}^\Delta(x) \cap \Gamma$ to x. Define u_1 to be the unique point such that $\sigma(x-u_1) = \sigma(\Delta)/8$ and $\|x-w\| = \|u_1 - w\|$. Let $\theta_1 = \theta_1(x)$ be the angle u_1xw and $\phi_1 = \pi/2 - \tan^{-1}(1/8) + \tan^{-1}(1/20)$. It follows that

$$\pi/4 < \theta_1(x) \le \phi_1 < \pi/2.$$

For $\lambda \ge 0$ define

$$I_1 = I_1(x, \lambda) = xu_1 \cap D(w, \|x-w\| - \lambda d^{3-\varepsilon}).$$

Let $u_1 \le l_1 \le r_1 \le x$ be the endpoints of I_1 if it is non-empty. Let $\lambda_1 = \min\{\cos\phi_1(1-\sin\phi_1), (\cos\phi_1)^2/(1+15^{3-\varepsilon})\} = (\cos\phi_1)^2/(1+15^{3-\varepsilon})$ and $C_1 = 2/\cos\phi_1$.

LEMMA 4.2. *If* $0 < \lambda \le \lambda_1$, $x \in R_\Delta^+ \backslash S$ *then*

(i) $I_1 = I_1(x, y) \ne \varnothing$;

(ii) $\|r_1 - x\| < C_1 \lambda d^{3-\varepsilon}$;

(iii) $\|l_1 - x\| > C_1 \lambda (15d)^{3-\varepsilon}$.

(iv) *If* w *is a closest point of* $C_{20}^+(x) \cap \Gamma$ *to* x, $\|w-w'\| < \lambda d^{3-\varepsilon}$ *and* $x' \in I_1$ *then*

$$\|x' - w'\| < \|x - w\|.$$

Proof. By the law of cosines, since $\pi/4 < \theta_1(x) \le \phi_1 < \pi/2$, and $w \in \Gamma$ then

$$\|u_1 - x\| = 2\|x-w\| \cos\theta_1 \ge 2\|x-w\| \cos\phi_1 \ge 2d\cos\phi_1.$$

Thus we may apply (4.1) to $\phi = \phi_1$, $A = 15^{3-\varepsilon}$, $g(x) = d^{3-\varepsilon}$ and $u = u_1$.

Q.E.D.

Type 2. Let $x \in R_\Delta^+ \setminus S$ and $\|x-p\| < |\Delta|$. Then $p \in [C_{1/8}^\Delta(x) \setminus C_{1/7}^\Delta(x)] \cap \Gamma$. Let u_2 be such that $\sigma(x-u_2)=\sigma(\Delta)/8$ and $\|x-p\| = \|u_2-p\|$. Let $\theta_2=\theta_2(x)$ be the angle u_2xp. Define $\phi_2=\pi/4$. By simple geometry $0<\theta_2<\phi_2$.

For $\lambda \geq 0$ define

$$I_2 = I_2(x,\lambda) = xu_2 \cap D(p, \|x-p\| - \lambda d^{3-\varepsilon}).$$

Let $u_2 \leq l_2 \leq r_2 \leq x$ be the endpoints of I_2 if it is non-empty. Let $\lambda_2 = \min\{\cos\phi_2(1-\sin\phi_2), (\cos\phi_2)^2/(1+15^{3-\varepsilon}10)\} = (\cos\phi_2)^2/(1+15^{3-\varepsilon}10)$ and $C_2 = 2/\cos\phi_2$.

LEMMA 4.3. *If* $0<\lambda \leq \lambda_2$, $x \in R_\Delta^+ \setminus S$ *and* $\|x-p\| < |\Delta|$ *then*

(i) $I_2 = I_2(x,\lambda) \neq \varnothing$;
(ii) $\|r_2 - x\| < C_2 \lambda d^{3-\varepsilon}$;
(iii) $\|l_2 - x\| > 10C_2\lambda(15d)^{3-\varepsilon}$.
(iv) *If* $x' \in I_2$ *and* $\|p-p'\| < \lambda d^{3-\varepsilon}$ *then*

$$\|x'-p'\| < \|x-p\|.$$

Proof. By the law of cosines, since $0<\theta_2(x)<\phi_2<\pi/2$, and since $p \in \Gamma$ we have

$$\|u_2-x\| = 2\|x-p\| \cos\theta_2 \geq 2\|x-p\|\cos\phi_2 \geq 2d\cos\phi_2.$$

Apply (4.1) to $\phi=\phi_2$, $A=15^{3-\varepsilon}10$, $g(x)=d^{3-\varepsilon}$ and $u=u_2$.

Q.E.D.

Type 1 *or* 2. Points in $D(p,|\Delta|) \setminus D(p,|\Delta|/2)$ could be of Type 1 or 2. For $x \in R_\Delta^+ \setminus S$ define

$$I_0(x,\lambda) = \begin{cases} I_1(x,\lambda), & \|x-p\| \geq |\Delta| \\ I_1(x,\lambda) \cap I_2(x,\lambda), & |\Delta|/2 < \|x-p\| < |\Delta| \\ I_2(x,\lambda) & \|x-p\| \leq |\Delta|/2 \end{cases}$$

Let $q<l_0<r_0<x$ be the endpoints of $I_0(x,\lambda)$ if it is non-empty. Let $C_0=\max\{C_1,C_2\}$ and $\lambda_0 = \min\{\lambda_1,\lambda_2\}$.

LEMMA 4.4. *If* $0<\lambda\leq\lambda_0$ *and* $x \in R_\Delta^+ \setminus S$ *then*

(i) $I_0 = I_0(x,\lambda) \neq \varnothing$;
(ii) $\|r_0-x\| < C_0\lambda d^{3-\varepsilon}$;
(iii) $\|l_0-x\| > C_0\lambda(15d)^{3-\varepsilon}$.
(iv) *If* $x' \in I_0$, *w is a closest point of* Γ *to x and* $\|w-w'\| < \lambda d^{3-\varepsilon}$ *then*

$$\|x'-w'\| < \|x-w\|.$$

Proof. (i)–(iii). Since $\phi_1 = \pi/2 + \tan^{-1}(1/20) - \tan^{-1}(1/8)$ and $\phi_2=\pi/4$ it follows that

$$C_2 < C_1 < 10C_2. \qquad 4.5.$$

Hence $C_0=C_1$.

By (4.2) $I_1(x,\lambda) \neq \varnothing$ for all $x \in R_\Delta^+ \setminus S$, $0<\lambda\leq\lambda_0$. If $\|x-p\|<|\Delta|$ then by (4.3), $I_2(x,\lambda) \neq \varnothing$. (i) will follow from (iii). Moreover,

$$\|r_0-x\| = \begin{cases} \|r_1-x\|, & \|x-p\| \geq |\Delta| \\ \max\{\|r_1-x\|, \|r_2-x\|\}, & |\Delta|/2 < \|x-p\| < |\Delta| \\ \|r_2-x\|, & \|x-p\| \leq |\Delta|/2 \end{cases}$$

$$< \lambda d^{3-\varepsilon} \begin{cases} C_1, & \|x-p\| \geq |\Delta| \\ \max\{C_1, C_2\}, & |\Delta|/2 < \|x-p\| < |\Delta| \\ C_2, & \|x-p\| \leq |\Delta|/2 \end{cases}$$

$$\leq C_0 \lambda d^{3-\varepsilon}.$$

$$\|l_0 - x\| = \begin{cases} \|l_1 - x\|, & \|x-p\| \geq |\Delta| \\ \min\{\|l_1 - x\|, \|l_2 - x\|\}, & |\Delta|/2 < \|x-p\| < |\Delta| \\ \|l_2 - x\|, & \|x-p\| \leq |\Delta|/2 \end{cases}$$

$$> \lambda(15d)^{3-\varepsilon} \begin{cases} C_1, & \|x-p\| \geq |\Delta| \\ \min\{C_1, 10C_2\}, & |\Delta|/2 < \|x-p\| < |\Delta| \\ 10C_2, & \|x-p\| \leq |\Delta|/2 \end{cases}$$

$$\geq C_0 \lambda (15d)^{3-\varepsilon}.$$

(iv) If $w \in C^{\Delta}_{20}(x)$ then $\|x-p\| > |\Delta|/2$. So $x' \in I_0$ implies $x' \in I_1$. Apply (4.2) (iv) to obtain the result. If $w = p$ then $\|x-p\| < |\Delta|$. So $x' \in I_0$ implies $x' \in I_2$. Apply (4.3) (iv).

Q.E.D.

Remark. The "safe" interval $I_0(x, \lambda)$ has been defined for $x \in R^+_\Delta \backslash S$. We can define $I_0(x, \lambda)$ for $x \in R^-_\Delta \backslash S$ by rotating R_Δ 180° about q. (The roles of p and q should be reversed.) We assume then, without proof, the key estimate (4.4) for all $x \in R \backslash S$.

Type 3. We have a similar set of estimates for points in $U \backslash S$.
Let $\mathbf{B} = \Gamma \cup S$.

LEMMA 4.6. *There exists a constant $A_1 > 1$ such that if $x \in U \backslash R$ then*

$$d(x, B) \leq d(x, \Gamma) \leq A_1 d(x, B).$$

Proof. Since $\Gamma \subset B$, we have $d(x, B) \leq d(x, \Gamma)$.

Let $x \in U \backslash R$. Suppose a closest point u of B to x lies in $B \backslash \Gamma$. (If $u \in \Gamma$, we are done.) Then $u \in \partial S_\Delta$, say, for some $\Delta = \Delta(p, q)$. Let $x' \in \partial R_\Delta$ be a closest point to u. Then $d(x, B) = \|x-u\| \geq \|u - x'\|$ since $x \notin R_\Delta$. The simple shape of $R_\Delta \backslash S$ implies $\|u-x'\| \geq 2\sin(\eta/2)\min\{\|u-p\|, \|u-q\|\}$ where $\eta = \tan^{-1}(1/8) - \tan^{-1}(1/10)$. Putting these inequalities together, we get

$$d(x, B) \geq 2\sin(\eta/2)\min\{\|u-p\|, \|u-q\|\}.$$

On the other hand,

$$d(x, \Gamma) \leq \min\{\|x-p\|, \|x-q\|\} \leq \|x-u\| + \min\{\|u-p\|, \|u-q\|\}$$
$$= d(x, B) + \min\{\|u-p\|, \|u-q\|\}.$$

Hence $d(x, \Gamma) \leq d(x, B)(1 + 1/(2\sin(\eta/2)))$. The result follows for $A_1 = 1 + 1/(2\sin(\eta/2))$.

Q.E.D.

Now let $x \in U \backslash S$ and w be a closest point of Γ to x. Then $w \in C^{\Delta}_{1/8}(x)$ if $x \in U^+$ and $w \in C^{\nabla}_{1/8}(x)$ if $x \in U^-$. Let u_3 satisfy $|\sigma(u_3 - x)| = \infty$ and $\|w - x\| = \|u_3 - w\|$. Let $\theta_3 = \theta_3(x)$ be the angle $u_3 x w$. Let $\phi_3 = \pi/2 - \tan^{-1}(1/8)$. Then $0 \leq \theta_3 < \phi_3 < \pi/2$.

For $\lambda > 0$ define

$$I_3 = I_3(x, \lambda) = xu_3 \cap D(w, \|x - w\| - \lambda d^{3-\varepsilon}).$$

Let $u_3 < l_3 < r_3 < x$ be the endpoints of $I_3(x, \lambda)$ if it is non-empty. Let

$$\lambda_3 = (\cos\phi_3)^2/(1 + (15A_1)^{3-\varepsilon}) \text{ and } C_3 = 2/\cos\phi_3.$$

LEMMA 4.7. *If* $0 < \lambda \leq \lambda_3$ *and* $x \in U \backslash S$ *then*

(i) $I_3 = I_3(x, \lambda) \neq \varnothing$;
(ii) $\|r_3 - x\| < C_3 \lambda d^{3-\varepsilon}$;
(iii) $\|l_3 - x\| > C_3 \lambda (15A_1 d)^{3-\varepsilon}$.
(iv) *If* $x' \in I_3$ *and* $\|w - w'\| < \lambda d^{3-\varepsilon}$ *then*

$$\|x' - w'\| < \|x - w\|.$$

Proof. Since $\|u_3 - x\| = 2\|x - w\| \cos\theta_3 \geq 2d\cos\phi_3$ we may apply (4.1) to

$$\phi = \phi_3,\ A = (15A_1)^{3-\varepsilon},\ g(x) = d^{3-\varepsilon} \text{ and } u = u_3.$$

Q.E.D.

§5. DIFFEOMORPHISMS OF THE ANNULUS

A Denjoy homeomorphism of the circle is a homeomorphism with irrational rotation number and a wandering interval. Any C^2 diffeomorphism with irrational rotation number is topologically conjugate to a rotation so that a Denjoy homeomorphism is never C^2.

If $C \subset S^1$ is a Cantor set and f is a homeomorphism of C which extends to a Denjoy homeomorphism of S^1, we call f a Denjoy homeomorphism of C.

The results of Denjoy Fractals [5] imply

Step 1. *For* β *and* $(1 - \varepsilon)$ *sufficiently small there exist a* β*-diamond circle* Q *with uniformity* $\kappa = 3$ *and a* $C^{3-\varepsilon}$ *mapping* $f\colon U \to \mathbf{R}^2$ *satisfying*;

(i) $f|\Gamma$ *is conjugate to a Denjoy homeomorphism*;

$$Df|\Gamma = Id, \quad D^2 f|\Gamma = 0;$$

(ii) $\|f(x) - f(y) - (x - y)\| < C\|x - y\|^{3-\varepsilon}$ *for all* $x, y \in \Gamma$ *where* $x, y \in \Gamma$.

Remark. This step (and only this step) cites notation and results from [5].

Proof. Let β_0 satisfy Lemma 1.9. Choose $(1 - \gamma) < \beta_0 C_1/4C_3$ where C_1 and C_3 are defined in [5, (3.7)]. Let $a = (1 - \gamma)/4$ and $\alpha = \sqrt{2} - 1 = 1/(2 + (1/(2 + \ldots)))$. Denote $Q = h(S^1)$, the curve in $\mathbf{R}^2$ determined by the constants α, γ and a as in [5, (3.1)]. We show that Q is a β-diamond curve for $\beta = (1 - \gamma)(2C_3/C_1)$.

For $I = (x, y) \subset S^1$, let $T(I)$ be the β-diamond with endpoints $h(x)$ and $h(y)$. Let $\Delta_n = h(\rho^{-1}\langle n\alpha\rangle)$, where ρ is a continuous map semi-conjugating a Denjoy homeomorphism to a rotation through 2α. Let S_n be the set S_{Δ_n} defined in §1. Recall the open intervals W_k of the circle in [5, (1.8)]. Define N_k to be the chain containing all the diamonds $T(I)$, $I \in W_k$, and all the diagonals Δ_n associated to an endpoint $\langle n\alpha\rangle$ of I. Let $I = (\langle m\alpha\rangle, \langle n\alpha\rangle) \in W_k$. Since $\tau(m)$ and $\tau(n)$ are minimal over I [5, (3.9)] applies and $S_p \subset T(I)$ for all $\langle p\alpha\rangle \in I$. Since h is continuous, $h(\rho^{-1}(I)) \subset T(I)$. Therefore

$$Q \subset N_k.$$

Since $W_{k-1} \subset W_k$ we have

$$N^{k-1} \subset N^k.$$

We need to prove that N^{k-1} is simple and N^k refines N^{k-1} for $k \geq 1$. This follows if $|\Delta| \leq \beta |T(I)|/2$ for $\Delta \in \Gamma(N^k) \setminus \Gamma(N^{k-1})$ and $T(I) \in \Xi(N^k)$, $k \geq 1$. This is a consequence of [5, (3.7)]: $C_1 |I|^\gamma < H(\rho^{-1}(I)) \leq |T(I)|$. Since $t_{k-1} < |m|$, it follows that $|\Delta_m| = \sqrt{2}a/m^\gamma < C(1-\gamma)|I|^\gamma$ for some $C > 0$. Choosing $(1-\gamma) \leq \beta C_1/2C$ we have

$$|\Delta_m| < \beta |T(I)|/2.$$

Thus each chain N^k is simple and refines N^{k-1}. [5, (3.7)] readily implies

$$\limsup \{ |T(I)| : T \in \Xi(N^k) \} = 0$$

Hence

$$Q = \cap N^k.$$

Since there are no more than 3 W^k-intervals in any W^{k-1}-interval, the uniformity constant is $\kappa = 3$.

It follows that Q is a β-diamond circle with uniformity κ.

By [5, (4.6) and (4.7)] for $\varepsilon = 2 - (1/\gamma)$ there exists a $C^{3-\varepsilon}$ mapping $f: U \to \mathbf{R}^2$ satisfying the conditions of Step 1.

Step 2. For β and $(1-\varepsilon)$ sufficiently small, there exist a β-diamond circle Q with uniformity $\kappa = 3$ and a $C^{3-\varepsilon}$ mapping $F: U \to \mathbf{R}^2$ satisfying

(i) *$F|Q$ is conjugate to a Denjoy homeomorphism with Γ its non-wandering set; $D^kF|\Gamma = D^kf|\Gamma$ for $k = 0, 1, 2$;*
(ii) *F embeds $S_\Delta^\pm$ in $S_{F\Delta}^\pm$ for $\Delta \in \Gamma(Q)$;*
(iii) *$d(F(x), F(\Delta)) < d(x, \Delta)$ for $\Delta \in \Gamma(Q)$ and $x \in S_\Delta \setminus \Delta$.*

Proof. Let f and Q be defined as in Step 1. We denote a member of $\Gamma(Q)$ by Δ_n where $|\Delta_n| = a/n^\gamma$; $f(\Delta_n) = \Delta_{n+2}$; its endpoints are p_n and q_n.

We define $F|S_n = F_n$ in coordinates (u_n, v_n). Let v_n denote the unit vector $(\sigma(\Delta_n)/\sqrt{2}, 1/\sqrt{2})$ and u_n the downward pointing unit vector with slope $\sigma(\Delta_n)/8$, so v_n and u_n are parallel to the sides of S_n.

We define F_n in (u_n, v_n) coordinates.

Fix a C^∞ bump function $\phi: \mathbf{R}^1 \to \mathbf{R}^+$ such that,

(i) supp $(\phi) \subset (0, 1)$;
(ii) $\int \phi(s)\, ds = 1$;
(iii) $\|\phi\| < 3/2$.

Let λ, $\mu > 0$ be constants with $1/3 \leq \mu/\lambda \leq 1$ and set

$$k(x) = x + (\mu - \lambda) \int_0^{x/\lambda} \phi(t)\, dt.$$

Clearly, k is a C^∞ function and

$$\|k' - 1\| \leq |\mu - \lambda| \lambda^{-1} \|\phi\| < |\mu/\lambda - 1|(3/2) \leq 1.$$

Therefore k is a diffeomorphism from $[0, \lambda]$ onto $[0, \mu]$. The first derivative of k is identically one near $x = 0$ and $x = \lambda$. Its higher derivatives are estimated as

$$\|D^j k\| \leq |\mu - \lambda| \lambda^{-j} \|D^{j-1}\phi\|$$

Take $\lambda=|\Delta_n|$ and $\mu=|\Delta_{n+2}|$, $(n\geq 0)$ and call the resulting k function k_n. Then $k_n(0)=0$, $k_n(|\Delta_n|)=|\Delta_{n+2}|$ and $Dk_n(a)=1$, $D^2k_n(a)=0$, for $a=0$ or $|\Delta_n|$.

Define h: $\mathbf{R}^1\to[-1,1]$ so that $h(0)=0$, $Dh(0)=1$, $D^2h(0)=0$; $|h(x)|<|x|$, $|h(|\Delta_{n+2}|)|<|h(|\Delta_n|)|(n\geq 0)$; $h(x)>0$ for $x>0$ and $h(x)<0$ for $x<0$.

In coordinates (u_n, v_n), define $h_n(x, y)=(h(x), k_n(y))$.

Define $F_n(r)=h_n(r-p_n)+p_{n+2}$.

It follows easily that

5.1. *There exists $C>0$ such that,*

(i) $DF_n(r_n)=Id$ *and* $D^2F_n(r_n)=0$ *for* $r=p$ *or* q; $DF_n(r)(u_n)=Id$, $D^2F_n(x)(u_n)=0$ *for all* $x\in\Delta_n$;

(ii) $\|D^3F_n\|<C||\Delta_n|-|\Delta_{n+2}||/|\Delta_n|^3$

(iii) F_n *maps* r_n *to* r_{n+2} *for* $r=p, q$; F_n *embeds* $S_n^{\pm}$ *in* $S_{n+2}^{\pm}$; $d(F_n(x), \Delta_{n+2})<d(x, \Delta_n)$ *for* $x\in S_n\setminus\Delta_n$.

Define $\theta_1|\Gamma=D^if$ on Γ; $\theta_i|S_n=D^iF_n$, $0\leq i\leq 2$.

5.2. *There exist constants C_S and C_Γ such that*

(i) $\theta_1|\Gamma=Id$; $\theta_2|\Gamma=0$;

(ii) C_Γ *is a* $C^{3-\varepsilon}$ *bound for* $\{\theta_i|\Gamma\}$;

(iii) C_S *is a* $C^{3-\varepsilon}$ *bound for* $\{\theta_i|S\}$.

Proof. (ii) follows since f is $C^{3-\varepsilon}$ and Γ is compact.

(iii) By (5.1) (ii) $\|D^3F_n\|<C[|\Delta_n|-|\Delta_{n+2}|]/|\Delta_n|^3$. Hence there exists a constant C_S such that

$$\|D^2F_n(x)-D^2F_n(y)\|<\|D^3F_n\|\,\|x-y\|<C_S\|x-y\|^{1-\varepsilon}\qquad \text{if } \gamma<1/(2-\varepsilon).$$

Apply Theorem 2.1 to conclude that C_S is a $C^{3-\varepsilon}$ bound for $\{\theta_i|S\}$.

Now apply Proposition 2.3 to obtain a $C^{3-\varepsilon}$ extension F of $\{\theta_i\}$ to U. This completes Step 2.

Step 3. *There is a $C^{3-\varepsilon}$ map $K:U\to\mathbf{R}^2$ and a closed annular neighborhood V of Q in U such that K satisfies the conditions for F of Step 2 and the additional condition*

$$d(K(x),\Gamma)<d(x,\Gamma) \text{ for } x\in(V\cap R)\setminus S.$$

Proof. Let F satisfy the conditions of Step 2. Recall λ_0 and C_0 of (4.4).

There exists a smooth closed neighborhood V_0 of Γ with λ_0 a $C^{3-\varepsilon}$ bound for $F|V_0$. We choose V_0 so that it can be extended to a smooth, closed annular neighborhood V of Q and $V\setminus V_0\subset S$. (See Fig. 8.)

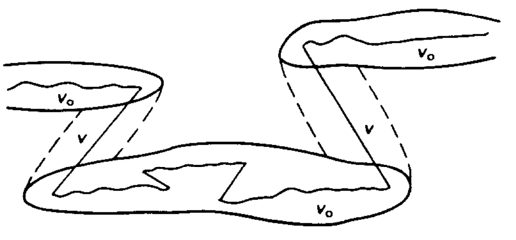

Fig. 8.

5.3. *If* $x \in V_0$ *and* w *is a closest point of* Γ *to* x *then,*

(i) $\|F(x)-F(w)-(x-w)\| \leq \lambda_0 d^{3-\varepsilon}$;
(ii) $\|DF(x)-Id\| \leq \lambda_0 d^{2-\varepsilon}$;
(iii) $\| D^2 F(x) \| \leq \lambda_0 d^{1-\varepsilon}$.

Proof. Note that $w \in V_0$ since $\Gamma \subset V_0$. These estimates follow since λ_0 is a $C^{3-\varepsilon}$ bound for $F|V_0$, $d = d(x, \Gamma) = \|x-w\|$ and $DF|\Gamma = Id$, $D^2F|\Gamma = 0$.

Apply (3.1) to $B = Q$, $C = C_0\lambda_0$ and $r = 3$. If $x \in R\backslash S$ then $d(x, Q) = d(x, \Gamma) = d$ by (1.9). Then there exists a $C^{3-\varepsilon}$ function h_1: $V \to \mathbf{R}^+$ with,

$$C_0\lambda_0 d^{3-\varepsilon} \leq h_1(x) \leq C_0\lambda_0(15d)^{3-\varepsilon} \textit{ for } x \in R\setminus S; \tag{5.4}$$

h_1 vanishes to the 2nd order on Q. It is uniformly bounded on R.

Let u_Δ be the downward pointing unit vector with slope $\sigma(\Delta)/8$. Define h_Δ: $U \to \mathbf{R}^2$ by

$$h_\Delta(x) = x \pm h_1(x)u_\Delta.$$

("$+$" is used if $x \in U^+$ and "$-$" if $x \in U^-$)

For $n = 0, 1, 2$, define θ_n: $Q \cup (R \cap V) \to L^n_s(\mathbf{R}^2, \mathbf{R}^2)$ by $\theta_n | R_\Delta = D^n h_\Delta | R_\Delta$ and $\theta_n | Q = D^n(Id)$. The function h_Δ is uniformly bounded by a constant C_R on R. Clearly 0 is a $C^{3-\varepsilon}$ bound for $\theta_n|Q$. Apply Proposition 2.3 to $B = Q \cup (R \cap V)$, $\{R_i\} = \{R_\Delta \cap V\colon \Delta \in \Gamma(Q)\}$, $\Lambda = \Gamma$, θ_n, $C_\Lambda = 0$ and C_R.

Therefore:

5.5. *There exists a* $C^{3-\varepsilon}$ *mapping* K_1: $U \to \mathbf{R}^2$ *such that for* $n = 0, 1, 2$,

(i) $D^n K_1 | Q = D^n(Id)$, $0 \leq n \leq 2$;
(ii) $D^n K_1 | R_\Delta = D^n h_\Delta$.

Define K: $U \to \mathbf{R}^2$ by $K = F + K_1 - Id$

5.6. (i) $D^n K | Q = D^n F | Q$, $0 \leq n \leq 2$;
(ii) K *maps* S_Δ *into* $S_{F\Delta}$; $d(K(x), F(\Delta)) < d(x, \Delta)$ *for* $x \in S_\Delta \setminus \Delta$;
(iii) $d(K(x), \Gamma) < d(x, \Gamma)$ *if* $x \in (V \cap R)\setminus S$.

Proof. (i) This follows from (5.5) (i).

(ii) By definition $K | S^+_\Delta = F + h_1(x)u_\Delta$. Since $h_1 > 0$, this implies $d(K(x), F(\Delta)) < d(F(x), F(\Delta))$ for $x \in S^+_\Delta$. By assumption $d(F(x), F(\Delta)) < d(x, \Delta)$; F maps $S^\pm_\Delta$ into $S^\pm_{F\Delta}$. Therefore $K(S^\pm_\Delta) \subset S^\pm_{F\Delta}$;

(iii) Since $x \in V \setminus S$, $x \in V_0$. Thus we may apply (5.3). Let $x' = K_1(x)$. Let w be a point of Γ closest to x and $w' = x + F(w) - F(x)$. Then $\| w - w' \| < \lambda_0 d^{3-\varepsilon}$ by (5.3) (i). Then $I_0(x, \lambda_0) \neq \varnothing$ by (4.4) (i). Assume $x \in R_\Delta$. Hence, $K_1(x) = h_\Delta(x)$ by (5.5) (ii). Use (4.4) (ii), (iii) and (5.4) to conclude

$$\|x'-x\| = h_1(x) \leq C_0\lambda_0(15d)^{3-\varepsilon} < \|l_0 - x\|,$$
$$\|x'-x\| = h_1(x) \geq C_0\lambda_0 d^{3-\varepsilon} > \|r_0 - x\|.$$

Recall the interval I_0 of (4.4). Since $\sigma(x'-x) = \sigma(u_\Delta) = \sigma(I_0)$ these two inequalities imply $x' \in I_0$. Therefore we may apply (4.4) (iv) and the assumption $F(\Gamma) = \Gamma$:

5.7. *If* $x \in (V \cap R)\setminus S$, $d(K(x), \Gamma) \leq \| K(x) - F(w) \| = \| x' - w' \| < \| x - w \| = d(x, \Gamma)$.

This completes Step 3.

Step 4. *There is a* $C^{3-\varepsilon}$ *map* $G: U \to \mathbf{R}^2$ *and a closed annular neighborhood* W *of* Q *in* U *such that* G *satisfies the conditions for* F *of Step* 2 *and the additional condition*:

$$\textit{If } x\in W \textit{ then either } G(x)\in S \textit{ or } d(G(x),\Gamma)<d(x,\Gamma).$$

Proof. Let K and V satisfy the conditions of Step 3. There are three special regions for K. There is S which is very well behaved. We will not change anything in S. $R\setminus S$ is also in control. All points in R get mapped closer to Γ. But we will have to change K in $R\setminus S$ in order to gain control over $V\setminus R$. We will apply very similar techniques as in Step 3 to move points in $V\setminus R$ closer to Γ. The effect on $R\setminus S$ is so slight that points in it are moved closer yet to Γ.

Recall λ_3 and C_3 of (4.7). Let W_0 be a smooth closed neighborhood of Γ with λ_3 a $C^{3-\varepsilon}$ bound for $K|W_0$. Choose W_0 so that it can be extended to a smooth, closed annular neighborhood W of Q, $W\subset V$ and $W\setminus W_0\subset S$.

Let $x\in W^+\setminus S$. If w is a closest point to x in Γ then $w\in C^{\Delta}_{1/8}(x)$ by (1.9). The slightest motion of x straight downwards brings x closer to w. We use (4.7) to take advantage of this.

Recall A_1 of (4.6). Apply (3.1) to $B=S$, and $C=C_3A_1^{3-\varepsilon}\lambda_3$. Then there exists a $C^{3-\varepsilon}$ function $h_2: U\to\mathbf{R}^1$ such that,

$$Cd(x,B)^{3-\varepsilon}\le h_2(x)\le C(15d(x,B))^{3-\varepsilon},$$

h_2 vanishes to the 2nd order at S. It follows from (4.7) that

$$C_3\lambda_3 d^{3-\varepsilon}\le h_2(x)\le C_3\lambda_3(15A_1d)^{3-\varepsilon}\ \textit{for } x\in W\setminus R. \qquad 5.8.$$

Let u denote the unit vector $(-1,0)$. Define $G: U\to\mathbf{R}^2$ by

$$G(x)=K(x)\pm h_2(x)u.$$

("$+$" is used if $x\in U^+$ and "$-$" if $x\in U^-$.)

5.9. (i) $D^nG|Q=D^nF|Q$, $0\le n\le 2$;

(ii) G *maps* S_Δ *into* $S_{F\Delta}$; $d(G(x),F(\Delta))<d(x,\Delta)$ *for* $x\in S_\Delta\setminus\Delta$;

(iii) *If* $x\in W\setminus S$ *either* $G(x)\in S$ *or* $d(G(x),\Gamma)<d(x,\Gamma)$.

Proof. (i) Since h_2 vanishes to the 2nd order on Q, $D^nG|Q=D^nK|Q$. Apply (5.6) (i).

(ii) $G=K$ on S. Apply (5.6) (ii).

(iii) Let $x\in W^+\setminus R$ and $x'=x+h_2(x)u$. Then $x\in W_0$. Let w be a point of Γ closest to x and $w'=x+K(w)-K(x)$. By (5.6) (i) $DK(w)=DF(w)=Id$ and $D^2K(w)=D^2F(w)=0$. Thus,

$$\|w-w'\|=\|K(x)-K(w)-DK(w)(x-w)-\tfrac{1}{2}D^2K(w)(x-w)^2\|<\lambda_3 d^{3-\varepsilon}.$$

Recall the interval I_3 of (4.7). Then

$I_3=I_3(x,\lambda_3)\neq\varnothing$. By (4.7) (ii), (iii) and (5.8)

$$\|x-x'\|=h_2(x)\le C_3\lambda_3(15A_1d)^{3-\varepsilon}<\|l_3-x\|,$$
$$\|x-x'\|=h_2(x)\ge C_3\lambda_3 d^{3-\varepsilon}>\|r_3-x\|.$$

Since $\sigma(x-x')=\sigma(u)=\sigma(I_3)$ these two inequalities imply $x'\in I_3$. Therefore we may apply (4.7) (iv) to conclude

$$d(G(x),\Gamma)\le\|G(x)-F(w)\|=\|x'-w'\|<\|x-w\|=d(x,\Gamma)\text{ for }x\in W^+\setminus R.$$

Suppose $x\in(W\cap R^+)\setminus S$. Note that $0<h_2(x)<d(K(x),K(w))$ (by (5.8), (5.3) and (5.6)).

If $w=p$ then by simple geometry either $G(x)\in S$ or

$$\|G(x)-F(w)\|=\|K(x)+h_2(x)u-F(w)\|<\|K(x)-F(w)\|.$$

By (5.7) this is bounded by $\|x-w\|$.

If $w \neq p$ then $w \in \Gamma \cap C^{\Delta}_{20}(x)$ by (1.9). It follows from (5.6) (i) that

$$K(w) = F(w) \in \Gamma \cap C^{\Delta}_{1/8}(K(x)).$$

Simple geometry and (5.7) imply

$$d(G(x), \Gamma) \leq \|G(x) - F(w)\| = \|K(x) + h_2(x)u - F(w)\| \leq \|K(x) - F(w)\| < \|x - w\| = d(x, \Gamma).$$

This completes Step 4.

Definition. Let W be a neighborhood of Q. Then Q is *uniformly attracting* if $G^n(W) \subset W$ for sufficiently large $n \geq 0$ and $\cap G^n(W) = Q$.

Q is *Lyapunov stable* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $n \geq 0$, $G^n(N_\delta(Q)) \subset N_\varepsilon(Q)$.

This next lemma is standard.

LEMMA 5.10. *Let $G: X \to X$ be a continuous endomorphism of a compact metric space X, and W a closed neighborhood of a closed subset Q. Suppose Q is Lyapunov stable and every point in W is attracted to Q. Then Q is uniformly attracting.*

Proof. Choose $\varepsilon > 0$ such that $N_\varepsilon(Q) \subset W$. Let $\delta > 0$ be supplied by the Lyapunov assumption. Every point $p \in W$ is attracted to Q, so for each $p \in W$, there exists n_p with

$$G^{n_p}(p) \in N_\delta(Q).$$

Continuity implies there exists a neighborhood W_p of p with $G^{n_p}(W_p) \subset N_\delta(Q)$. W is compact so there exist $p_1, \ldots, p_m$ with $W = \cup W_{p_i}$. Take $N = \max(n_{p_1}, \ldots, n_{p_m})$. Then $G^n(W) \subset N_\varepsilon Q$ for all $n \geq N$.

Q.E.D.

Step 5. Let F and Q be given as in Step 2. There is a $C^{3-\varepsilon}$ map $G: U \to \mathbf{R}^2$ and a closed annular neighborhood N of Q in U such that,

(i) *$D^k G | Q = D^k F | Q$ for $k = 0, 1, 2$;*
(ii) *$G | N$ is an embedding;*
(iii) *Q is uniformly attracting under G.*

Proof. Let G and W satisfy the conditions of Step 4. First we claim that G is Lyapunov stable. Since $Q \cup S$ has a quasi-structure, there exists a constant $A_2 > 1$ such that for all $x \in N_\delta(Q) \cap S_\Delta$, $\Delta \in \Gamma(Q)$,

$$d(x, \Delta) \leq A_2 d(x, Q) \leq A_2 \delta.$$

Let $\lambda > 0$ be given. Without loss of generality, we assume $N_\lambda(Q) \subset W$. Choose $\delta < \lambda / A_2$ and consider $p \in N_\delta(Q)$.

Suppose first that the entire forward G-orbit of p lies in $W \backslash S$. Then for $n \geq 0$,

$$d(G^n(p), \Gamma) \leq d(p, \Gamma) = d(p, Q) < \delta,$$

so $G^n(p) \in N_\delta(Q) \subset N_\lambda(Q)$. At the other extreme suppose p lies in S. Then $p \in S_{\Delta_m}$ for some m. Then its entire forward orbit lies in S and

$$d(G^n(p), \Delta_{m+n}) \leq d(p, \Delta_m) \leq A_2 \delta < \lambda,$$

shows that $G^n(p) \in N_\lambda(Q)$. Finally suppose that $p \in W \backslash S$ and for some smallest $k \geq 1$, $G^k(p) \in S_{\Delta_m}$.

Then, as shown above, $G^n(p)\in N_\delta(Q)$, for $0\le n\le k$. And, for $k\le n$,

$$d(G^n(p),\ \Delta_{m+n-k})\le d(G^k(p),\Delta_m)\le A_2\delta<\lambda.$$

Thus, $G^n(p)\in N_\lambda(Q)$. This proves $G^n(N_\delta(Q))\subset N_\lambda(Q)$ for all $n\ge 0$ and verifies Lyapunov stability.

From (5.9) (i) $D^nG|Q=D^nF|Q$, $G|Q$ is $1-1$ and DG_x is invertible for all $x\in Q$. Hence, for some compact neighborhood W_1 of Q, G embeds W_1. By Lyapunov stability, there exists $\delta>0$ such that $G^n(N_\delta(Q))\subset W_1$ for all $n\ge 0$. Let

$$W_2=\cup G^n(N_\delta(Q)),\quad n\ge 0.$$

Then $cl(W_2)$ is compact, it is contained in W_1 and is carried into itself by G.

We claim Q pointwise attracts $cl(W_2)$. Let $p\in cl(W_2)$. If $p\in S$ then $G^n(p)\in S_\Delta$ and $|S_\Delta|\to 0$ as $n\to\infty$, so $G^n(p)\to Q$. If $p\in W\backslash S$ and $G^k(p)\in S$ for some k then the same argument holds: $G^n(p)\to Q$ as $n\to\infty$. Suppose $p\in W\backslash S$ and $G^n(p)\in W\backslash S$ for all $n\ge 0$. Either $G^n(p)\to 0$ or else there is a subsequence $G^{n_k}(p)$ converging to some $r\notin Q$. This r does not belong to S because all points of S are attracted to Q. Consequently $r\in W\backslash S$. Since $d(G(r),\Gamma)<d(r,\Gamma)$ we have,

$$d(G(r'),\Gamma)\le d(r',\Gamma)-\mu,$$

for some $\mu>0$ and all r' near r. For large k, $G^{n_k}(p)$ is near p so

$$\begin{aligned} d(G^{n_{k+1}}(p),\Gamma)&<d(G^{n_{k+1}-1}(p),\Gamma)<\ \dots\ <\\ &<d(G^{n_k+1}(p),\Gamma)=d(G(G^{n_k}(p)),\Gamma)\\ &\le d(G^{n_k}(p),\Gamma)-\mu.\end{aligned}$$

Hence

$$d(G^{n_k}(p),\Gamma)-d(G^{n_{k+1}}(p),\Gamma)\ge\mu$$

which contradicts convergence of $d(G^n(p),\Gamma)$ as $n\to\infty$. Therefore $G^n(p)\to Q$. Pointwise attraction is verified.

By (5.10) applied to $G\colon cl(W_2)\to cl(W_2)$ we see that G uniformly attracts $cl(W_2)$. Let N be an annular neighborhood of Q in $cl(W_2)$. Since $N\subset W_2$, G^n is defined on N and embeds it for all $n\ge 0$. Moreover $G^n|N$ converges uniformly to Q. Hence for some n_0, $G^n_0(N)\subset N$.

This completes Step 5.

Step 6. *Let F and Q be given as in Step 2. There is a $C^{3-\varepsilon}$ map $G\colon U\to\mathbf{R}^2$ and a closed annular neighborhood N of Q in U such that,*

(i) $D^kG|\Gamma=D^kF|\Gamma$ *for* $k=0,1,2$;
(ii) $G|N$ *is an embedding*;
(iii) Q *is uniformly semi-stable under* H.

This last condition means that Q attracts from one side and repells from the other; i.e.,

$$G^{\pm n}(N^\pm)\subset N^\pm$$

for sufficiently large $n\ge 0$ and

$$\cap G^{\pm n}(N^\pm)=Q$$

Proof. Knowing only the special data of f at the Cantor set Γ in Step 1, it is possible to make extensions of $f|\Gamma$ to U for which Q is uniformly attracting, repelling, or semi-stable. We chose to present the attracting case for its most natural notation. To make a semi-stable example, merely define $(F_n)^{-1}|S_\Delta^-$ in Step 2 to be attracting and with rotation number -2α

instead of 2α. That is, define k_n^- by substituting $\lambda = |\Delta_n|$ and $\mu = |\Delta_{n-2}|$ into the formula for k. Define $h_n^-(x, y) = (h(x), k_n^-(y))$ and $(F_n)^{-1}(r) = h_n^-(r - p_n) + p_{n-2}$. $F_n|S_\Delta^+$ is defined exactly as in Step 2. Since k_n, k_n^- and h are C^2-tangent to the identity at the origin, F_n is $C^{3-\varepsilon}$. Except for minor, trivial remarks, Steps 3–5 are identical.

This completes Step 6.

Next is a standard result from differential topology.

LEMMA 5.11. *If A_1 and A_2 are $C^{r+\alpha}$ smooth annuli and f_0: $\partial A_1 \to \partial A_2$ is a $C^{r+\alpha}$ diffeomorphism preserving orientation of each boundary component then there exists a $C^{r+\alpha}$ diffeomorphism f: $A_1 \to A_2$ extending f_0. Moreover, if f_0 is already extended to a neighborhood of ∂A_1 then f can be made to agree with a restriction of this extension.*

Proof. A_1 and A_2 are diffeomorphic to the standard annulus A. The problem then becomes to extend a given f_0: $\partial A \to \partial A$ to a diffeomorphism $A \to A$. The map $f_0|\partial^+ A$ is a diffeomorphism f^+: $S^1 \to S^1$ and likewise $f_0|\partial^- A$ is $f^-: S^1 \to S^1$. Both f^+ and f^- preserve orientation. Therefore they are isotopic as diffeomorphisms $S^1 \to S^1$. Suspend the isotopy to give a diffeomorphism f: $A \to A$.

Step 7. There is a $C^{3-\varepsilon}$ diffeomorphism H: $cl(U) \to cl(U)$ such that;

(i) *$H|\partial U$ is the identity;*
(ii) *$H|U$ has no periodic points;*
(iii) *The outer boundary $\partial^+ U$ is a repellor;*
(iv) *The inner boundary $\partial^- U$ is an attractor;*
(v) *There is an orbit asymptotic to both $\partial^+ U$ and $\partial^- U$;*
(vi) *There is an orbit bounded away from ∂U.*

Proof. From Step 6 we have an annular neighborhood N of Q and an embedding $G: N \to U$ with

$$G^{\pm n}(N^\pm) \subset N^\pm \text{ for } n \geq n_0.$$

Let $H_1 = G^{n_0}$. H_1 carries $\partial^+ N$ strictly inside N and $\partial^- N$ strictly outside N.

Since $\cap G^{\pm n}(N^\pm) = Q$, there exist $x_0 \in N^+ \setminus S$ and $y_0 \in \mathrm{int}(S_\Delta^-)$ with $H_1(x_0) \in \mathrm{int}(S_\Delta^+)$ and $H_1(y_0) \in N^- \setminus S$.

Let ξ be a C^∞ smooth diffeomorphism which is the identity outside S_Δ and sends $H_1(x_0)$ to y_0.

Define $H_2 = \xi \circ H_1: N \to U$. It is a $C^{3-\varepsilon}$ embedding, preserving orientation of each boundary component. In two iterations it sends $x_0 \in N^+ \setminus S$ to $N^- \setminus S$.

Lemma 5.11 may be used to extend H_2 to a diffeomorphism H_3 of U fixing both boundary components. Let V_1 be the annulus bounded by $\partial^+ U$ and $\partial^+ N$. Note that H_3 maps $\partial^+ U$ onto itself and $\partial^+ N$ inside N. Thus H_3 may be easily perturbed to H_4 so that $\partial^+ U$ is a repeller and if x, $H_4(x) \in V_1$ then $d(H_4(x), \partial^+ N) < d(x, \partial^+ N)$. (We put no restrictions on H_4 if $x \in V_1$, $H_4(x) \notin V_1$). Similarly perturb H_4 to H_5 in the annulus V_2 bounded by $\partial^- N$ and $\partial^- U$ so that $\partial^- U$ is a repeller of H_5^{-1}. If x, $H_5(x) \in V_2$ then $d(H_5^{-1}(x), \partial^- N) < d(x, \partial^- N)$. Let $H = H_5$.

It is immediate that $H|\partial U$ is the identity.

We verify (ii)–(vi);

(ii) Assume $x \in U^+ \setminus Q$. If $x \in S_\Delta^+$ then $H^n(x) \in S_{F^n\Delta}^+$ until it possibly lies in the support of ξ. Then $H^{n+1}(x)$ might be in U^-. If so, it stays in U^-. So $H^n(x) \neq x$ for all $n \geq 1$. If $H^{n+1}(x)$ is not in U^- then it stays in S^+, never returning to S_Δ^+. Suppose $x \in U^+ \setminus S$. There are two possibilities. If $x \in N \setminus S$ then $H = G$ and $d(H(x), \Gamma) < d(x, \Gamma)$. N is mapped into itself, so this inequality continues unless $H^n(x)$ ever enters S^+. Hence x is not periodic. Otherwise, $x \in U^+ \setminus N$. If $H(x) \in U^+ \setminus N$ then $d(H(x), \partial^+ N) < d(x, \partial^+ N)$. This inequality continues until $H^n(x) \in N$. Then the orbit never comes back into $U^+ \setminus N$. Hence x is not a periodic point. If $x \in U^-$ repeat the argument using H^{-1}.

(iii) and (iv) This is built into the definition of H.

(v) Consider x_0. Observe that $H^{-1}(x_0) \in U^+ \setminus S$ implies $H^{-n}(x_0)$ is asymptotic to $\partial^+ U$. Also $H^n(y_0) \in U^- \setminus S$ implies $H^n(y_0)$ is asymptotic to $\partial^- U$. Finally note that $H(x_0) = y_0$.

(vi) Γ is an invariant set for F and thus for H. Every orbit in Γ is bounded away from ∂U.

This completes step 7.

Step 8. *There is a $C^{3-\varepsilon}$ vector field on S^3 which has no zeros and no closed integral curves.*

Proof. Since Steps 1–7 were all **Z**-equivariant we may project H to a diffeomorphism of the annulus A and then to the two-sphere. This final $C^{3-\varepsilon}$ diffeomorphism of S^2 has the north pole as a repeller, the south pole an attractor, one orbit asymptotic to both poles and one orbit is bounded away from the poles. It has no periodic points.

Step 8 follows from Theorem A.

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Department of Mathematics,
University of California,
Berkeley,
U.S.A.