# Lecture 2. A non-trivial error correcting code.

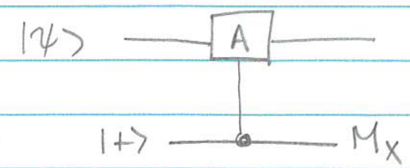
key points from last lecture :

- 3 qubit code protecting against bit flip error
- used two ancillas which measured $Z_1Z_2$ & $Z_2Z_3$ evalues
- acceptable "code space" $\overline{|\psi\rangle} = \alpha|000\rangle + \beta|111\rangle$ was in +1 eigenspaces of all "stabilizers" ← to be defined.
- (if $Z_1Z_2$ or $Z_2Z_3$ measured $-1$ eigenvalue then we had a good guess for what went on)
- (we saw how to non-destructively measure eigenvalues)
- (continuous errors were discretised)

goal for today,

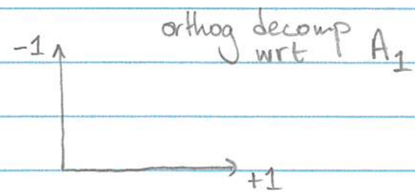
- see this problem in a more mathematical framework "stabilizer formalism"
- give a nice example [[7,1,3]] CSS code,
  - will see two examples of "logical gates"
  - will get a first smell of "transversality"
- a consequence of this abstraction will be (for this lecture only) a loss of "geometry" or relation to the physical layout of qubits.

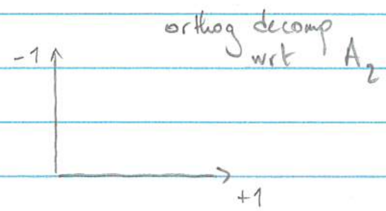
# Commentary from lecture 1



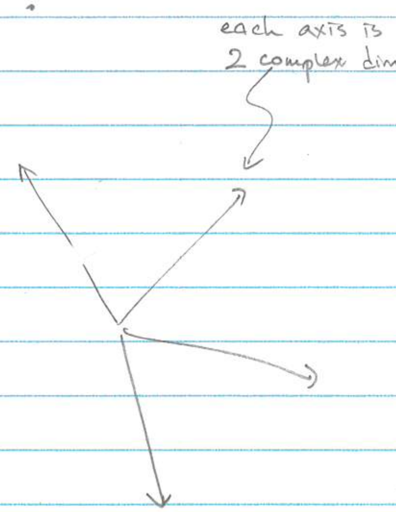

- $-M_X = -\boxed{H}- M_Z = -\boxed{H}-\boxed{\nearrow}$
- Example of $A = "Z_i Z_j"$ allowed me to measure "parity" between qubits $i, j$. It is *not* the same as measuring in $Z$ basis qubits $i, j$.
- this is called "funky phase kickback" ... apparently.

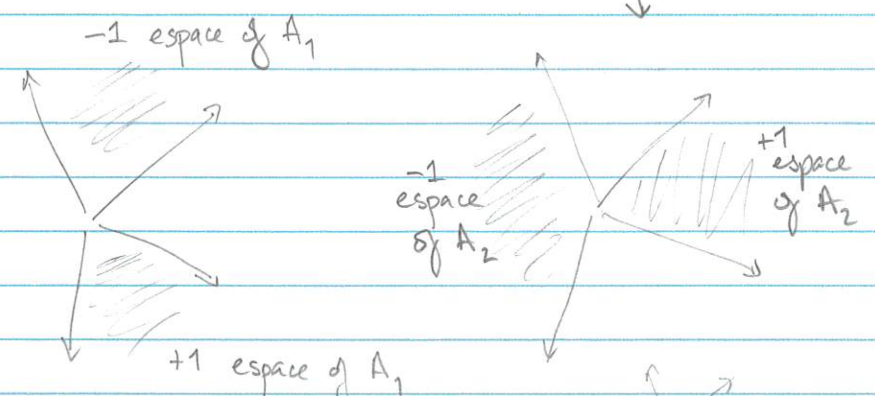
Here is the picture I have in my head :



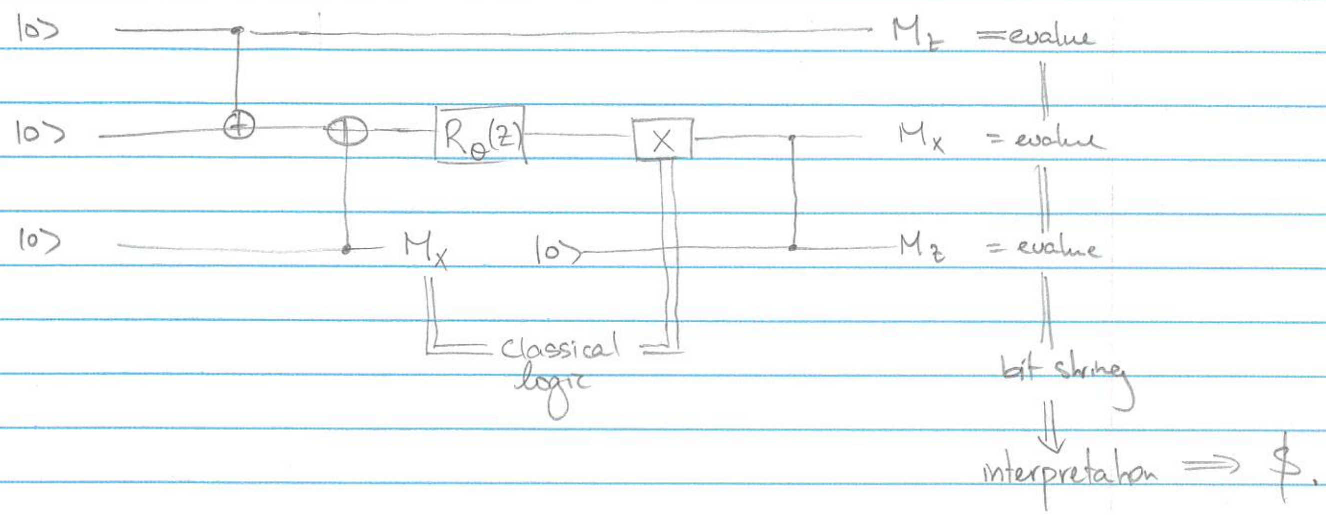

$\mathbb{C}^{2^3} = \mathbb{C}^8$ :




INTERSECTION OF +1 ESPACES ≡ CODE SPACE :

## What's the big picture from a circuit point of view ?




- on the top line, I have a long time where I need to preserve the state of my qubit.
- this is the same as wanting to apply the identity operator
- maybe noise happens
- yesterday, today, 3rd lecture are showing us ways to solve this problem.

- sooner or later, we need to look at all the other parts of this problem.
  - 2q gates, CNOT
  - X, Y, Z operators
  - Phase, H,
  - Arbitrary rotation gates

- but also
  - state preparation
  - measurement.

- I'll atleast talk about how to do the computation part in the 4th lecture.

# Stabilizer formula

three examples (degrees of freedom)

| | | | |
|---|---|---|---|
| 1 qubit | $\langle Z \rangle \xrightarrow{H} \langle HZH^\dagger \rangle = \langle X \rangle$ | $\vert 0\rangle \xrightarrow{H} \vert +\rangle$ | $Z\vert 0\rangle = \vert 0\rangle$, $X\vert +\rangle = \vert +\rangle$ |

$$\langle Z \rangle \longleftrightarrow |0\rangle \qquad Z|0\rangle = |0\rangle$$

$$\langle HZH^\dagger \rangle = \langle X \rangle \longleftrightarrow |+\rangle \qquad X|+\rangle = |+\rangle$$

2 qubits

$$\langle XX, ZZ \rangle \longleftrightarrow |\Phi^+\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$\downarrow IX \qquad\qquad \downarrow IX$$

$$\langle XX, -ZZ \rangle \longleftrightarrow |\Psi^+\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

2 qubits, 1 degree of freedom

$$\langle ZZ \rangle \longleftrightarrow |\psi\rangle = \alpha|00\rangle + \beta|11\rangle$$

Def$^n$. Pauli group for 1 qubit $\mathcal{P}_1 = \{\pm 1, \pm i\} \times \{I, X, Y, Z\}$

Pauli group for $n$ qubits $\mathcal{P}_n = \mathcal{P}_1^{\otimes n}$

Def$^n$. An $n$-qubit stabilizer group $\mathcal{S}$ is an abelian subgroup of $\mathcal{P}_n$ st $-I \notin \mathcal{S}$

Lemma All elements $S$ of a stabilizer $\mathcal{S}$ have eigenvalues $\pm 1$.

Def$^n$ $\mathcal{C}_n$ = Clifford group $= \{ U \in \mathcal{U}(2^n) \mid UPU^\dagger = P' \text{ for } P, P' \in \mathcal{P}_n \}$

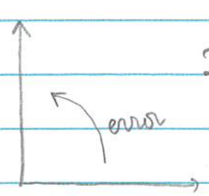
Heisenberg picture $\qquad\qquad$ Schrödinger picture

$$
\begin{tikzcd}
\langle S_i \rangle \arrow[rr, leftrightarrow, "S_i|\psi\rangle = |\psi\rangle \ \forall i"'] \arrow[d, "U"] & & |\Psi\rangle \arrow[d, "U"] \\
\langle S_i' \rangle \arrow[rr, leftrightarrow, "S_i'|\psi'\rangle = |\psi'\rangle"'] & & |\psi'\rangle
\end{tikzcd}
$$

$S_i' = U S_i U^{\dagger}$ $\qquad$ $|\psi'\rangle = U|\psi\rangle$

Remember our 3 qubit code example

$$
\begin{tikzcd}
\langle ZZI, IZZ \rangle \arrow[r, phantom, "\sim"] \arrow[d, "IIX"] & |\psi\rangle = \alpha|000\rangle + \beta|111\rangle \arrow[d, "IIX"] \\
\langle ZZI, -IZZ \rangle \arrow[r, phantom, "\sim"] & |\psi\rangle = \alpha|001\rangle + \beta|110\rangle
\end{tikzcd}
$$



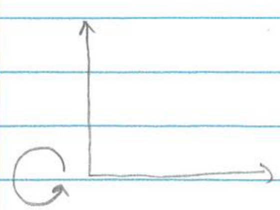

$2^3$ dimensional space

2 dimensional space of '+1 espace of all stabilizers'

OR

$$
\begin{tikzcd}
\langle ZZI, IZZ \rangle \arrow[r, phantom, "\sim"] \arrow[d, "XXX"] & |\psi\rangle = \alpha|000\rangle + \beta|111\rangle \arrow[d, "XXX"] \\
\langle ZZI, IZZ \rangle \arrow[r, phantom, "\sim"] & |\psi\rangle = \beta|000\rangle + \alpha|111\rangle
\end{tikzcd}
$$

logical X



# A proper error correcting code.

We could look at a 5 qubit code with 4 stabilizer generators

$S_1$ XZZX1
$S_2$ 1XZZX
$S_3$ X1XZZ
$S_4$ ZX1XZ

which you can check would correct single X, Z errors.
eg. if X error occured on 3rd qubit,
then measuring $S_1, S_2$ would give −1 evalues

And there are also logical operators $X_L$ = XXXXX
$Z_L$ = ZZZZZ

But the following 7 qubit code is way snazzier

$S_1$ = 111 XXXX
$S_2$ = 1XX 11XX
$S_3$ = X1X 1X1X
$S_4$ = 111 ZZZZ
$S_5$ = 1ZZ 11ZZ
$S_6$ = Z1Z 1Z1Z

- CSS ≡ only X's or only Z's occur on stabilizers
- Also corrects single qubit pauli errors
- Logical X, Z (all Xs, all Zs)

But also : · logical H ($H^{\otimes 7}$) — preserves stabilizers
— transforms $X_L, Z_L$ appropriately

& · logical phase S ( $(ZS)^{\otimes 7}$ )

& · logical CNOT