DIRAC OPERATOR VIA AMBIENT METRIC

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Abstract. For an even dimensional Riemannian manifold, we consider its Lorentzian cone. We relate the Dirac operator on the cone to the Dirac operator on the original manifold.

1. Clifford algebra

Consider the complexified Clifford algebra $\mathbb{C}\mathrm{l}_n$ for $n = 2m$ which is isomorphic to the algebra of complex matrices of square dimension 2^m. Set $\Sigma = \mathbb{C}^{2^m}$ and introduce the representation $\chi : \mathbb{C}\mathrm{l}_n \to \mathrm{End}(\Sigma)$. Let $\rho : \mathrm{Spin}_n \to \mathrm{End}(\Sigma)$ be the restriction of χ to $\mathrm{Spin}_n \subset \mathbb{C}\mathrm{l}_n^0 \subset \mathbb{C}\mathrm{l}_n$.

The complex volume element belongs to $\mathbb{C}\mathrm{l}_n^0$ and is central, we use it to construct projections which split $\Sigma = \Sigma^+ \oplus \Sigma^+$ and $\rho = \rho^+ \oplus \rho^-$. Multiplication by the complexified volume element $\omega^{\mathbb{C}}$ gives the conjugation map: for $\psi = \psi^+ + \psi^- \in \Sigma$, we set $\overline{\psi} = \psi^+ - \psi^-$.

Consider the complexified Clifford algebra $\mathbb{C}\mathrm{l}_{1+n}$. The standard representation is a direct sum of two irreducible representations $\tilde{\chi} = \tilde{\chi}^+ \oplus \tilde{\chi}^- : \mathbb{C}\mathrm{l}_{n+1} \to \mathrm{End}(\Sigma) \oplus \mathrm{End}(\Sigma)$. When these are restricted to $\mathrm{Spin}_{1,n} \subset \mathbb{C}\mathrm{l}_{n+1}^0$ they are isomorphic. We thus set $\tilde{\rho} = \chi^+|_{\mathrm{Spin}_{1,n}}$

2. Geometry

Consider a spin manifold (M, g) of dimension $n = 2m$ and build the Lorentzian cone $(\tilde{M}, \tilde{g})$ where $\tilde{M} = \mathbb{R}_s^+ \times M$ and $\tilde{g} = -ds^2 + s^2 g$. Denote the projection $\pi : \tilde{M} \to M$.

The respective orthonormal frame bundles are denoted P and $\tilde{P}$ (We will not distinguish notation between the orthonormal frame bundle and its universal covering associated with the spin structure). Then a local section of P is written

$$u \in \Gamma(M, P), \qquad u : M \times \mathbb{R}^n \to M \times \otimes^n \mathrm{T}M : u_x : (e_i)_{1\le i\le n} \mapsto (X_i)_{1\le i\le n}$$

For such a frame, there exists a distinguished orthonormal frame for $\tilde{M}$:

$$\tilde{u} : \tilde{M} \times \mathbb{R}^{1+n} \to \tilde{M} \times \otimes^{1+n}\mathrm{T}\tilde{M} : \tilde{u}_{(s,x)} : (e_0, e_i)_{1\le i\le n} \mapsto (\partial_s, s^{-1}X_i)_{1\le i\le n}$$

The Levi-Civita connections are easily related

$$\tilde{\nabla}_{\partial_s}\partial_s = 0 \qquad \tilde{\nabla}_{s^{-1}X_i}\partial_s = s^{-2}X_i$$
$$\tilde{\nabla}_{\partial_s}s^{-1}X_i = 0 \qquad \tilde{\nabla}_{s^{-1}X_i}s^{-1}X_j = s^{-1}\delta_{ij}\partial_s + s^{-2}\nabla_{X_i}X_j$$

3. Spin geometry

We combine the previous two sections. Let $\Sigma M = P \times_\rho \Sigma$ be the spinor bundle of M. A spinor may be written

$$\Psi \in \Gamma(M;\Sigma M), \qquad \Psi = [u,\psi], \qquad u \in \Gamma(M,P), \psi \in \Gamma(M,\Sigma)$$

We identify a spinor $\tilde{\Psi}$ of $\tilde{M}$ with a family of spinors $(\Psi_s)_{s\in\mathbb{R}^+}$ on M. Using the frame u and its associated frame $\tilde{u}$ we have

$$\tilde{\Psi} = \pi^*(\Psi_s)_{s\in\mathbb{R}^+}, \qquad [\tilde{u},\psi] = [u,\psi_s]_{s\in\mathbb{R}^+}$$

and for a fixed s_0 we have the equality

$$\Gamma(\{s_0\}\times M;\Sigma) \ni \psi|_{s_0} = \psi_{s_0} \in \Gamma(M;\Sigma)$$

Next, we identify Clifford multiplication by $\mathrm{T}\tilde{M}$ with Clifford multiplication by $\mathrm{T}M$. In short, the formulae are

$$\partial_s \tilde{\cdot}\, \tilde{\Psi} = \pi^*(\overline{\Psi}_s)_{s\in\mathbb{R}^+}$$
$$(s^{-1}X_i)\,\tilde{\cdot}\, \tilde{\Psi} = \pi^*(X_i\cdot\Psi_s)_{s\in\mathbb{R}^+}$$

In light of the formula for calculating the covariant derivative of a spinor, we introduce the endomorphism Θ on ΣM :

$$\Theta = \sum_{i=1}^n X_i \cdot \theta(X_i)$$
$$= \sum_{i=1}^n X_i \cdot \left(\frac{1}{2}\sum_{j<k} g(\nabla_{X_i}X_j, X_k)X_j\cdot X_k\cdot\right)$$

The connection acts on a spinor in the following way

$$\nabla_X\Psi = [u, X\psi] + \theta(X)\Psi$$

The Dirac operator on $\Gamma(M;\Sigma M)$ is

$$\mathcal{D}\Psi = \sum_{i=1}^n X_i\cdot\nabla_{X_i}\Psi$$
$$= \left(\sum_{i=1}^n X_i\cdot[u,X_i\psi]\right) + \Theta\Psi$$

4. Calculation of Dirac operator

To calculate $\tilde{\theta}$, the ambiant version of θ, we use the relationship between $\tilde{\nabla}$ and ∇. After a few lines of caculations we obtain

$$
\begin{aligned}
\tilde{\theta}(\partial_s) &= 0 \\
\tilde{\theta}(s^{-1}X_i) &= \frac{1}{2}s^{-1}\partial_s\tilde{\cdot}(s^{-1}X_i)\tilde{\cdot} + \frac{1}{2}s^{-1}\sum_{j<k} g(\nabla_{X_i}X_j, X_k)(s^{-1}X_j)\tilde{\cdot}(s^{-1}X_k)\tilde{\cdot}
\end{aligned}
$$

Note that the first term in the second line leads to the following calculation when we multiply by $(s^{-1}X_i)\tilde{\cdot}$

$$
\begin{aligned}
\frac{1}{2}s^{-1}\sum_{i=1}^{n}(s^{-1}X_i)\tilde{\cdot}\partial_s\tilde{\cdot}(s^{-1}X_i)\tilde{\cdot}\tilde{\Psi} &= \frac{1}{2}s^{-1}\pi^*\left(\sum_{i=1}^{n} X_i\cdot\overline{X_i\cdot\Psi_s}\right)_{s\in\mathbb{R}^+} \\
&= \frac{n}{2}s^{-1}\pi^*(\overline{\Psi}_s)_{s\in\mathbb{R}^+}
\end{aligned}
$$

since $X_i\cdot\omega^{\mathbb{C}}\cdot X_i = \omega^{\mathbb{C}}$ when the dimension is even. Continuing on to the calculation of $\tilde{\Theta}$ we end up obtaining

$$
\tilde{\Theta}\tilde{\Psi} = \frac{n}{2}s^{-1}\pi^*(\overline{\Psi}_s)_{s\in\mathbb{R}^+} + s^{-1}\pi^*(\Theta\Psi_s)_{s\in\mathbb{R}^+}
$$

The calculation of the Dirac operator is

$$
\begin{aligned}
\tilde{\mathcal{D}}\tilde{\Psi} &= \partial_s\tilde{\cdot}[\tilde{u},\partial_s\psi] + \left(\sum_{i=1}^{n}(s^{-1}X_i)\tilde{\cdot}[\tilde{u},(s^{-1}X_i)\psi]\right) + \tilde{\Theta}\tilde{\Psi} \\
&= \pi^*[u,\overline{\partial_s\psi_s}]_{s\in\mathbb{R}^+} + s^{-1}\pi^*\left(\sum_{i=1}^{n} X_i\cdot[u,X_i\psi_s]\right)_{s\in\mathbb{R}^+} + \tilde{\Theta}\tilde{\Psi} \\
&= \pi^*[u,\overline{(\partial_s+\tfrac{n}{2}s^{-1})\psi_s}]_{s\in\mathbb{R}^+} + s^{-1}\pi^*(\mathcal{D}\Psi_s)_{s\in\mathbb{R}^+}
\end{aligned}
$$

So multiplication on the left by s gives

$$
s\tilde{\mathcal{D}}\tilde{\Psi} = \pi^*\left(\mathcal{D}\Psi_s + \overline{(s\partial_s+\tfrac{n}{2})\Psi_s}\right)_{s\in\mathbb{R}^+}
$$

Writing $\tilde{\Psi}$ as a column of two spinors $\pi^*(\Psi_s^{\pm})_{s\in\mathbb{R}^+}$, the ambiant Dirac operator, correctly conjugated, takes the form

$$
s^{\frac{n}{2}+1}\tilde{\mathcal{D}}s^{-\frac{n}{2}} = \pi^*\begin{bmatrix} s\partial_s & \mathcal{D}^- \\ \mathcal{D}^+ & -s\partial_s \end{bmatrix}
$$

5. Square of the Dirac operator

If we use the two commutation relations $[\tilde{\mathcal{D}},s] = \left[\begin{smallmatrix} s & 0 \\ 0 & -s \end{smallmatrix}\right]$ and $[\mathcal{D},s\partial_s]=0$, we find that

$$
s^{\frac{n}{2}+2}\tilde{\mathcal{D}}^2 s^{-\frac{n}{2}} = \pi^*\begin{bmatrix} \mathcal{D}^2+(s\partial_s)^2-s & 0 \\ 0 & \mathcal{D}^2+(s\partial_s)^2+s \end{bmatrix}
$$