RUELLE ZETA FUNCTION AS PRODUCT OF ZETA FUNCTIONS OVER FORMS

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Abstract. The standard calculation turning a Ruelle zeta function into a product of zeta functions over differential forms.

1. The Calculation

Let Σ be a negatively curved closed manifold of dimension $n+1$. The Selberg zeta function is

$$\zeta_S(\lambda) := \prod_{\gamma^\#} \prod_{k \in \mathbb{N}_0} \left(1 - e^{-(\lambda+k)\ell_{\gamma^\#}}\right)$$

and the Ruelle zeta function is

$$\zeta_R(\lambda) := \prod_{\gamma^\#} \left(1 - e^{-\lambda \ell_{\gamma^\#}}\right)$$

which converge for $\operatorname{Re} \lambda \gg 1$. Here, the product is over all primitive geodesics, denoted $\gamma^\#$, and $\ell_{\gamma^\#}$ denotes the length of a given primitive geodesic $\gamma^\#$. Closed geodesics (not necessarily primitive) are denoted simply as γ.

Denoting by $\mathcal{P}_\gamma$ the linearised Poincaré map associated with a closed geodesic γ, we remark

$$|\det(I - \mathcal{P}_\gamma)| = (-1)^n \det(I - \mathcal{P}_\gamma)$$

as $\dim E_s = n$. Where E_s is the stable subbundle of T^*M and $M = S^*\Sigma$ is the unit cotangent bundle of the original manifold Σ. Linear algebra (of endomorphisms on $\mathbb{C}^{2n}$) implies

$$\det(I - \mathcal{P}_\gamma) = \sum_{k=0}^{2n} (-1)^k \operatorname{tr} \wedge^k \mathcal{P}_\gamma.$$

The Ruelle zeta function may be developed as:

$$\begin{aligned}
\log \zeta_R(\lambda) &= \sum_{\gamma^\#} \log(1 - e^{-\lambda \ell_{\gamma^\#}}) \\
&= -\sum_{\gamma^\#} \sum_{k=1}^{\infty} \frac{1}{k} e^{-\lambda k \ell_{\gamma^\#}} \\
&= -\sum_{\gamma} \frac{\ell_{\gamma^\#}}{\ell_\gamma} e^{-\lambda \ell_\gamma} \\
&= \left(\frac{(-1)^n \sum_{k=0}^{2n} (-1)^k \operatorname{tr} \wedge^k \mathcal{P}_\gamma}{|\det(I - \mathcal{P}_\gamma)|} \right) \left(-\sum_\gamma \frac{\ell_{\gamma^\#}}{\ell_\gamma} e^{-\lambda \ell_\gamma} \right) \\
&= \sum_{k=0}^{2n} (-1)^{n+k+1} \sum_\gamma \frac{\operatorname{tr} \wedge^k P_\gamma}{|\det(I - \mathcal{P}_\gamma)|} \frac{\ell_{\gamma^\#}}{\ell_\gamma} e^{-\lambda \ell_\gamma} \\
&= \sum_{k=0}^{2n} (-1)^{n+k} \log \zeta_k(\lambda)
\end{aligned}$$

where

$$\log \zeta_k(\lambda) = -\sum_\gamma \frac{\operatorname{tr} \wedge^k P_\gamma}{|\det(I - \mathcal{P}_\gamma)|} \frac{\ell_{\gamma^\#}}{\ell_\gamma} e^{-\lambda \ell_\gamma}.$$

The following function is meromorphic:

$$\frac{\zeta_k'(\lambda)}{\zeta_k(\lambda)} = \frac{d}{d\lambda} \log \zeta_k(\lambda) = \sum_\gamma \frac{e^{-\lambda \ell_\gamma} \ell_{\gamma^\#} \operatorname{tr} \wedge^k P_\gamma}{|\det(I - \mathcal{P}_\gamma)|}$$

Restricting to the case that Σ is a surface implies

$$\zeta_R(\lambda) = \frac{\zeta_1(\lambda)}{\zeta_0(\lambda)\zeta_2(\lambda)}$$

and the zero of ζ_R at $\lambda = 0$ has multiplicity given by $m_1(0) - m_0(0) - m_2(0)$ where $m_k(\lambda)$ is the multiplicity of the zero of $\zeta_k(\lambda)$.