Kitaev in the sky with diamonds

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Rigetti Computing

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summary

A classical problem for probability distributions

A quantum problem for states

A quantum problem for channels

Kitaev's a pretty smart guy

In physics, an example of this design science approach is Kitaev's notion of a topological quantum computer. This is one of the most radical new ideas of the past hundred years. Rather than building a computer out of component parts, the aspiration is to create a novel phase of matter that wants to compute. Fluids want to flow; solids want to maintain a stable shape; topological quantum computers want to compute. Indeed, not only do they want to compute, they want to quantum compute, and to do so in a way that protects the quantum state against the effects of noise!

- Michael Nielsen

Kitaev sometimes has his head in the clouds

The results of Sec. 5 suggest that fermions have slightly more computational power than qubits. The logarithmic slowdown in simulation of fermions seems to be inevitable in the general case. However, in the physical world fermions (e.g. electrons) interact locally not only in the sense that the interaction is pairwise, but also in the geometric sense: a particle can not instantly jump to another position far away. Such physical interactions might be easier to simulate. In this section we study an abstract model of geometrically local interactions. The result is that geometrically local gates can indeed be simulated without any substantial slowdown, i.e. the simulation cost is constant. Therefore one can speculate that, in principle, electrons might not be fundamental particles but, rather, excitations in a (nonperturbative) system bosons. Of course, this is only a logical possibility which may or may not be true.

- Sergey Bravyi, Alexei Kitaev

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A natural norm on the space of transformations is given by

$$\|T\|_1 = \sup_{X \neq 0} \frac{\|TX\|_1}{\|X\|_1} = \sup_{X \neq 0} \frac{\|T^* X\|}{\|X\|}.$$

Unfortunately, this norm is unstable relative to the tensor product. Example: we consider the transformation $T : |i\rangle\langle j| \mapsto |j\rangle\langle i|$ $(i, j = 0, 1)$. It is clear that $\|T\|_1 = 1$, however $\|T \otimes I_{\mathcal{B}}\|_1 = 2$. (Apply the transformation $T \otimes I_{\mathcal{B}}$ to the operator $X = \sum_{i,j} |i,i\rangle\langle j,j|$.) This is why we shall use another norm on the space of transformations $T(\mathcal{N}, \mathcal{M})$:

$$\|T\|_\diamond = \inf\left\{ \|A\| \, \|B\| : \mathrm{Tr}_{\mathcal{F}}(A \cdot B^\dagger) = T \right\}, \qquad A, B \in \mathbf{L}(\mathcal{N}, \mathcal{M} \otimes \mathcal{F}). \qquad (21)$$

Here $\mathcal{F}$ is an arbitrary unitary space of dimension $\geqslant (\dim \mathcal{N})(\dim \mathcal{M})$. The following result implies that this is indeed a norm.

Proposition 3.9. $\|T\|_\diamond = \|T \otimes I_{\mathcal{G}}\|_1 \geqslant \|T\|_1$, *where* $\dim \mathcal{G} \geqslant \dim \mathcal{N}$.

- Kitaev. 1997

5.3 The Diamond Metric on Quantum Gates

The natural norm on the space of super-operators is

$$\|T\|_1 \;=\; \sup_{X\neq 0} \frac{\|TX\|_1}{\|X\|_1}$$

Unfortunately, this norm is not stable with respect to tensoring with the identity. Counterexample: $T : |i\rangle\langle j| \mapsto |j\rangle\langle i| \;\; (i, j = 0, 1)$. It is clear that $\|T\|_1 \leq 1$. However $\|T \otimes I_{\mathcal{B}}\|_1 \geq 2$. (Apply the super-operator $T \otimes I_{\mathcal{B}}$ to the operator $X = \sum_{i,j} |i, i\rangle\langle j, j|$). For this reason, we have to define another norm on super-operators

definition 8 *Let $T : \mathbf{L}(\mathcal{N}) \to \mathbf{L}(\mathcal{M})$ and $A, B \in \mathbf{L}(\mathcal{N}, \mathcal{M} \otimes \mathcal{F})$, where $\mathcal{F}$ is an arbitrary Hilbert space of dimensionality* $\geq (\dim \mathcal{N})(\dim \mathcal{M})$.

$$\|T\|_\diamondsuit \;=\; \inf\Big\{\|A\|\,\|B\| \,:\, \mathrm{Tr}_{\mathcal{F}}(A \cdot B^\dagger) = T\Big\}$$

This definition seems very complicated. However it is worthwhile using this norm because it satisfies very nice properties, and provides powerful tools for proofs regarding quantum errors. Here are some properties which are satisfied by the diamond norm. The first property is that the diamond norm is the stabilized version of the “naive” norm $\|\cdot\|_1$. The proof of this is complicated and non-trivial. It implies also that $\|\cdot\|_\diamondsuit$ is a norm.

- Aharonov, Kitaev, Nisan. 1997

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and dependent on the outcome k, Antoinette prepares p_k and gives it to Bonaparte. Bonaparte then observes $x \in X$, and he is asked to guess which binary variable had been chosen by Antoinette.

A classical problem: Bayes' theorem

$$P(C = 0|X = x)$$

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$$P(C = 0|X = x) = \frac{\lambda p_0(x)}{\lambda p_0(x) + (1 - \lambda)p_1(x)}$$

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$$P(C = 0|X = x) = \frac{\lambda p_0(x)}{\lambda p_0(x) + (1 - \lambda)p_1(x)}$$

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Bonaparte should choose the binary variable $\{0,1\}$ dependent on whichever probability is larger.

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Bonaparte should choose the binary variable $\{0, 1\}$ dependent on whichever probability is larger.
More interestingly, how often will he get this choice right?

A classical problem: how good is your guess?

$$\mathbf{zero} = \{x \in X | \text{Bonaparte should guess } p_0 \text{ as distribution from Antoinette}\} \subset X$$

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[0] two remarks: $\lambda = 1/2$, $p_0 = p_1$, and also p_0, p_1 distinct

A quantum problem on states

Consider two density matrices ρ_1, ρ_2 on a complex vector space X. Consider a binary space $C = \{0, 1\}$ where Antoinette chooses

- 0 with probability λ;
- 1 with probability $1 - \lambda$

and dependent on the outcome k, Antoinette prepares ρ_k and gives it to Bonaparte. Bonaparte's goal is to measure the system to best determine which binary variable had been chosen by Antoinette.

Holevo-Helstrom theorem

$$P(\text{Success}) = \lambda \sum_{x \in \mathbf{zero}} p_0(x) + (1-\lambda) \sum_{x \in \mathbf{one}} p_1(x)$$

Holevo-Helstrom theorem

Bonaparte introduces a measurement whose labels tell him whether to predict the binary variable was 0 or 1:

$$\mu : \{0, 1\} \to \mathrm{Pos}(X)$$

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Bonaparte introduces a measurement whose labels tell him whether to predict the binary variable was 0 or 1:

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Holevo-Helstrom gives a bound on Bonaparte's predictive power:

$$\lambda \langle \mu(0) \,|\, \rho_0 \rangle + (1 - \lambda) \langle \mu(1) \,|\, \rho_1 \rangle \leq \frac{1}{2} + \frac{1}{2} \|\lambda \rho_0 + (1 - \lambda) \rho_1\|_1$$

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In fact, HH tells Bonaparte how to attain this limit.

HH has a pretty easy proof (like all nice theorems)

- Begin by making two mixed "states" $\rho_{\pm} = \lambda\rho_0 \pm (1-\lambda)\rho_1$.

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&= \frac{1}{2} \langle \mu(0) + \mu(1) \,|\, \rho_+ \rangle + \frac{1}{2} \langle \mu(0) - \mu(1) \,|\, \rho_- \rangle \\
&\leq \frac{1}{2} + \frac{1}{2} \|\mu(0) - \mu(1)\|_\infty \|\rho_-\|_1 \\
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&= \frac{1}{2}\,\langle \mu(0) + \mu(1) \,|\, \rho_+\rangle + \frac{1}{2}\,\langle \mu(0) - \mu(1) \,|\, \rho_-\rangle \\
&\leq \frac{1}{2} + \frac{1}{2}\|\mu(0) - \mu(1)\|_\infty \|\rho_-\|_1 \\
&\leq \frac{1}{2} + \frac{1}{2}\|\lambda\rho_0 - (1-\lambda)\rho_1\|_1
\end{aligned}$$

- To get equality, use a Jordan-Hahn decomposition of $\rho_- = P - Q$ and set

$$\mu(0) = \Pi_{\mathrm{im}\,P}, \qquad \mu(1) = 1 - \Pi_{\mathrm{im}\,P}.$$

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- flips a biased coin $C = \{0, 1\}$ whose bias is $\lambda, 1 - \lambda$,
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- returns the state $\Phi_k(\sigma)$ to Bonaparte.

And it's now Bonaparte's turn to guestimate what value the coin took by cunningly measuring $\Phi_k(\sigma)$

Werner-Holevo channels

Naively, from our state discrimination exercise, Bonaparte chooses σ to maximise

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for $\rho_k = \Phi_k(\sigma)$.

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but ...

With $n = \dim(X)$ and $\lambda = \frac{n+1}{2n}$ and

$$\Phi_0(\sigma) = \frac{1}{n+1}\left(\mathrm{tr}(\sigma)1_X + \sigma^t\right), \qquad \Phi_1(\sigma) = \frac{1}{n-1}\left(\mathrm{tr}(\sigma)1_X - \sigma^t\right)$$

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we find, for all states σ, that

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we find, for all states σ, that

$$\|\lambda\Phi_0(\sigma) - (1-\lambda)\Phi_1(\sigma)\|_1 = \frac{1}{n}$$

Werner-Holevo channels

However if we couple X with itself and produce a MES:

$$\tau = \frac{1}{n} \sum_{i,j=1}^{n} |i\rangle \langle j| \otimes |i\rangle \langle j|$$

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$$\rho_k = (\Phi_k \otimes 1_{L(X)})(\tau)$$

are orthogonal.

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and get Antoinette to use the channels $\Phi_k \otimes 1_{L(X)}$, then the states that Bonaparte receives

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are orthogonal. So they can be discriminated perfectly (irrespective of λ) and

$$\|\lambda(\Phi_0 \otimes 1_{L(X)}(\tau) - (1-\lambda)(\Phi_1 \otimes 1_{L(X)})(\tau)\|_1 = 1$$

Diamond norm... finally

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&= \|\Phi \otimes 1_{L(X)}\|_{\text{operator norm}} \\
&= \|\Phi \otimes 1_{L(X)}\|_{L(X \otimes X) \to L(Y \otimes X)} \\
&= \|\Phi \otimes 1_{L(X)}\|_{(L(X \otimes X), \|\cdot\|_1) \to (L(Y \otimes X), \|\cdot\|_1)}
\end{aligned}$$

Diamond norm. . . finally

Consider a channel $\Phi : L(X) \to L(Y)$. You might write $\Phi \in T(X, Y)$.

$$\|\Phi\|_\diamond = \|\Phi \otimes 1_{L(X)}\|_{L(X\otimes X)\to L(Y\otimes X)}$$

where $L(X \otimes X), L(Y \otimes X)$ both use the 1-norm $\|\cdot\|_1$

Holevo-Helstrom theorem for channels

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Equality may be attained if $\dim(Z) \geq \dim(X)$.

tl;dr

why diamond

. . .

because entanglement