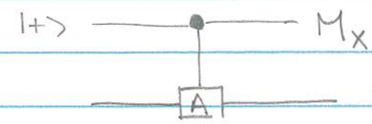
Indirect measurement.

Consider

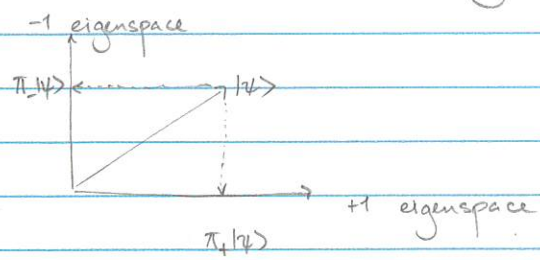


where $A^2 = 1$, bottom line is arbitrary number of qubits

So A acts on $|\psi\rangle \in \mathbb{C}^{2^n}$

$A^2 = 1 \implies (A+1)(A-1) = 0 \implies$ eigenvalues of A are ± 1

let's draw $\mathbb{C}^{2^n}$:



What are the projectors $\pi_\pm$? $\quad \frac{I \pm A}{2}$

Proof: Is $\pi_+|\psi\rangle = \frac{1+A}{2}|\psi\rangle$ really a +1 eigenvector of A?

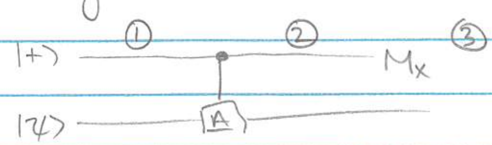
$$A\left(\pi_+|\psi\rangle\right) = A\left(\frac{1+A}{2}\right)|\psi\rangle = \left(\frac{A+I}{2}\right)|\psi\rangle = \left(\pi_+|\psi\rangle\right) \quad \square$$

One annoying thing is that $\pi_\pm|\psi\rangle$ aren't of unit length...

Let's write $|\psi\rangle = \alpha\,\underbrace{|\psi_+\rangle + \beta\,|\psi_-\rangle}_{\text{unit length}}$ where $|\psi_\pm\rangle = \pi_\pm|\psi\rangle \big/ \langle\psi|\pi_\pm\psi\rangle$

and $\alpha, \beta = \langle\psi|\pi_\pm\psi\rangle$

Analyse indirect measurement in three stages



Stage ①

$$|+\rangle \otimes |\psi\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle \right) \otimes |\psi\rangle$$

$$= \frac{1}{\sqrt{2}} |0\rangle \otimes |\psi\rangle + \frac{1}{\sqrt{2}} |1\rangle \otimes |\psi\rangle$$

Stage ②

$$\frac{1}{\sqrt{2}} |0\rangle \otimes I|\psi\rangle + \frac{1}{\sqrt{2}} |1\rangle \otimes A|\psi\rangle$$

$$= \frac{1}{\sqrt{2}} |0\rangle \otimes \left(\alpha |\psi_+\rangle + \beta |\psi_-\rangle \right) + \frac{1}{\sqrt{2}} |1\rangle \left(\alpha |\psi_+\rangle - \beta |\psi_-\rangle \right)$$

$$= \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle \right) \otimes \alpha |\psi_+\rangle + \frac{1}{\sqrt{2}} \left(|0\rangle - |1\rangle \right) \otimes \beta |\psi_-\rangle$$

$$= \alpha |+\rangle \otimes |\psi_+\rangle + \beta |-\rangle \otimes |\psi_-\rangle$$

Stage ③

Measure X will either see $|+\rangle$ or $|-\rangle$ and the probabilities are $|\alpha|^2$, $|\beta|^2$ respectively.

if X measures $|+\rangle$ then $|\psi\rangle$ is definitely in $|\psi_+\rangle$
$|-\rangle$ $|\psi_-\rangle$

Remark that if I wanted to stay in $|\psi_+\rangle$ state, but my ancilla was read out as $|-\rangle$ (so I am actually in $|\psi_-\rangle$) then hopefully I have a procedure / program that I can apply so that

$$|\psi_-\rangle \xrightarrow{\text{program}} |\psi_+\rangle$$

How do we quantumly error correct?

Early ideas in quantum information were just translations of classical ideas into quantum notation. But error correction can't even begin due to no cloning.

majority vote $\quad 0 \longrightarrow 000$

$\quad\quad\quad\quad\quad\quad\quad 1 \longrightarrow 111$

error with order $p \longrightarrow$ error with order p^2

$$|\psi\rangle \xrightarrow{?} |\psi\rangle|\psi\rangle|\psi\rangle$$

Challenges

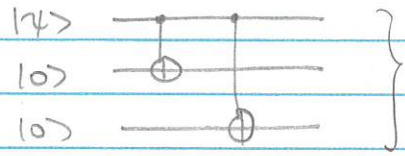
- no cloning theorem
- measuring data in order to determine errors will destroy superpositions
- must correct Y, Z errors, not just X errors
- must correct infinite set of unitary errors (and channels which decohere state)

(unitary channels, pauli errors, dephasing depolarizing, amplitude damping, photon loss, multiqubit channels)

- the most difficult ~ errors during computation ~ leading to notion of fault tolerance (only to be touched upon at end)

A quantum code to protect against bit flips.

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \xrightarrow{\text{encoding}} \alpha|000\rangle + \beta|111\rangle = |\overline{\psi}\rangle$



$\alpha|000\rangle + \beta|111\rangle$

(this seems to have circumvented no-cloning problem)

Pretend we get a bit flip error on 2nd qubit

$X_2|\overline{\psi}\rangle = \alpha|010\rangle + \beta|101\rangle$

the subtle point is to realise the middle qubit is different from qubits 1,3 irrespective of state $|\psi\rangle$

rather than doing measurements on qubits 1,2,3 (which would destroy superposition) we can measure pairwise parity between qubits

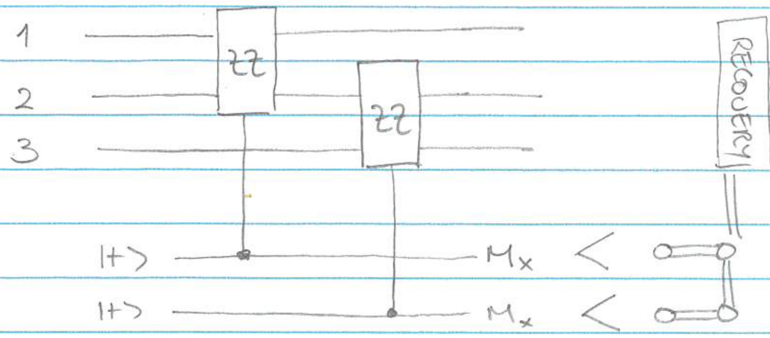
(this seems to circumvent destroying superposition problem)

measure parity between 1,2 $=$ measure Z_1Z_2 eigenvalue

$|+\rangle \Longrightarrow$ same parity

$|-\rangle \Longrightarrow$ opposite parity

do this also for parity between qubits 2,3



in the case that error X_2 happened, we observe that

$$Z_1 Z_2 \left(X_2 |\bar{\psi}\rangle \right) = - \left(X_2 |\bar{\psi}\rangle \right)$$
$$Z_2 Z_3 \left(X_2 |\bar{\psi}\rangle \right) = - \left(X_2 |\bar{\psi}\rangle \right)$$

so our wavefunction is -1 evalue of Z_1Z_2 & Z_2Z_3
so both M_x measurements will return $|-\rangle$ state

Use this to work out where a potential X_k error occurred :

Measurement result of ancillas 1,2 $\Longrightarrow$ position of bit flip

$\vert+\rangle$	$\vert+\rangle$	no error
$\vert+\rangle$	$\vert-\rangle$	X_1 error
$\vert-\rangle$	$\vert+\rangle$	X_3 error
$\vert-\rangle$	$\vert-\rangle$	X_2 error

Abstractly we have a table

Syndrome	Error	Recovery
00	I I I	I I I
01	X I I	X I I
10	I I X	I I X
11	I X I	I X I

(now go up and update code word above)

We could go and look at phase flip errors (this would be worth while) but I'm more interested in the continuous version of a bit flip error:

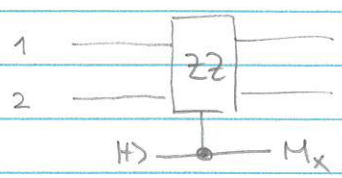
$$R_\theta(X) = \exp(-i\theta X)$$
$$= \cos\theta \, I - i \sin\theta \, X$$

Suppose I got this on the first qubit:

Codeword $|\bar{\psi}\rangle = \alpha|000\rangle + \beta|111\rangle$

$$\longrightarrow R_\theta(X_1)|\bar{\psi}\rangle = \cos\theta \, |\bar{\psi}\rangle - i\sin\theta \, X_1|\bar{\psi}\rangle$$

Note that $Z_1 Z_2 |\bar{\psi}\rangle = \underline{+1} \, |\bar{\psi}\rangle \qquad Z_1 Z_2 \left(X_1|\bar{\psi}\rangle\right) = \underline{-1} \, \left(X_1|\bar{\psi}\rangle\right)$



two options: either measure $|+\rangle$ w/ prob $\cos^2\theta$
or measure $|-\rangle$ $\sin^2\theta$

If measure $|+\rangle$ then wavevector projects to $|\bar{\psi}\rangle$:)
$|-\rangle$ $X_1|\bar{\psi}\rangle$:(

This has DISCRETIZED the error! So we can return to previous problem where we just had Pauli-bit-flip errors!!!

seems to have circumvented problem of infinite set of unitary errors.