Structures de Clifford paires et résonances quantiques

Charles Hadfield

Thèse de doctorat

soutenue le 19 juin 2017

Directeurs de thèse	:	C. Guillarmou	Paris-Sud
		A. Moroianu	Paris-Sud
Rapporteurs	:	M. Herzlich	Montpellier
		A. Vasy	Stanford
Examinateurs	:	O. Biquard	ENS
		E. Delay	Avignon
		P. Gauduchon	Polytechnique
		J. Hilgert	Paderborn

Remerciements

J'exprime mes profonds remerciements à mes directeurs de thèse Colin et Andrei. Le sentier sinueux qui a mené à leur rencontre a débouché sur un travail de trois ans bien fécond. Je les remercie pareillement de leur volonté de partager avec moi leurs connaissances mathématiques aussi bien que de la façon détendue avec laquelle ils me les ont transmises.

Je tiens à remercier Marc Herzlich d'avoir accepté d'être rapporteur de thèse et de sa lecture soigneuse du manuscrit. Également, je remercie András Vasy d'avoir accepté d'être rapporteur et de son intérêt porté au résultat principal de la thèse ainsi qu'à mon parcours professionel.

Messieurs Olivier Biquard, Erwann Delay, Paul Gauduchon, et Joachim Hilgert m'ont fait l'honneur de participer au jury de soutenance ; je les en remercie profondément.

Je remercie la FSMP pour la bourse accordée en 2013 qui a ouvert la porte à cette aventure française. Je remercie aussi l'ENS, de m'avoir accueilli au début de ma thèse et d'avoir créé un environnement propice à la recherche.

En ce qui concerne l'administration, scolaire et préfectorale, je dois beaucoup à Bénédicte. Sans elle, je ne me serais pas sorti du labyrinthe.

Les trois conférences à Peyresq resteront longuement dans ma mémoire. Je remercie donc Frédéric de les avoir organisées et d'avoir partagé sa curiosité scientifique.

Enfin, à tous ceux qui m'ont laissé squatter chez eux ces trois dernières années, c'est un plaisir de passer du temps avec vous.

Résumé

Ce manuscrit se compose de deux parties indépendantes.

La première partie de cette thèse étudie les structures de Clifford paires. Pour une variété riemannienne munie d'une structure de Clifford paire, nous introduisons l'espace de twisteurs en généralisant la construction d'un tel espace dans le cas d'une variété quaternion-hermitienne. Nous construisons une structure presque-complexe sur l'espace de twisteurs et considérons son intégrabilité lorsque la structure de Clifford est parallèle. Dans certains cas, nous pouvons aussi le fournir d'une métrique kählerienne ou, correspondant à une structure presque-complexe alternative, d'une métrique "nearly Kähler". Dans un second temps, nous introduisons une structure appelée Clifford-Weyl sur une variété conforme. Il s'agit d'une structure de Clifford paire qui est parallèle par rapport au produit tensoriel d'une connexion métrique sur le fibré de Clifford et une connexion de Weyl. Nous démontrons que la connexion de Weyl est fermée sauf dans certains cas génériques de basse dimension où nous arrivons à décrire des exemples explicites où les structures de Clifford-Weyl sont non-fermées.

La seconde partie de cette thèse étudie des résonances quantiques. Au-dessus d'une variété assymptotiquement hyperbolique paire, nous considérons le laplacien de Lichnerowicz agissant sur les sections du fibré des formes multilinéaires symétriques. Lorsqu'il s'agit de formes bilinéaires symétriques, nous obtenons une extension méromorphe de la résolvante dudit laplacien à l'ensemble du plan complexe si la variété est Einstein. Cela définit les résonances quantiques pour ce laplacien. Pour les formes multinéaires symétriques en général, une telle extension méromorphe est possible si la variété est convexe-cocompacte. Dans les deux cas, nous devons restreindre le laplacien aux sections qui sont de trace et de divergence nulles. Nous utilisons ce deuxième résultat afin d'établir une correspondance classique-quantique pour les variétés hyperboliques convexes-cocompactes. La correspondance identifie le spectre du flot géodésique (les résonances de Ruelle) avec les spectres des laplaciens agissant sur les tenseurs symétriques qui sont de trace et de divergence nulles (les résonances quantiques).

Abstract

We study independently even Clifford structures on Riemannian manifolds and quantum resonances on asymptotically hyperbolic manifolds.

In the first part of this thesis, we study even Clifford structures. First, we introduce the twistor space of a Riemannian manifold with an even Clifford structure. This notion generalises the twistor space of quaternion-Hermitian manifolds. We construct almost complex structures on the twistor space and check their integrability when the even Clifford structure is parallel. In some cases we give Kähler and nearly-Kähler metrics to these spaces. Second, we introduce the concept of a Clifford-Weyl structure on a conformal manifold. This consists of an even Clifford structure parallel with respect to the tensor product of a metric connection on the Cifford bundle and a Weyl structure on the manifold. We show that the Weyl structure is necessarily closed except for some "generic" low-dimensional instances, where explicit examples of non-closed Clifford-Weyl structures are constructed.

In the second part of this thesis, we study quantum resonances. First, we consider the Lichnerowicz Laplacian acting on symmetric 2-tensors on manifolds with an even Riemannian conformally compact Einstein metric. The resolvent of the Laplacian, upon restriction to trace-free, divergence-free tensors, is shown to have a meromorphic continuation to the complex plane. This defines quantum resonances for this Laplacian. For higher rank symmetric tensors, a similar result is proved for convex cocompact quotients of hyperbolic space. Second, we apply this result to establish a direct classical-quantum correspondence on convex cocompact hyperbolic manifolds. The correspondence identifies the spectrum of the geodesic flow with the spectrum of the Laplacian acting on trace-free, divergence-free symmetric tensors. This extends the correspondence previously obtained for cocompact quotients.

Contents

1. Introduction

1.1 Even Clifford Structures

The first part of this thesis considers even Clifford structures, recently introduced by Moroianu and Semmelmann [MS11]. In this thesis, we construct a twistor space for Riemannian manifolds carrying such a structure, this is joint work with Gerardo Arizmendi. Next, we consider the conformal nature inherent in the original definition of even Clifford structures, this is joint work with Andrei Moroianu.

We briefly introduce the setting, with precise notions given in Chapter 2. On a Riemannian manifold (M, g) with a (locally defined) oriented Euclidean vector bundle (E, h), an even Clifford bundle is the added data of an algebra bundle morphism

$$\varphi : \mathrm{Cl}^0(E, h) \to \mathrm{End}(\mathrm{T}M)$$

from the even Clifford bundle of (E, h) into endomorphisms of the tangent bundle of the manifold. Moreover, this Clifford morphism, is demanded to send the subbundle $\Lambda^2 E \subset \mathrm{Cl}^0(E, h)$ into the the subbundle of skew endomorphisms $\mathrm{End}^-(\mathrm{T}M)$. The structure is parallel if the bundle map is parallel with respect to a metric connection ∇^E of E coupled to the Levi-Civita connection ∇. The classification of Riemannian manifolds carrying a parallel rank Clifford structure found in [MS11] is recalled in Section 2.2.

This (parallel) structure may be considered as a generalisation of quaternion-Kähler geometry (rather than hyper-Kähler geometry). Indeed in the general setting, if we assume E is of rank r and denote by $\{\xi_i\}_{1\leq i\leq r}$ a local orthonormal frame for (E, h), then the bundle morphism provides locally defined almost complex structures

$$J_{ij} := \varphi(\xi_i \cdot \xi_j) \in \mathrm{End}^- \mathrm{T}M, \qquad J_{ij}^2 = -1_{\mathrm{T}M}$$

where $\cdot$ denotes Clifford multiplication. Restricting to rank $r = 3$, we obtain a 3-dimensional subbundle of $\mathrm{End}\,\mathrm{T}M$ spanned by $\{J_{ij}\}_{1\leq i<j\leq 3}$ where the almost complex structures J_{12}, J_{13}, J_{23} play the respective roles of the locally defined almost complex structures I, J, K present in almost Hermitian geometry. The quaternion relationship $I \circ J = K$ in this case is a consequence of the Clifford algebra, in particular

$$(\xi_1 \cdot \xi_2) \cdot (\xi_1 \cdot \xi_3) = -(\xi_1 \cdot \xi_1) \cdot (\xi_2 \cdot \xi_3) = \xi_2 \cdot \xi_3$$

and the fact that φ is an algebra morphism. That the globally defined subbundle $\mathrm{span}\{I, J, K\}$ of $\mathrm{End}\,\mathrm{T}M$ be preserved by the Levi-Civita connection is precisely the condition that the Clifford morphism be parallel with respect to a connection on E which is metric.

Even Clifford structures are in some sense dual to spin structures on manifolds. If (M, g) is a

Riemannian manifold with spin structure, then the spin structure provides the spinor bundle which is a representation space for the even Clifford algebra. For even Clifford structures, it is rather the tangent bundle that plays the role of the representation space for the Clifford algebra of the auxiliary bundle E. A nice illustration of this duality appears in Section 4.4 when considering conformal manifolds of dimension 8 and rank 8 Euclidean bundles.

We refer to the original article [MS11] for alternative approaches to the concept of Clifford structures on Riemannian manifolds previously considered in the literature. Since the publication of [MS11], several articles have recently appeared on this topic. An alternative definition of even Clifford structures, as structure group reductions is given by Arizmendi, García-Pulido, and Herrera [AGH16, AH15]. These authors have considered even Clifford structures with large automorphism groups with Santana [AHS16]. They have also obtained rigidity and vanishing results [GH16]. Moroianu and Pilca have begun a classification of homogeneous even Clifford structures [MP13], and Parton, Picinni, and Vuletescu have considered relations with symmetric spaces and the Severi varieties. [PP15, PPV15].

Twistors

The notion of a twistor space was first introduced by Penrose in [Pen77]. Following the ideas of Penrose, the twistor construction for a 4-dimensional Riemannian manifold was developed by Atiyah, Hitchin, and Singer [AHS78]. This was later generalised for even dimensional manifolds by O'Brian and Rawnsley [OR85]. The twistor space Z of an even dimensional Riemannian manifold M admits a natural almost complex structure, and it is well known that such a twistor space is complex if and only if the manifold is self-dual for $\dim(M) = 4$ and locally conformally flat for $\dim(M) \geq 6$ [AHS78, OR85]. A converse theorem (the so called reverse Penrose construction) in dimension 4 has been used to construct half-conformally flat Einstein manifolds.

Before continuing to other generalisations, we recall the twistor space construction in the setting of an oriented Riemannian manifold (M^4, g), details of which may be found in Besse [Bes08]. One observes that the Hodge $*$-operation acts on 2-forms as an involution and thus may be used to decompose 2-forms into self-dual and anti-self-dual forms. The twistor space $\pi : Z \to M$ is then taken to be the unit sphere bundle of the 3-dimensional real vector bundle of anti-self-dual forms. The fibres are thus 2-spheres. Using the metric, one identifies 2-forms and skew-adjoint endomorphisms of the tangent bundle. Next, the Levi-Civita connection splits the tangent bundle of Z such that $TZ = V + H$ where $V = \ker(\pi_*)$ is the tangent bundle along the fibres and H identifies, at each point, with the tangent space on the base manifold below via π_*. As each point of the twistor space is a complex structure on the tangent space below, one may partly define the complex structure on H. The remainder of the definition, defining the complex structure on V, is possible after identifying the fibres with $\mathbb{CP}^1$. Distinct from this construction is the notion that the manifold is half-conformally flat, a condition on the decomposition of the Weyl tensor also involving the Hodge $*$-operation. It is under precisely this hypothesis that one may show that the almost complex structure of the twistor space is integrable.

In another generalisation, the twistor space Z of quaternion-Kähler manifolds was defined in [Sal82]. This is an $\mathbb{S}^2$-bundle of pointwise Hermitian structures compatible with the quaternionic structure. It is well known that this bundle admits two almost complex structures $\mathcal{J}$ and $\widetilde{\mathcal{J}}$, one of which is always integrable and the other is never integrable [ES83]. Moreover, the manifold Z admits two Einstein metrics h and $\widetilde{h}$ such that $(Z, \mathcal{J}, h)$ is Kähler-Einstein [BB82, Sal82] and $(Z, \widetilde{\mathcal{J}}, \widetilde{h})$ is nearly Kähler [AGI98].

In [Fri01], the twistor space was defined in the context of weak Spin(9) structures on 16-dimensional

Riemannian manifolds, which correspond to rank 9 even Clifford structures [MS11]. Additionally, this twistor construction was studied for $\mathbb{R}^{16}$, which carries a parallel flat even Clifford structure, the Cayley plane $F_4/\mathrm{Spin}(9)$, which carries a parallel non-flat even Clifford structure, and $\mathbb{S}^1 \times \mathbb{S}^{15}$, which carries a non-parallel even Clifford structure. In the first two cases, the twistor space admits a Kähler metric and in the last case the twistor space is a complex manifold which does not admit a Kähler metric.

We generalise these constructions to even Clifford structures of arbitrary rank $r \geq 3$, noting that ranks 3 and 9 constitute two of the aforementioned constructions. In this context, the $\mathbb{S}^2$-fibre of pointwise Hermitian structures present in the quaternion-Kähler setting is replaced with $\widetilde{\mathrm{Gr}}(2, r)$, the Grassmannian of oriented 2-planes in $\mathbb{R}^r$. The twistor space admits an almost complex strucutre. Indeed, the Levi-Civita connection splits the tangent bundle of the twistor space into horizontal and vertical subbundles, and Clifford multiplication provides the action of the almost complex structure $\mathcal{J}$ on the vertical subbundle. In the spirit of [BB82], we prove

Theorem 1. *Let M be a Riemannian manifold of dimension $n \neq 8$ carrying a parallel even Clifford structure of rank $r > 4$, then the almost complex structure $\mathcal{J}$ on Z is integrable.*

Analogous to theorems in [Sal82] and [AGI98] we also prove

Theorem 2. *The twistor space $(Z, \mathcal{J})$ of a Riemannian manifold of dimension $n \neq 8$ with a parallel even Clifford structure of rank $r > 4$ and* $\mathrm{Ric} > 0$ *admits a Kähler metric.*

By appealing to the classification of parallel even Clifford structures recalled in Chapter 2, the these theorems may be summarised by the following tables.

r	M	$\dim(M)$	fibre of Z	type of Z
3 and 4	QK manifold	$4k$	$\mathbb{S}^2$	complex, Kähler if $\mathrm{Ric} > 0$
4	$M_1 \times M_2$, M_i QK	$4(n_1 + n_2)$	$\mathbb{S}^2 \times \mathbb{S}^2$	complex, Kähler if $\mathrm{Ric}(M_i) > 0$
5	QK	8	$\mathrm{Sp}(2)/\mathrm{U}(2)$	complex if locally symmetric
6	Kähler	8	$\mathrm{U}(4)/\mathrm{U}(2) \times \mathrm{U}(2)$	complex if Bochner tensor $\equiv 0$
7	Spin(7) holonomy	8	$\widetilde{\mathrm{Gr}}(2, 7)$	not complex
8	Riemannian	8	$\mathrm{SO}(8)/\mathrm{U}(4)$	complex if Weyl tensor $\equiv 0$

Table 1: Twistor spaces for low rank and dimension 8

r	M	$\dim(M)$	fibre of Z	type of Z
5	$\mathrm{Sp}(k+2)/(\mathrm{Sp}(k) \times \mathrm{Sp}(2))$	$8k$, $k \geq 2$	$\mathrm{Sp}(2)/\mathrm{U}(2)$	Kähler
6	$\mathrm{SU}(k+4)/\mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(4))$	$8k$, $k \geq 2$	$\mathrm{U}(4)/\mathrm{U}(2) \times \mathrm{U}(2)$	Kähler
8	$\mathrm{SO}(k+8)/(\mathrm{SO}(k) \times \mathrm{SO}(8))$	$8k$, $k \geq 2$	$\mathrm{SO}(8)/\mathrm{U}(4)$	Kähler
9	$\mathrm{F}_4/\mathrm{Spin}(9)$	16	$\widetilde{\mathrm{Gr}}(2, 9)$	Kähler
10	$\mathrm{E}_6/(\mathrm{Spin}(10) \cdot \mathrm{U}(1))$	32	$\widetilde{\mathrm{Gr}}(2, 10)$	Kähler
12	$\mathrm{E}_7/(\mathrm{Spin}(12) \cdot \mathrm{SU}(2))$	64	$\widetilde{\mathrm{Gr}}(2, 12)$	Kähler
16	$\mathrm{E}_8/\mathrm{Spin}^+(16)$	128	$\widetilde{\mathrm{Gr}}(2, 16)$	Kähler
arbitrary	Cl^0_r representation	$d_r k$	$\widetilde{\mathrm{Gr}}(2, r)$	Kähler

Table 2: Twistor spaces for higher rank

For the non-compact dual spaces of these symmetric spaces, the twistor space is only complex as the negative curvature obstructs the construction of an appropriate metric on the fibres. Above, d_r denotes the dimension of the irreducible representations of Cl^0_r.

Finally, an observation of Nagy [Nag02] in the setting of Riemannian submersions with totally geodesic fibres provides a slick way of defining an alternative almost complex structure which is not integrable, and a metric providing a nearly Kähler geometry.

Theorem 3. *The twistor space Z of a Riemannian manifold with a parallel even Clifford structure of rank $r \geq 3$ and $\mathrm{Ric} > 0$, admits an almost complex structure $\widetilde{\mathcal{J}}$ and a metric $\widetilde{h}$ such that $(Z, \widetilde{\mathcal{J}}, \widetilde{h})$ is nearly Kähler.*

Clifford-Weyl structures

Recall the setting of an even Clifford structure as a Riemannian manifold (M, g) carrying a (locally defined) Euclidean vector bundle (E, h) together an algebra bundle map $\varphi : \mathrm{Cl}^0(E, h) \to \mathrm{End}(\mathrm{T}M)$ mapping $\Lambda^2 E$ into the bundle of skew-symmetric endomorphisms $\mathrm{End}^-(\mathrm{T}M)$. As $\mathrm{End}^-(\mathrm{T}M)$ depends only on the conformal structure of g, even Clifford structures may be considered more naturally defined on conformal, rather than Riemannian, manifolds.

Studying even Clifford structures on Riemannian manifolds becomes natural when we demand the Clifford morphism φ to be parallel with respect to certain connections. Indeed given a metric connection ∇^E on E, the Riemmanian structure of the base manifold naturally demands that φ be parallel with respect to this connection coupled with the Levi-Civita connection.

On a conformal manifold, where no distinguished connection exists on the tangent bundle, we may consider Weyl connections (which are in one-to-one correspondence with covariant derivatives of the weight bundle of the manifold). We introduce in Defintion 4.1 the notion of a Clifford-Weyl structure on a conformal manifold (M, c) to be the tuple $(E, h, \varphi, \nabla^E, D)$ where (E, h, φ) satisfy the conditions of an even Clifford structure and, now, the metric connection ∇^E along with the choice of a Weyl connection D provide the coupled connection $\nabla^E \otimes D$ with respect to which the Clifford morphism is parallel.

Immediately the natural question to ask is under what conditions does this problem locally reduce to a problem in Riemannian geometry, i.e. under what conditions is D closed? We show that there are six instances (called generic) where the presence of a Clifford-Weyl structure need not force the Weyl connection to be closed, and that in all other cases (called non-generic), the associated Weyl structure of a Clifford-Weyl structure is automatically closed. More precisely, in the non-generic setting we prove

Theorem 4. *Suppose a conformal manifold of dimension n carries a rank $r \geq 2$ Clifford-Weyl structure such that (n, r) is different from $(2, 2)$, $(4, 2)$, $(4, 3)$, $(4, 4)$ and $(8, 8)$. Then the associated Weyl connection is closed. The same conclusion holds if $(n, r) = (8, 4)$, provided that the restriction of the Clifford morphism φ to $\Lambda^2 E$ is not injective.*

The proof of this theorem is performed in three stages. The first case when $r = 2$ is a standard result in Hermitian geometry. It is presented explicitly in Proposition 4.2. This proposition turns out to be useful as it inspires the beginning of the proof of the theorem in the large rank setting $r \geq 5$. In this large rank setting, the proof shows a similarity to techniques present in the proof of [MS11, Proposition 2.10]. (The statement of this proposition is announced in this thesis as Proposition 2.7.) We progressively develop restrictions on the curvature of the Weyl connection until a trick using the skew-symmetry of the Hermitian structures J_{ij} forces the curvature to both commute and anti-commute with these Hermitian structures. Finally one needs to deal with the remaining ranks $r = 3, 4$. The rank 3 setting corresponds to Hermitian Weyl geometry where the statement is standard [Orn01]. A proof using the Kraines form may be relatively easily extended to the rank 4 setting using knowledge of the irreducible representations of the relative Clifford algebras.

The cases excluded by this theorem are somehow generic, and they are treated in

Theorem 5. *(i) Let D be a Weyl structure on an oriented conformal manifold (M, c) of dimension 2, 4 or 8. Then (M, c) carries a Clifford-Weyl structure of rank $r = 2$ for $n = 2$, $r = 3$ or $r = 4$ for $n = 4$ and $r = 8$ for $n = 8$, whose associated Weyl structure is D.*

(ii) Let D be a Weyl structure on a conformal manifold (M^n, c). Then there exists a Clifford-Weyl structure of rank 2 on (M, c) with associated Weyl structure D if and only if D preserves a complex structure compatible with c. If $n = 4$, every complex structure J compatible with c is preserved by a unique Weyl structure D^J, which is closed if and only if J is locally conformally Kähler.

(iii) Let D be a Weyl structure on a conformal manifold (M^8, c). Then there exists a Clifford-Weyl structure of rank 4 whose Clifford morphism $\varphi : \mathrm{Cl}^0(E, h) \to \mathrm{End}(\mathrm{T}M)$ is injective upon restriction to $\Lambda^2 E$, if and only if D is the adapted Weyl structure of a conformal product structure on (M, c) with 4-dimensional factors (cf. [BM11]).

The proof of this theorem, given in Section 4.4, is constructive, showing in each of these cases that there are examples of Clifford-Weyl structures with non-closed associated Weyl structures.

1.2 Asymptotically Hyperbolic Manifolds

The second part of this thesis is principally an analysis of quantum resonances on asymptotically hyperbolic manifolds. A consequence of this analysis is a correspondence between quantum resonances and Ruelle resonances on convex cocompact quotients of hyperbolic space.

More generally, the mathematical study of scattering resonances encompasses several areas of research as explained in the recent survey article by Zworski [Zwo17]. As mentioned in that article, scattering resonances generalise eigenvalues to systems where energy can dissipate or scatter to infinity. They appear under different names in a variety of settings: as quantum resonances in quantum scattering theory; in obstacle scattering as scattering poles; in general relativity, the resonant states associated with gravitational waves are quasi-normal modes; and as Ruelle resonances in the presence of an Anosov flow.

Quantum resonances

The geometric setting of asymptotically hyperbolic manifolds (X, g), modelled on convex cocompact quotients of hyperbolic space, dates to work of Mazzeo and Melrose [Maz88, MM87] from a spectral-theory perspective, and to work of Fefferman and Graham [FG85] from a conformal geometry perspective. Introducing a spectral parameter $\lambda \in \mathbb{C}$, we consider the (positive) Laplacian on functions Δ giving the operator

$$\Delta - \tfrac{n^2}{4} + \lambda^2$$

where $\dim(X) = n + 1$. For $\mathrm{Re}\,\lambda > \frac{n}{2}$, this operator has an inverse on $L^2(X)$, written

$$\mathcal{R}_\lambda = (\Delta - \tfrac{n^2}{4} + \lambda^2)^{-1}$$

and an immediate question is posed: (under what conditions) does this operator extend meromorphically beyond $\mathrm{Re}\,\lambda = \frac{n}{2}$ and eventually to $\mathbb{C}$? Such a meromorphic extension then implicitly defines quantum resonances as the poles of said extension. The meromorphic extension with finite rank poles of the resolvent of the Laplacian on functions is obtained in [MM87] excluding certain exceptional points in $\mathbb{C}$. Precisely, the set of exceptional points is $-\frac{n}{2} - \frac{1}{2}\mathbb{N}$. Refining the definition of asymptotically hyperbolic manifolds by introducing a notion of evenness, Guillarmou [Gui05] provides the meromorphic extension to all of $\mathbb{C}$ and shows that for such an extension, said evenness is essential. Indeed, in the same article, examples are given where the extension of the resolvent contains essential singularities at the points $-\frac{n}{2} - \frac{1}{2} - \mathbb{N}_0$. A first hint at the importance of this evenness property appears in the work of Guillopé and Zworski [GZ95] which provides the meromorphic extension to all of $\mathbb{C}$ when the geometry at infinity is strictly that of convex cocompact hyperbolic manifolds. By shifting viewpoint and studying a Fredholm problem, rather than using Melrose's pseudodifferential calculus on manifolds with corners, Vasy [Vas13a, Vas13b] is also able to recover the result of [Gui05] detailing the meromorphic extension. This technique is presented in a very accessible article of Zworski [Zwo16] in a microlocal language (non-semiclassical). This alternative method is more appropriate when one considers vector bundles. Effectively contained in [Vas13a], the meromorphic extension is explicitly obtained in [Vas17] for the resolvent of the Hodge Laplacian upon restriction to coclosed forms (or excluding top forms, for closed forms). Such a restriction is natural in light of works in a conformal setting [AG11, BG05], i.e. the boundary of the asymptotic space. In fact, from the conformal geometry viewpoint, Vasy's method of placing the asympotically hyperbolic manifold in an ambient manifold equipped with a Lorentzian metric is very much in the spirit of both the tractor calculus [BEG94] as well as the ambient metric construction [FG12].

This thesis will be interested in defining quantum resonances for symmetric tensors (of the cotangent bundle), rather than functions, above the asymptotically hyperbolic space (X, g). Here the natural operator is the Lichnerowicz Laplacian [HMS16] which differs from the rough Laplacian via a zeroth order curvature correction. There are several reasons why such resonances are interesting. With knowledge of the asymptotics of the resolvent of the Laplacian on functions, it is possible to construct the Poisson operator, the Scattering operator, and study in a conformal setting, the GJMS operators and the Q-curvature of Branson [DGH08, Chapters 5,6]. Simply from a geometric point of view, it should be particularly interesting to extend these results and constructions to the case of symmetric 2-cotensors above a conformal manifold which would then be related to metrics on the extended "bulk" Poincaré-Einstein manifold. Again considering symmetric 2-cotensors, the Lichnerowicz Lapalcian plays a fundamental role in problems involving deformations of metrics and their Ricci tensors [Biq00, Del99, GL91] as well as to linearised gravity [Wan09]. Spectral analysis of the Lichnerowicz Laplacian [Del02, Del07] as well as the desire to build a scattering operator emphasise the importance of considering this Laplacian acting on more general spaces than that of L^2 sections. From the viewpoint of gravitational waves, the recent work [BVW15] studies decay rates of solutions to the wave equation (acting on the trivial bundle) on Minkowski space with metrics similar to the ambient metric present in this thesis. It is very natural to consider this problem on symmetric 2-cotensors acted upon by the Lichnerowicz d'Alembertian. These motivations will not be addressed in this thesis. Another reason is that the meromorphic extension of the resolvent of the Laplacian acting on symmetric tensors is a key ingredient for relating Ruelle resonances and quantum resonances in the setting of convex cocompact hyperbolic manifolds. This correspondence will be addressed later in this introduction.

We now turn our attention to the contents of Chapters 5 and 6. What follows is the five theorems related to quantum resonances and an outline of their proof.

Let $\overline{X}$ be a compact manifold with boundary $Y = \partial\overline{X}$. That (X, g) is asymptotically hyperbolic means that, locally near Y in $\overline{X}$, there exists a chart $[0, \varepsilon)_\rho \times Y$ such that on $(0, \varepsilon) \times Y$, the metric

g takes the form

$$g = \frac{d\rho^2 + h}{\rho^2}$$

where h is a family of Riemannian metrics on Y, depending smoothly on $\rho \in [0, \varepsilon)$. That g is even means that h has a Taylor series about $\rho = 0$ in which only even powers of ρ appear. Above X, we consider the set of symmetric cotensors of rank m, denoting this vector bundle $\mathcal{E}^{(m)} = \mathrm{Sym}^m \mathrm{T}^* X$. On symmetric tensors, there exist two common Laplacians. The (positive) rough Laplacian $\nabla^*\nabla$ and the Lichnerowicz Laplacian Δ, originally defined on 2-cotensors [Lic61], but easily extendible to arbitrary degree [HMS16]. On functions, these two Laplacians coincide, on one forms, the Lichnerowicz Laplacian agrees with the Hodge Laplacian, and in general, for symmetric m-cotensors, the Lichnerowicz Laplacian differs from the rough Laplacian by a zeroth order curvature operator

$$\Delta = \nabla^*\nabla + q(\mathrm{R}).$$

We construct the Lorentzian cone $M = \mathbb{R}_s^+ \times X$ with metric $\eta = -ds \otimes ds + s^2 g$ (and call s the Lorentzian scale). Pulling $\mathcal{E}^{(m)}$ back to M we naturally see $\mathcal{E}^{(m)}$ as a subbundle of the bundle of all symmetric cotensors of rank m above M, this larger bundle is denoted $\mathcal{F} = \mathrm{Sym}^m \mathrm{T}^* M$. On $\mathcal{F}$ we consider the Lichnerowicz d'Alembertian $\Box$. Up to symmetric powers of $\frac{ds}{s}$ we may identify $\mathcal{F}$ with the direct sum of $\mathcal{E}^{(k)} = \mathrm{Sym}^k \mathrm{T}^* X$ for all $k \leq m$. Indeed by denoting $\mathcal{E} = \oplus_{k=0}^m \mathcal{E}^{(k)}$ the bundle of all symmetric tensors above X of rank not greater than m, we are able to pull back sections of this bundle and see them as sections of $\mathcal{F}$:

$$\pi_s^* : C^\infty(X; \mathcal{E}) \to C^\infty(M; \mathcal{F}).$$

A long calculation gives the structure of the Lichnerowicz d'Alembertian with respect to this identification. It is seen that $s^2 \Box$ decomposes as the Lichnerowicz Laplacian Δ acting on each subbundle of $\mathcal{E}^{(k)}$ for $0 \leq k \leq m$ however these fibres are coupled via off-diagonal terms consisting of the symmetric differential d and its adjoint, the divergence δ. (There are also less important couplings due to the trace Λ and its adjoint L.) Also present in the diagonal are terms involving $s\partial_s$ and $(s\partial_s)^2$. By conjugating by $s^{-\frac{n}{2}+m}$ we obtain an operator

$$\mathbf{Q} = \nabla^*\nabla + (s\partial_s)^2 + \mathbf{D} + \mathbf{G}$$

where $\mathbf{D}$ is of first order consisting of the symmetric differential and the divergence, while $\mathbf{G}$ is a smooth endomorphism on $\mathcal{F}$. By appealing to the b-calculus of Melrose [Mel93], we can push this operator acting on $\mathcal{F}$ above M to a family of operators (holomorphic in the complex variable λ) acting on $\mathcal{E}$ above X of the form

$$\mathcal{Q}_\lambda = \nabla^*\nabla + \lambda^2 + \mathcal{D} + \mathcal{G}$$

where $\mathcal{D}$ is of first order consisting of the symmetric differential and the divergence, while $\mathcal{G}$ is a smooth endomorphism on $\mathcal{E}$. Explicitly, in matrix notation writing

$$u = \begin{bmatrix} u^{(m)} \\ \vdots \\ u^{(0)} \end{bmatrix}, \qquad u \in C^\infty(X; \mathcal{E}), u^{(k)} \in C^\infty(X; \mathcal{E}^{(k)})$$

the operator $\mathcal{Q}_\lambda$ takes the following form

$$\begin{bmatrix} \Delta+\lambda^2-c_m-\mathrm{L}\,\Lambda & 2b_{m-1}\,\mathrm{d} & -b_{m-2}b_{m-1}\,\mathrm{L} & & & \\ -2b_{m-1}\,\delta & \ddots & \ddots & \ddots & & \mathbf{0} \\ -b_{m-2}b_{m-1}\,\Lambda & \ddots & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & -b_0b_1\,\mathrm{L} \\ \mathbf{0} & & \ddots & \ddots & \ddots & 2b_0\,\mathrm{d} \\ & & & -b_0b_1\,\Lambda & -2b_0\,\delta & \Delta+\lambda^2-c_0-\mathrm{L}\,\Lambda \end{bmatrix}$$

for constants

$$b_k=\sqrt{m-k}, \qquad c_k=\tfrac{n^2}{4}+m(n+2k+1)-k(2n+3k-1)$$

and operators: Δ the Lichnerowicz Laplacian; δ the divergence; d the symmetric differential; Λ the trace; L the adjoint of the trace. (The operator $\mathcal{Q}_\lambda$ naively does not appear self-adjoint for $\lambda\in i\mathbb{R}$ since δ is the adjoint of d. The sign discrepancy is due to the Lorentzian signature of η. The operator is indeed self-adjoint for $\lambda\in i\mathbb{R}$ as detailed in Proposition 6.24.) This family of operators has the following meromorphic family of inverses.

Theorem 6. *Let (X^{n+1},g) be even asymptotically hyperbolic. Then the inverse of*

$$\mathcal{Q}_\lambda \ \textit{acting on}\ L^2_s(X;\mathcal{E})$$

written $\mathcal{Q}_\lambda^{-1}$ has a meromorphic continuation from $\operatorname{Re}\lambda\gg 1$ to $\mathbb{C}$,

$$\mathcal{Q}_\lambda^{-1}: C_c^\infty(X;\mathcal{E})\to\rho^{\lambda+\frac{n}{2}-m}\bigoplus_{k=0}^{m}\rho^{-2k}C^\infty_{\mathrm{even}}(\overline{X};\mathcal{E}^{(k)})$$

with finite rank poles.

In order to demonstrate Theorem 6, Vasy's technique is to consider a slightly larger manifold X_e as well as the ambient space $M_e=\mathbb{R}^+\times X_e$. Using two key tricks near the boundary $Y=\partial\overline{X}$: the evenness property allows us to introduce the coordinate $\mu=\rho^2$ and twisting the Lorentzian scale with the boundary defining function gives (what is termed the Euclidean scale) $t=s/\rho$, it is seen that the ambient metric η may be extended non-degenerately past $\mathbb{R}^+\times Y$ to M_e. This is the main content of Chapter 5. On $\mathrm{Sym}^m\mathrm{T}^*M_e$ we construct analogous to $\mathbf{Q}$, an operator $\mathbf{P}$ replacing appearances of s by t which, on M is easily related to $\mathbf{Q}$. Again the b-calculus provides a family of operators $\mathcal{P}$ on $\oplus_{k=0}^m\mathrm{Sym}^k\mathrm{T}^*X_e$ above X_e. Section 6.5 shows precisely how this family of operators fits into a Fredholm framework giving a meromorphic inverse, and very quickly also provides Theorem 6.

Consider $u\in C^\infty(X;\mathcal{E})$. Although the trace operator Λ acting on each subbundle $\mathcal{E}^{(k)}$ gives a notion of u being trace-free, it is more natural to consider the ambient trace operator from $\mathcal{F}$, denoted Λ_η (Subsection 6.1). Pulling u back to M, we have $\pi_s^*u\in C^\infty(M;\mathcal{F})$ and we may consider the condition that $\pi_s^*u\in\ker\Lambda_\eta$. Avoiding extra notation for this subbundle of $\mathcal{E}$ (consisting of symmetric tensors above X which are trace-free with respect to the ambient trace operator Λ_η) we will simply refer to its sections using the notation

$$C^\infty(X;\mathcal{E})\cap\ker(\Lambda_\eta\circ\pi_s^*)$$

On this subbundle, the operator $\mathcal{Q}_\lambda$ takes the following form

$$\begin{bmatrix} \Delta+\lambda^2-c'_m & 2b_{m-1}\,\mathrm{d} & & & 0 \\ -2b_{m-1}\,\delta & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 2b_0\,\mathrm{d} \\ 0 & & & -2b_0\,\delta & \Delta+\lambda^2-c'_0 \end{bmatrix}$$

with the modified constants

$$c'_k = c_k - (m-k)(m-k-1).$$

Note that if $u = u^{(m)} \in C^\infty(X;\mathcal{E}^{(m)})$ then $u \in \ker\Lambda$ if and only if $\pi_s^* u \in \ker\Lambda_\eta$. Again, a similar meromorphic extension of the inverse may be obtained.

Theorem 7. *Let (X^{n+1}, g) be even asymptotically hyperbolic. Then the inverse of*

$$\mathcal{Q}_\lambda \;\; \textit{acting on} \;\; L^2_s(X;\mathcal{E}) \cap \ker(\Lambda_\eta \circ \pi_s^*)$$

written $\mathcal{Q}_\lambda^{-1}$ has a meromorphic continuation from $\operatorname{Re}\lambda \gg 1$ *to* $\mathbb{C}$,

$$\mathcal{Q}_\lambda^{-1} : C_c^\infty(X;\mathcal{E}) \cap \ker(\Lambda_\eta \circ \pi_s^*) \to \rho^{\lambda+\frac{n}{2}-m}\left(\bigoplus_{k=0}^{m} \rho^{-2k} C^\infty_{\mathrm{even}}(\overline{X};\mathcal{E}^{(k)})\right) \cap \ker(\Lambda_\eta \circ \pi_s^*)$$

with finite rank poles.

In order to uncouple the Lichnerowicz Laplacian acting on $\mathcal{E}^{(m)}$ and obtain the desired meromorphic extension of the resolvent, we need to restrict further from simply trace-free tensors to trace-free divergence-free tensors. Equivalently, we must be able to commute the Lichnerowicz Laplacian with both the trace operator and the divergence operator. The first commutation is always possible giving the preceding structure of $\mathcal{Q}_\lambda$ however, unlike in the setting of differential forms (where the Hodge Laplacian always commutes with the divergence), such a commutation on symmetric tensors depends on the geometry of (X, g). For $m = 2$ the condition is that the Ricci tensor be parallel, while for $m \geq 3$, the manifold must be locally isomorphic to hyperbolic space.

Theorem 8. *Let (X^{n+1}, g) be even asymptotically hyperbolic and Einstein. Then the inverse of*

$$\Delta - \frac{n(n-8)}{4} + \lambda^2 \;\; \textit{acting on} \;\; L^2(X;\mathcal{E}^{(2)}) \cap \ker\Lambda \cap \ker\delta$$

written $\mathcal{R}_\lambda$ has a meromorphic continuation from $\operatorname{Re}\lambda \gg 1$ *to* $\mathbb{C}$,

$$\mathcal{R}_\lambda : C_c^\infty(X;\mathcal{E}^{(2)}) \cap \ker\Lambda \cap \ker\delta \to \rho^{\lambda+\frac{n}{2}-2} C^\infty_{\mathrm{even}}(\overline{X};\mathcal{E}^{(2)}) \cap \ker\Lambda \cap \ker\delta$$

with finite rank poles.

Theorem 9. *Let* (X^{n+1}, g) *be a convex cocompact quotient of* $\mathbb{H}^{n+1}$*. Then the inverse of*

$$\Delta - \frac{n^2 - 4m(n+m-2)}{4} + \lambda^2 \text{ acting on } L^2(X; \mathcal{E}^{(m)}) \cap \ker \Lambda \cap \ker \delta$$

written $\mathcal{R}_\lambda$ *has a meromorphic continuation from* $\operatorname{Re}\lambda \gg 1$ *to* $\mathbb{C}$,

$$\mathcal{R}_\lambda : C_c^\infty(X; \mathcal{E}^{(m)}) \cap \ker \Lambda \cap \ker \delta \to \rho^{\lambda + \frac{n}{2} - m} C^\infty_{\text{even}}(\overline{X}; \mathcal{E}^{(m)}) \cap \ker \Lambda \cap \ker \delta$$

with finite rank poles.

Note that on $\mathbb{H}^{n+1}$, the difference between the Lichnerowicz Laplacian and the rough Laplacian is $q(\mathrm{R}) = -m(n+m-1)$. Thus by introducing a spectral parameter $s = \lambda + \frac{n}{2}$ (not to be confused with the Lorentzian scale), the previous operator $\Delta - c_m + \lambda^2$ may be equivalently written

$$\nabla^*\nabla - s(n-s) - m$$

in the spirit of [DFG15].

A semiclassical analysis of this problem provides high energy estimates which are useful in the setting of analysis of gravitational waves.

Theorem 10. *Suppose that* X *is an even asymptotically hyperbolic manifold which is non-trapping. Then the meromorphic continuation, written* $\mathcal{Q}_\lambda^{-1}$ *of the inverse of* $\mathcal{Q}_\lambda$ *initially acting on* $L^2_s(X; \mathcal{E})$ *has non-trapping estimates holding in every strip* $|\operatorname{Re}\lambda| < C, |\operatorname{Im}\lambda| \gg 0$*: for* $s > \frac{1}{2} + C$

$$\|\rho^{-\lambda - \frac{n}{2} + m} \, \mathcal{Q}_\lambda^{-1} f\|_{H^s_{|\lambda|^{-1}}(X;\mathcal{E})} \leq C|\lambda|^{-1} \|\rho^{-\lambda - \frac{n}{2} + m - 2} f\|_{H^{s-1}_{|\lambda|^{-1}}(X;\mathcal{E})}.$$

If X *is furthermore Einstein, then restricting to symmetric 2-cotensors, the meromorphic continuation* $\mathcal{R}_\lambda$ *of the inverse of*

$$\Delta - \frac{n(n-8)}{4} + \lambda^2$$

initially acting on $L^2(X; \mathcal{E}^{(2)}) \cap \ker \Lambda \cap \ker \delta$ *has non-trapping estimates holding in every strip* $|\operatorname{Re}\lambda| < C, |\operatorname{Im}\lambda| \gg 0$*: for* $s > \frac{1}{2} + C$

$$\|\rho^{-\lambda - \frac{n}{2} + 2} \, \mathcal{R}_\lambda f\|_{H^s_{|\lambda|^{-1}}(X;\mathcal{E}^{(2)})} \leq C|\lambda|^{-1} \|\rho^{-\lambda - \frac{n}{2}} f\|_{H^{s-1}_{|\lambda|^{-1}}(X;\mathcal{E}^{(2)})}.$$

Classical-Quantum correspondence

An application of these meromorphic extensions is given in Chapter 7.

On a closed hyperbolic surface, Selberg's trace formula [Sel56] establishes a connection between eigenvalues of the Laplacian (on functions) and closed geodesics via the Selberg zeta function. In the convex cocompact setting this result is established by Patterson and Perry [PP01] where quantum resonances play the role of eigenvalues. Although these results indicate a correspondence between classical and quantum phenomena, it is the result of Faure and Tsujii [FT13, Proposition 4.1] that establishes a direct link between eigenvalues of the Laplacian on a closed hyperbolic surface and Ruelle resonances of the generator of the geodesic flow on the unit tangent bundle. In the convex cocompact setting, the link between quantum resonances and Ruelle resonances has recently been established by Guillarmou, Hilgert, and Weich [GHW16].

The result of [FT13] on closed hyperbolic surfaces has been extended to closed hyperbolic manifolds of arbitrary dimension by Dyatlov, Faure, and Guillarmou [DFG15]. Interestingly, in this higher dimensional setting, the correspondence is no longer simply between Ruelle resonances and the spectrum of the Laplacian acting on functions, but rather the spectrums of the Laplacian acting on symmetric tensors (precisely, those tensors which are trace-free and divergence-free).

Chapter 7 establishes this correspondence in the convex cocompact setting (for manifolds of dimension at least 3). We detail this theorem and sketch its resolution.

Let X be a convex cocompact quotient of hyperbolic space of $\mathbb{H}^{n+1}$ (with $n \geq 2$) supplied with the hyperbolic metric. Let A denote the generator of the geodesic flow (a tangent vector field on the unit tangent bundle SX). The construction of Ruelle resonances for the operator $A + \lambda$ by Faure and Sjöstrand [FS11] when the base manifold is compact has been extended to the open setting by Dyatlov and Guillarmou [DG16] and applies in this current setting. Specifically, for $\operatorname{Re}\lambda > 0$, the operator $A + \lambda$ is invertible as an operator on L^2 sections and admits a meromorphic extension $\mathcal{R}_{A,0}(\lambda) : C_c^\infty(SX) \to \mathcal{D}'(SX)$ with poles of finite rank. The poles being the Ruelle resonances. So let $\lambda_0 \in \mathbb{C}$ be a pole of the resolvent $\mathcal{R}_{A,0}(\lambda)$. Necessarily, we have $\operatorname{Re}\lambda_0 \leq 0$. Due to certain restrictions on the Poisson isomorphism of [DFG15] we impose the constraint that $\lambda_0 \notin -\frac{n}{2} - \frac{1}{2}\mathbb{N}_0$. In this introduction we will assume for simplicity that the pole of $\mathcal{R}_{A,0}(\lambda)$ at λ_0 is simple and consider an associated Ruelle resonant state, $u \in \mathcal{D}'(SX)$. Such resonant states are characterised by the equation $(A + \lambda_0)u = 0$ subject to a wave front condition on u detailed in Section 7.2. We write

$$u \in \operatorname{Res}_{A,0}(\lambda_0).$$

A non-trivial idea contained in [DFG15] is the construction of horosphere operators that generalise the horocycle vector fields present for hyperbolic surfaces. Specifically, we note that the tangent bundle $\mathrm{T}X$ over X may be pulled back to a bundle over SX which decomposes canonically into a line bundle spanned by A and the perpendicular n-dimensional bundle denoted $\mathcal{E}$. By [DFG15], there exists a differential operator

$$\mathrm{d}_- : C^\infty(SX; \operatorname{Sym}^m\mathcal{E}) \to C^\infty(SX; \operatorname{Sym}^{m+1}\mathcal{E})$$

which may be morally thought of as a symmetric differential along the negative horospheres. Moreover, this operator enjoys the commutation relation

$$[A, \mathrm{d}_-] = -\,\mathrm{d}_-$$

where it is easy to extend the vector field A to a first-order differential operator on the tensor bundle $\mathcal{E}$. As (tensor valued) Ruelle resonances are also restricted to $\operatorname{Re}\lambda \leq 0$, this commutation relation implies the existence of $m \in \mathbb{N}_0$ such that $v := (\mathrm{d}_-)^m u \neq 0$ and $\mathrm{d}_-\, v = 0$. Moreover, $(A+\lambda_0+m)v = 0$ As the vector bundle $\mathcal{E}$ carries a natural metric, we have a notion of a trace operator Λ and its adjoint L acting on $\operatorname{Sym}^m\mathcal{E}$. We may thus decompose v into trace-free components

$$v = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \mathrm{L}^k\, v^{(m-2k)}, \qquad v^{(m-2k)} \in \mathcal{D}'(SX; \operatorname{Sym}^{m-2k}\mathcal{E}) \cap \ker(\Lambda) \cap \ker(A + \lambda_0 + m).$$

Integrating over the fibres of $SX \to X$ allows $v^{(m-2k)}$ to be pushed to a symmetric $(m-2k)$-tensor on X

$$\varphi^{(m-2k)} := \pi_{0*}v^{(m-2k)} \in C^\infty(X; \mathrm{Sym}^{m-2k}\mathrm{T}^*X).$$

and the properties of the Poisson transform imply that

$$\varphi^{(m-2k)} \in \ker(\nabla^*\nabla + (\lambda_0 + m)(n + \lambda_0 + m) - (m - 2k))$$

In fact, it is also trace-free, divergence-free, and satisfies precise asymptotics at the boundary. Lemma 7.12 gives a classification of quantum resonant states from which we conclude that $\varphi^{(m-2k)}$ is indeed a quantum resonant state associated with the resonance $\lambda_0 + m + n$. We write

$$\varphi^{(m-2k)} \in \mathrm{Res}_{\Delta,m-2k}(\lambda_0 + m + n).$$

Thanks to properties of the Poisson operator (mostly detailed in [DFG15]) this path may be reversed and we obtain an isomorphism between quantum resonances and Ruelle resonances. Two aspects of the proof render the isomorphism considerably labour intensive. First, one needs to deal with inverting the horosphere operators. Second, one needs to consider the possibility that the Ruelle resonance is not a simple pole, but rather, there may exist generalised Ruelle resonant states.

Theorem 11. *Let $X = \Gamma\backslash\mathbb{H}^{n+1}$ be a smooth oriented convex cocompact hyperbolic manifold, and $\lambda_0 \in \mathbb{C}\backslash(-\frac{n}{2} - \frac{1}{2}\mathbb{N}_0)$. There exists a vector space linear isomorphism between Ruelle generalised resonant states*

$$\mathrm{Res}_{A,0}(\lambda_0)$$

and the following space of quantum generalised resonant states

$$\bigoplus_{m\in\mathbb{N}_0}\bigoplus_{k=0}^{\lfloor\frac{m}{2}\rfloor}\mathrm{Res}_{\Delta,m-2k}(\lambda_0 + m + n).$$

2. Even Clifford Structures

This chapter is structured as follows. Section 2.1 recalls the classification of even Clifford algebras. The standard reference for such algebras is [LM89]. Section 2.2 introduces even Clifford structures and states the classification found in [MS11].

2.1 Clifford Algebras

Consider $\mathbb{R}^r$ endowed with its canonical positive definite inner product, and denote by $\{\xi_i\}_{1\leq i\leq r}$ the standard orthonormal basis. Let Cl_r denote the Clifford algebra. One may define this algebra as the quotient of the tensor algebra of $\mathbb{R}^r$ by the ideal generated by elements of the form $v\otimes v+|v|^2$ for $v\in\mathbb{R}^r$. There is then a natural embedding of $\mathbb{R}^r$ in Cl_r and the Clifford algebra is generated by all products of the orthonormal basis subject to the relations

$$\xi_i\cdot\xi_j+\xi_j\cdot\xi_i=-2\delta_{ij},\qquad \text{for } 1\leq i,j\leq r.$$

Here, $\cdot$ denotes Clifford multiplication. The $\mathbb{Z}_2$-grading of the tensor algebra of $\mathbb{R}^r$ descends to the Clifford algebra giving it a $\mathbb{Z}_2$-grading and provides the even Clifford algebra denoted by Cl^0_r. (There is an algebra isomorphism between Cl^0_r and Cl_{r-1}.) The spin group $\mathrm{Spin}(r)\subset\mathrm{Cl}^0_r$ is

$$\mathrm{Spin}(r):=\{v_1\cdot\ldots\cdot v_k\mid v\in\mathbb{R}^r,|v|=1,k\in 2\mathbb{N}\}$$

and has Lie algebra

$$\mathfrak{spin}(r):=\mathrm{span}\{\,\xi_i\cdot\xi_j\mid 1\leq i<j\leq n\}\simeq\mathfrak{so}(r)\simeq\Lambda^2\mathbb{R}^r.$$

Clifford algebras may be explicitly described as matrix algebras over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ [LM89, Section 1.4]. We announce this description for even Clifford algebras. Let d_r denote the dimension of an irreducible representation of Cl^0_r and denote by ε_r the number of distinct irreducible representations.

r mod 8	Cl^0_r	d_r	ε_r
2	$\mathbb{C}(\frac{1}{2}d_r)$	$2^{\frac{r}{2}}$	1
3	$\mathbb{H}(\frac{1}{4}d_r)$	$2^{\frac{r+1}{2}}$	1
4	$\mathbb{H}(\frac{1}{4}d_r)\oplus\mathbb{H}(\frac{1}{4}d_r)$	$2^{\frac{r}{2}}$	2
5	$\mathbb{H}(\frac{1}{4}d_r)$	$2^{\frac{r+1}{2}}$	1
6	$\mathbb{C}(\frac{1}{2}d_r)$	$2^{\frac{r}{2}}$	1
7	$\mathbb{R}(d_r)$	$2^{\frac{r-1}{2}}$	1
8	$\mathbb{R}(d_r)\oplus\mathbb{R}(d_r)$	$2^{\frac{r-2}{2}}$	2
9	$\mathbb{R}(d_r)$	$2^{\frac{r-1}{2}}$	1

Table 3: Representations of even Clifford algebras

An attractive and explicit realisation of such algebras which is geared toward problems involving even Clifford structures is developed in an article by Arizmendi and Herrera [AH15]. Following this article, we let $\tilde{\Delta}_r$ denote the irreducible representations of Cl_r^0 for $r \not\equiv 0 \mod 4$ and by $\tilde{\Delta}_r^{\pm}$ the two inequivalent irreducible representations in the case $r \equiv 0 \mod 4$. The classification of Clifford algebras implies that the representations are complex for $r \equiv 2, 6 \mod 8$ and quaternionic for $r \equiv 3, 4, 5 \mod 8$.

In preperation of even Clifford structures on manifolds, we consider a possible embedding

$$\varphi : \begin{cases} \mathfrak{spin}(r) & \rightarrow & \mathfrak{so}(n) \\ \xi_i \cdot \xi_j & \mapsto & J_{ij} \end{cases}$$

subject to the constraint that $J_{ij}^2 = -1$ under the identification $\mathfrak{so}(n) \simeq \mathrm{End}^-(\mathbb{R}^n)$. We consider separately the cases where Cl_r^0 has one or two inequivalent irreducible representations. If $r \not\equiv 0 \mod 4$ then the requirement that $J_{ij}^2 = -1$ forces the following decomposition

$$\mathbb{R}^n = \oplus^k \tilde{\Delta}_r = \tilde{\Delta}_r \otimes \mathbb{R}^k$$

for some $k \in \mathbb{N}$. Alternatively, if $r \equiv 0 \mod 4$ then the requirement that $J_{ij}^2 = -1$ forces the following decomposition

$$\mathbb{R}^n = (\tilde{\Delta}_r^+ \otimes \mathbb{R}^{k_+}) \oplus (\tilde{\Delta}_r^- \otimes \mathbb{R}^{k_-})$$

for some $k_\pm \in \mathbb{N}_0$ with at least one of $k_\pm$ non-zero. In particular we obtain a restriction on the possible values of n. Specifically, if $r \not\equiv 0 \mod 4$ then $n = d_r k$ and if $r \equiv 0 \mod 4$ then $n = d_r(k_+ + k_-)$.

2.2 Even Clifford Structures

Let (M^n, g) be a Riemannian manifold with Levi-Civita connection ∇. Consider an oriented rank r Euclidean bundle (E, h) which provides the Clifford bundle $\mathrm{Cl}(E, h)$ over M. When working locally, we denote by $\{e_i\}_{1 \leq i \leq n}$ a local orthonormal frame for $\mathrm{T}M$ and by $\{\xi_i\}_{1 \leq i \leq r}$ a local oriented orthonormal frame for (E, h).

The definition of a Clifford structure, as defined in [MS11] but of which we will not make use, then requires the further existence of a non-vanishing algebra bundle morphism, termed a Clifford morphism,

$$\varphi : \mathrm{Cl}(E, h) \rightarrow \mathrm{End}(\mathrm{T}M)$$

which maps E into the bundle of skew-symmetric endomorphisms $\mathrm{End}^-(\mathrm{T}M)$. We mention this construction as, unlike for the even structures that are of interest in this thesis, there is a clear requirement that E be globally defined and not merely a projective bundle.

We recall the notion of a projective bundle. We begin by remarking that isomorphism classes of oriented Euclidean rank r vector bundles (E, h) are in one-to-one correspondence with isomorphism classes of principal $\mathrm{SO}(k)$-bundles P over M. Metric covariant derivatives on E are then identified with principal $\mathrm{SO}(k)$-connections on P. We recall that the projective special orthogonal group is

$$\mathrm{PSO}(k) := \begin{cases} \mathrm{SO}(k) & \text{if } k \text{ is odd} \\ \mathrm{SO}(k)/\{\pm \mathrm{I}_k\} & \text{if } k \text{ is even} \end{cases}$$

and use the preceding one-to-one correspondence to motivate

Definition 2.1. *A locally defined oriented Euclidean rank r vector bundle over M is a principal $\mathrm{PSO}(r)$-bundle over M.*

The terminology is justified by the fact that the structure group of a principal $\mathrm{PSO}(r)$-bundle can be reduced to $\mathrm{SO}(r)$ over any contractible open neighborhood U of M, and thus gives rise to an oriented Euclidean rank r vector bundle over U.

If E is a locally defined oriented Euclidean rank r vector bundle over M and $\rho : \mathrm{PSO}(k) \to \mathrm{SO}(N)$ is a group morphism, one obtains a rank N oriented Euclidean vector bundle $\rho(E)$ over M by enlarging the structure group of E to $\mathrm{SO}(N)$ and considering the associated vector bundle. In particular, since the even-dimensional tensor powers of the standard representation of $\mathrm{SO}(r)$ on $\mathbb{R}^r$ descend to $\mathrm{PSO}(r)$, the even tensor powers of a locally defined oriented Euclidean vector bundle are globally defined vector bundles. And in our case, such a locally defined vector bundle leads to the globally defined even Clifford bundle $\mathrm{Cl}^0(E, h)$.

We can give the definition of even Clifford structures on Riemannian manifolds, which have been introduced in [MS11].

Definition 2.2. *A rank $r \geq 2$ even Clifford structure on a Riemannian manifold (M^n, g) is an oriented, locally defined, rank r Euclidean bundle (E, h) over M together with a non-vanishing algebra bundle morphism, called a Clifford morphism, $\varphi : \mathrm{Cl}^0(E, h) \to \mathrm{End}(\mathrm{T}M)$ which maps $\Lambda^2 E \subset \mathrm{Cl}^0(E, h)$ to the bundle of skew-symmetric endomorphisms $\mathrm{End}^-(\mathrm{T}M)$.*

Remark 2.3. For r even, this definition corresponds to what [MS11, Remark 2.5] terms projective even Clifford structures.

Definition 2.4. *An even Clifford structure (M, g, E, h, φ), is called parallel if there exists a metric connection ∇^E on (E, h) such that φ is connection preserving with respect to ∇^E and the Levi-Civita connection ∇ of (M, g).*

Explicitly, the even Clifford structure is parallel if, for every tangent vector $X \in \mathrm{T}M$ and section σ of $\mathrm{Cl}^0(E, h)$, that

$$\varphi(\nabla^E_X \sigma) = \nabla_X \varphi(\sigma).$$

An alternative and natural setting is to consider homogeneous (rather than parallel) even Clifford structures on homogeneous spaces [MP13].

Definition 2.5. *A parallel even Clifford structure $(E, h, \nabla^E, \varphi)$, is called flat if the connection ∇^E is flat.*

These definitions contain several geometries and provides a general framework to study them. For rank 2, an even Clifford structure provides an almost Hermitian structure (and vice versa) and it is parallel if and only if the corresponding almost Hermitian structure establishes a Kähler structure. For rank 3, an even Clifford structure provides a quaternion-Hermitian structure (and vice versa) and it is parallel if and only if the quaternion-Hermitian structure establishes a quaternion-Kähler strucutre.

The principal result of [MS11] is a complete classification of manifolds carrying parallel even Clifford structures. Before announcing this classification we state two preliminary results [MS11, Lemma 2.4, Proposition 2.10] which are often appealed to in this thesis. Our convention is that $\langle\cdot,\cdot\rangle$ denotes minus the trace of the product of two endomorphisms.

Lemma 2.6. *[MS11] Let (E, h, φ) be a rank r even Clifford structure and let $\{\xi_i\}_{1\leq i\leq r}$ be a local orthonormal frame on E. The local endomorphisms $J_{ij} := \varphi(\xi_i \cdot \xi_j) \in \mathrm{End}(\mathrm{T}M)$ are skew-symmetric for $i \neq j$ and satisfy*

$$\begin{cases} J_{ii} = -1_{\mathrm{T}_xM} & \text{for all } 1 \leq i \leq r, \\ J_{ij} = -J_{ji} \text{ and } J_{ij}^2 = -1_{\mathrm{T}_xM} & \text{for all } i \neq j, \\ J_{ij} \circ J_{ik} = J_{jk} & \text{for all } i, j, k \text{ mutually distinct}, \\ J_{ij} \circ J_{kl} = J_{kl} \circ J_{ij} & \text{for all } i, j, k, l \text{ mutually distinct.} \end{cases}$$

Moreover, if $r \neq 4$, then

$$\langle J_{ij}, J_{kl}\rangle = 0, \qquad \text{unless } i = j, k = l \text{ or } i = k \neq j = l \text{ or } i = l \neq k = j.$$

□

Proposition 2.7. *[MS11] Consider a complete simply connected Riemannian manifold (M^n, g) and suppose it carries a parellel non-flat even Clifford structure $(E, h, \nabla^E, \varphi)$ of rank $r \geq 3$. Then the following holds:*

(i) If $r = 4$ then (M, g) is a Riemannian product of two quaternion-Kähler manifolds.

(ii) If $r \neq 4$ and $n \neq 8$ then

(a) The curvature of ∇^E, viewed as a map from $\Lambda^2 M$ to $\mathrm{End}^-(E) \simeq \Lambda^2 E$ is a non-zero constant times the metric adjoint of the Clifford morphism φ.

(b) M is Einstein with non-vanishing scalar curvature and has irreducible holonomy.

(iii) If $r \neq 4$ and $n = 8$, then (a) implies (b).

□

The list of complete simply connected Riemannian manifolds M carrying a parallel rank r even Clifford structure is given in the tables below. For the sake of simplicity, the non-compact duals of the compact symmetric spaces have been omitted. As in the previous section, d_r denotes the dimension of the irreducible representations of Cl_r^0. In Table 5, the non-compact duals of the compact symmetric spaces have been omitted.

r	M	dimension of M
2	Kähler	$2k$, $k \geq 1$
3 and 4	hyper-Kähler	$4k$, $k \geq 1$
4	reducible hyper-Kähler	$4(n_+ + n_-)$, $n_\pm \geq 1$
arbitrary	Cl_r^0 representation space	multiple of d_r

Table 4: Manifolds with a flat even Clifford structure

r	M	dimension of M
2	Kähler	$2k,\ k \geq 1$
3	quaternion-Kähler (QK)	$4k,\ k \geq 1$
4	product of two QK manifolds	$4(n_+ + n_-),\ n_\pm \geq 1$
5	QK	8
6	Kähler	8
7	Spin(7) holonomy	8
8	Riemannian	8
5	$\mathrm{Sp}(k+2)/\mathrm{Sp}(k) \times \mathrm{Sp}(2)$	$8k,\ k \geq 2$
6	$\mathrm{SU}(k+4)/\mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(4))$	$8k,\ k \geq 2$
8	$\mathrm{SO}(k+8)/\mathrm{SO}(k) \times \mathrm{SO}(8)$	$8k,\ k \geq 2$
9	$\mathbb{O}\mathbb{P}^2 = \mathrm{F}_4/\mathrm{Spin}(9)$	16
10	$(\mathbb{C} \otimes \mathbb{O})\mathbb{P}^2 = \mathrm{E}_6/\mathrm{Spin}(10) \cdot \mathrm{U}(1)$	32
12	$(\mathbb{H} \otimes \mathbb{O})\mathbb{P}^2 = \mathrm{E}_7/\mathrm{Spin}(12) \cdot \mathrm{SU}(2)$	64
16	$(\mathbb{O} \otimes \mathbb{O})\mathbb{P}^2 = \mathrm{E}_8/\mathrm{Spin}^+(16)$	128

Table 5: Manifolds with a parallel non-flat even Clifford structure

3. Twistor Spaces

This chapter is structured as follows. Section 3.1 explains the construction of the twistor space of Riemannian manifolds with even Clifford structures and their almost complex structures. Section 3.2 checks the integrability of the almost complex structure and constructs in the appropriate cases, a Kähler metric on the twistor space. Section 3.3 provides an alternative almost complex structure on the twistor space leading to a nearly Kähler structure.

3.1 Twistor Space of an Even Clifford Structure

Let (M^n, g) be a manifold with an even Clifford structure (E, h, φ). Given $x \in M$, let $\{\xi_i\}_{1\leq i\leq r}$ be an orthonormal basis for the fibre E_x and define $J_{ij} := \varphi(\xi_i \cdot \xi_j)$ where $\cdot$ denotes Clifford multiplication. For each x we consider the subspace Z_x of $\mathrm{End}(\mathrm{T}_x M)$ where

$$Z_x := \left\{ J = \sum_{1\leq i<j\leq r} a_{ij} J_{ij} \;\middle|\; J^2 = -1_{\mathrm{T}_x M},\, a_{ij} \in \mathbb{R} \right\}$$

and define the twistor space of the even Clifford structure to be the disjoint union

$$Z := \bigsqcup_{x\in M} Z_x.$$

We will denote by π the projection onto M. This is a bundle of pointwise orthogonal complex structures. For a parallel Cl_3^0 structure this coincides with the definition of the twistor space of a quaternion-Kähler manifold, where the fibre is homeomorphic to $\mathbb{S}^2$. It is not hard to see that for a Cl_4^0 structure the fibre of the twistor space is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^2$, which corresponds to the isomorphism between $\mathrm{Spin}(4)$ and $\mathrm{Spin}(3) \times \mathrm{Spin}(3)$. In general the fibre at each point is isomorphic to $\widetilde{\mathrm{Gr}}(2, r)$, the Grassmannian of oriented 2-planes in $\mathbb{R}^r$, as we see from

Lemma 3.1. *Let $A \in \mathfrak{spin}(r) \subset \mathrm{Cl}_r^0$, then $A^2 = -1$ if and only if there exist $v_1, v_2 \in \mathbb{R}^r$ orthonormal vectors such that $A = v_1 \cdot v_2$.*

Proof. Let $A \in \mathfrak{spin}(r) \subset \mathrm{Cl}_r^0$ and $\{\xi_i\}_{1\leq i\leq r}$ an orthornormal basis for $\mathbb{R}^r$, then under a change of basis, we can suppose that $A = \sum_{i=1}^{\lfloor \frac{r}{2} \rfloor} a_i \xi_{2i-1} \cdot \xi_{2i}$. The condition $A^2 = -1$ yields the equations

$$\sum_{i=1}^{\lfloor \frac{r}{2} \rfloor} a_i^2 = 1, \qquad a_i a_j = 0, (i < j).$$

The solutions of these equations are the r-tuples $(\pm 1, 0, \ldots, 0)$, $(0, \pm 1, 0, \ldots, 0)$, $\ldots$, $(0, \ldots, \pm 1)$. Therefore $A = \pm \xi_{2i-1} \cdot \xi_{2i}$ for some $1 \leq i \leq \lfloor \frac{r}{2} \rfloor$. Conversely, if $A = v_1 \cdot v_2$ with v_1 and v_2

orthonormal, then $A^2 = v_1 \cdot v_2 \cdot v_1 \cdot v_2 = -v_1^2 v_2^2 = -1$, which proves the assertion. □

Remark 3.2. Another way to prove this is by using that an element A in $\Lambda^2\mathbb{R}^r$ is decomposable if and only if $A \wedge A = 0$.

Consider $\widetilde{\mathrm{Gr}}(2,r)$ as a Hermitian-symmetric space. Its complex structure can be given using Clifford multiplication. Let $z \in \widetilde{\mathrm{Gr}}(2,r)$, which for a suitable frame can be written as $z = \xi_1 \wedge \xi_2 = \xi_1 \cdot \xi_2$. The tangent space $\mathrm{T}_z\widetilde{\mathrm{Gr}}(2,r)$ can be identified with

$$\mathrm{span}\,\{\xi_i \cdot \xi_j \mid i \in \{1,2\}, j \in \{3,\dots,r\}\} = \left\{ \sum_{s=3}^{r} \alpha_s \xi_1 \cdot \xi_s + \beta_s \xi_2 \cdot \xi_s \,\middle|\, \alpha_s, \beta_s \in \mathbb{R} \right\}.$$

The complex structure is then given by $\tilde{J}_z(v) := z \cdot v$.

The Levi-Civita connection on M induces a connection on Z. For each $S \in Z$, the connection gives a splitting $\mathrm{T}_S Z = V_S \oplus H_S$ where $V_S = \ker(\pi_*)$ is isomorphic to $\mathrm{T}_{\pi(S)}\widetilde{\mathrm{Gr}}(2,r)$, the isomorphism given by the differential of the Clifford map, and H_S, the horizontal subspace, is isomorphic to $\mathrm{T}_{\pi(S)}M$. We recall the usual construction of almost complex structures on Z. Given $U \in V_S$ and $X \in H_S$ we define

$$\mathcal{J}(U+X)_S := \hat{J}(U) + \pi_*^{-1}(S\pi_*(X))$$

where $\hat{J}(U) := \varphi_* \tilde{J} \varphi_*^{-1}(U) = SU$.

This construction is inspired by the low rank setting. Indeed for $r = 4, 5$ we have:

- $r = 3$: In this case this is the construction of the almost complex structure for quaternion-Kähler manifolds, so $(Z, \mathcal{J})$ is a complex manifold, see [BB82, Sal82].
- $r = 4$: In this case the manifold is locally a product of two quaternion-Kähler manifolds [MS11]. Moreover $\widetilde{\mathrm{Gr}}(2,4)$ is isomorphic to $\mathbb{S}^2 \times \mathbb{S}^2$ as Kähler manifolds, so in this case the twistor space is the product of the twistor spaces of two quaternion-Kähler manifolds. In particular, it is a complex manifold.

From now on, we will suppose that (M, g) carries a parallel even Clifford structure of rank $r \geq 5$. We treat the 8-dimensional case first. In this case, the rank should be $5, 6, 7$ or 8 and the following holds:

- $r = 5$: In this case the manifold is known to be quaternion-Kähler in the case of positive curvature [MS11]. The twistor space has fibre isomorphic to $\mathrm{Sp}(2)/\mathrm{U}(2)$ and has been considered in [Bur90]. The twistor space is complex exactly when M is locally symmetric, hence locally isometric to either quaternionic projective space or to $\widetilde{\mathrm{Gr}}(2,6) \cong \mathrm{Gr}_{\mathbb{C}}(2,4)$.
- $r = 6$: In this case the manifold is known to be Kähler [MS11]. The twistor space has fibre $\widetilde{\mathrm{Gr}}(2,6)$ and has been considered in [OR85, Bur90]. The twistor is complex exactly when the Bochner tensor of M vanishes.
- $r = 7$: In this case the manifold has $\mathrm{Spin}(7)$ holonomy. The twistor space has fiber $\mathrm{SO}(7)/\mathrm{SO}(5) \times \mathrm{SO}(2)$. According to [Bur90], the twistor space in this case is never complex.
- $r = 8$: In this case the manifold is Riemannian. The twistor fibre is isomorphic to $\mathrm{SO}(8)/\mathrm{U}(4)$ which is the usual fibre of the twistor space defined for even dimensional Riemannian manifolds. As mentioned in the introduction, the twistor space is complex if and only if M is conformally flat.

3.2 Integrability of Almost Complex Structure

We will now assume that $n \neq 8$.

Lemma 3.3. *Let M be a Riemannian manifold of dimension $n \neq 8$ carrying a parallel even Clifford structure of rank $r > 4$, then for every $S \in Z$ and $X, Y \in \mathrm{T}_{\pi(S)}M$, the curvature R of the Levi-Civita connection satisfies*

$$[\mathrm{R}_{SX,SY}, S] - S[\mathrm{R}_{SX,Y}, S] - S[\mathrm{R}_{X,SY}, S] - [\mathrm{R}_{X,Y}, S] = 0.$$

Proof. It suffices to prove the proposition for $S = J_{12}$. If the parallel even Clifford structure is flat then the manifold is flat [MS11, Theorem 2.9]. So $\mathrm{R}_{X,Y} = 0$ for all $X, Y \in \mathrm{T}_{\pi(S)}M$. If the parallel even Clifford structure is not flat and $n \neq 8$, then the proof of Proposition 2.7, explicitly [MS11, Equation (15)], implies the existence of a non-zero constant κ such that

$$[\mathrm{R}_{X,Y}, J_{12}] = \kappa \sum_{s>2} g(J_{s1}X, Y)J_{s2} - g(J_{s2}X, Y)J_{s1}.$$

Using the properties of the endomorphisms J_{ij} in Lemma 2.6, specifically, $J_{ij} \circ J_{ik} = J_{jk}$ for i, j, k mutually distinct, the result follows upon summing the following four calculations.

$$\begin{aligned}
[\mathrm{R}_{J_{12}X,J_{12}Y}, J_{12}] &= \kappa \sum_{s>2} -g(J_{s1}X, Y)J_{s2} + g(J_{s2}X, Y)J_{s1} \\
-J_{12}[\mathrm{R}_{J_{12}X,Y}, J_{12}] &= \kappa \sum_{s>2} -g(J_{s2}X, Y)J_{s1} + g(J_{s1}X, Y)J_{s2} \\
-J_{12}[\mathrm{R}_{X,J_{12}Y}, J_{12}] &= \kappa \sum_{s>2} -g(J_{s2}X, Y)J_{s1} + g(J_{s1}X, Y)J_{s2} \\
-[\mathrm{R}_{X,Y}, J_{12}] &= \kappa \sum_{s>2} -g(J_{s1}X, Y)J_{s2} + g(J_{s2}X, Y)J_{s1}
\end{aligned}$$

□

Theorem 1. *Let M be a Riemannian manifold of dimension $n \neq 8$ carrying a parallel even Clifford structure of rank $r > 4$, then the almost complex structure $\mathcal{J}$ on Z is integrable.*

Proof. We proceed as in 14.68 of [Bes08]. For an arbitrary vector field W, we let $\mathcal{V}(W)$ denote the vertical part of W and $\mathcal{H}(W)$ the horizontal part of W. Let $N_{\mathcal{J}}$ be the Nijenhuis tensor of $\mathcal{J}$. Let U and V be vertical vector fields and X and Y basic horizontal vector fields.

Let us first check that $N_{\mathcal{J}}(U, V) = 0$. Since U and V are vertical, $\mathcal{J}(U)$ and $\mathcal{J}(V)$ are also vertical vector fields. Thus $N_{\mathcal{J}}(U, V) = N_{\hat{J}}(U, V) = 0$, since $\hat{J}$ is a complex structure.

Now we will check that $N_{\mathcal{J}}(X, U) = 0$. From the two facts that the horizontal transport of the horizontal distribution respects $\hat{J}$, and that $[X, U]$ is vertical if U is, we obtain $[X, \mathcal{J}U] = \mathcal{J}[X, U]$. This reduces the Nijenhuis tensor to $N_{\mathcal{J}}(X, U) = \mathcal{J}([\mathcal{J}(X), U]) - [\mathcal{J}(X), \mathcal{J}(U)]$. The vertical part of this vanishes by noting that both terms in $\mathcal{V}([\mathcal{J}(X), J(U)]) = \mathcal{J}(\mathcal{V}[J(X), U])$ are tensorial in X. Finally, for the horizontal part of $N_{\mathcal{J}}(X, U)$ observe first that $\pi_*([J(X), U]) = -U\pi_*X$ from which we obtain

$$\begin{aligned}
\pi_*(\mathcal{J}[\mathcal{J}(X), U]) &= \pi_*(\mathcal{J}\mathcal{H}[\mathcal{J}(X), U]) \\
&= \pi_*(\pi_*^{-1}S\pi_*[\mathcal{J}(X), U]) \\
&= -SU\pi_*X
\end{aligned}$$

By the same reasoning

$$\begin{aligned}\pi_*([\mathcal{J}(X), \mathcal{J}(U)]) &= -\mathcal{J}(U)\pi_* X \\ &= -SU\pi_* X\end{aligned}$$

and so $N_{\mathcal{J}}(X, U) = 0$.

Finally, we check that $N_{\mathcal{J}}(X, Y) = 0$. This is done by considering the horizontal and vertical components separately. For the horizontal component, we consider S in Z with $\pi(S) = x$ as a section, also denoted S, of Z about x and demand that $\nabla S = 0$ at x. This gives a local almost complex structure on a neighborhood of x which has an associated Nijenhuis tensor N_S. A direct calculation gives agreement, on the neighbourhood of x, between the two Nijenhuis tensors considered, explicitly,

$$\pi_*(N_{\mathcal{J}}(X, Y)_S) = N_S(\pi_*(X), \pi_*(Y)).$$

The tensor N_S is then seen to vanish at x as ∇ is torsion-free and, at x, ∇S vanishes. Studying the vertical component one recalls O'Neill's formulas for Riemannian submersions [Bes08, Chapter 9]. In particular, $\mathcal{V}[X, Y]_{\pi(S)} = -[\mathrm{R}_{\pi_* X, \pi_* Y}, S]$, which implies $\mathcal{V}(N_{\mathcal{J}}(X, Y)) = 0$ precisely by Lemma 3.3. □

Theorem 2. *The twistor space $(Z, \mathcal{J})$ of a Riemannian manifold of dimension $n \neq 8$ with a parallel even Clifford structure of rank $r > 4$ and $\mathrm{Ric} > 0$ admits a Kähler metric.*

Proof. In this case the manifold (M, g) is Einstein with $\mathrm{Ric} = \kappa(n/4 + 2r - 4)$ by Proposition 2.7. Using the condition that $\mathrm{Ric} > 0$, we choose a metric h on Z such that π is a Riemannian submersion with totally geodesic fibres isometric to $\widetilde{\mathrm{Gr}}(2, r)$ with Kähler metric and $\mathrm{Ric} = 2r\kappa$, so that the collection $\{J_{ij}\}$ forms a mutually orthogonal frame and $\|J_{ij}\|^2 = 1/\kappa$. Let U and V be vertical vector fields and X and Y basic horizontal vector fields. The theorem follows a similar argument to that given in 14.81 of [Bes08]. We consider separately the four cases coming from $(\nabla_E \mathcal{J})F$ where E, F may be horizontal or vertical.

First we show $\nabla_U \mathcal{J} = 0$. Restricting to its action on a vertical field, we immediately get $(\nabla_U \mathcal{J})V = 0$ as the fibre is Kähler and totally geodesic. In order to prove $(\nabla_U \mathcal{J})X = 0$, it suffices to consider only the horizontal component (again as the fibres are totally geodesic). By appropriately choosing a local orthonormal frame for E, we may assume that $S = J_{12}$ and $U = \lambda J_{s1}$ with $s > 2$. The Koszul formula and the relationship between the vertical component of the Lie bracket and the curvature mentioned in the previous proof give, at S,

$$\begin{aligned}2h(\nabla_U X, Y) &= -h([X, Y], U) \\ &= \lambda h([\mathrm{R}_{\pi_* X, \pi_* Y}, J_{12}], J_{s1}).\end{aligned}$$

Recalling Equation 3.2 we deduce $h(\nabla_U X, Y) = -\frac{1}{2}\lambda g(J_{s2}\pi_* X, \pi_* Y)$. Using this result, $\pi_*(\nabla_U X)_S = -\frac{1}{2}\lambda J_{s2}\pi_* X$, we obtain

$$\begin{aligned}\pi_*(\mathcal{J}\nabla_U X)_S &= -\tfrac{1}{2} J_{12}\lambda J_{s2}\pi_* X \\ &= \tfrac{1}{2} U\pi_* X.\end{aligned}$$

Similarly, one proves that

$$\pi_*(\nabla_U \mathcal{J} X) = \tfrac{1}{2} U\pi_* X,$$

from which we conclude $\pi_*((\nabla_U \mathcal{J})X) = 0$.

Second, we show $\nabla_X \mathcal{J} = 0$. Recall O'Neill's A tensor

$$A_E F := \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F$$

where E and F are arbitrary vectors. We show, as an initial calculation, that $A_X(\mathcal{J}Y) = \mathcal{J}(A_X Y)$ and $\mathcal{J}(A_X U) = A_X(\mathcal{J}U)$. In our situation we note the following decomposition into horizontal and vertical components

$$\begin{aligned}
\nabla_X U &= \mathcal{V}\nabla_X U + A_X U \\
\nabla_X Y &= A_X Y + \mathcal{H}\nabla_X Y.
\end{aligned}$$

By [Bes08, Proposition 9.24], we have $A_X Y = \frac{1}{2}\mathcal{V}[X,Y]$ so at $S = J_{12}$, we get $A_X Y = -\frac{1}{2}[\mathrm{R}_{\pi_* X, \pi_* Y}, J_{12}]$ and the claim that $A_X(\mathcal{J}Y) = \mathcal{J}(A_X Y)$ is equivalent to

$$[\mathrm{R}_{X', J_{12}Y'}, J_{12}] = J_{12}[\mathrm{R}_{X',Y'}, J_{12}]$$

where, for the sake of notation, we have denoted $X' = \pi_* X$ and $Y' = \pi_* Y$. Equation 3.2 gives the result as

$$\begin{aligned}
J_{12}[\mathrm{R}_{X',Y'}, J_{12}] &= J_{12}(\kappa \sum_{s>2} g(J_{s1}X', Y')J_{s2} - g(J_{s2}X', Y')J_{s1}) \\
&= \kappa \sum_{s>2} -g(J_{s1}X', Y')J_{s1} - g(J_{s2}X', Y')J_{s2}
\end{aligned}$$

and similarly

$$\begin{aligned}
[\mathrm{R}_{X', J_{12}Y'}, J_{12}] &= \kappa \sum_{s>2} g(J_{s1}X', J_{12}Y')J_{s2} - g(J_{s2}X', J_{12}Y')J_{s1} \\
&= \kappa \sum_{s>2} -g(J_{s2}X', Y')J_{s2} - g(J_{s1}X', Y')J_{s1}.
\end{aligned}$$

For the second claim, we use the skew symmetry of A, $h(A_X Y, U) = -h(Y, A_X U)$ to obtain

$$\begin{aligned}
h(\mathcal{J}A_X U, Y) &= h(U, A_X \mathcal{J}Y) \\
&= h(U, \mathcal{J}A_X Y) \\
&= h(A_X \mathcal{J}U, Y).
\end{aligned}$$

Therefore $\mathcal{J}(A_X U) = A_X(\mathcal{J}U)$.

We apply this result to $(\nabla_X \mathcal{J})U$ where

$$\begin{aligned}
(\nabla_X \mathcal{J})U &= \nabla_X(\mathcal{J}U) - \mathcal{J}\nabla_X U \\
&= \mathcal{V}\nabla_X(\mathcal{J}U) + A_X \mathcal{J}U - \mathcal{J}\mathcal{V}\nabla_X U - \mathcal{J}A_X U \\
&= \mathcal{V}\nabla_X(\mathcal{J}U) - \mathcal{J}\mathcal{V}\nabla_X U
\end{aligned}$$

Taking the inner product of each term with V and studying the respective Koszul formulas gives the result that $(\nabla_X \mathcal{J})U = 0$. By a similar calculation for $(\nabla_X \mathcal{J})Y$,

$$(\nabla_X \mathcal{J})Y = \mathcal{H}\nabla_X(\mathcal{J}Y) - \mathcal{J}\mathcal{H}\nabla_X Y.$$

As this is horizontal we may use a similar idea to that presented in the preceding proof. Specifically, we consider $S \in Z$ with $x = \pi(S)$ as a section over a neighborhood of x with $\nabla S = 0$ at x. Studying the appropriate Koszul formulas one concludes that, at x,

$$(\nabla_X \mathcal{J})Y = (\nabla_{\pi_* X} S)\pi_* Y.$$

The result now follows since, at x,

$$\begin{aligned}(\nabla_X \mathcal{J})Y &= \pi_*^{-1}(\nabla_{\pi_* X}(S\pi_* Y) - S\nabla_{\pi_* X}\pi_* Y) \\ &= \pi_*^{-1}((\nabla_{\pi_* X} S)\pi_* Y) = 0.\end{aligned}$$

□

A summary of these results is given in Tables 1 and 2 found in the introduction to this thesis.

3.3 Nearly Kähler Structure

A nearly Kähler manifold is an almost Hermitian manifold $(Z, \widetilde{\mathcal{J}}, \widetilde{h})$ with the property that, for all tangent vectors X, we have $(\widetilde{\nabla}_X \widetilde{\mathcal{J}})X = 0$ where $\widetilde{\nabla}$ is the Levi-Civita connection of $(Z, \widetilde{h})$.

We use the following observation of Nagy [Nag02] to construct nearly Kähler metrics on the twistor space. Consider a Riemannian submersion with totally geodesic fibres

$$F \to (Z, h) \to M$$

and let $\mathrm{T}Z = V \oplus H$ be the corresponding splitting of $\mathrm{T}Z$. Suppose that Z admits a complex structure $\mathcal{J}$ compatible with h and preserving V and H such that $(Z, \mathcal{J}, h)$ is a Kähler manifold. Consider now the Riemannian metric on Z defined by

$$\widetilde{h} := \begin{cases} \frac{1}{2}h & \text{on } V \times V \\ h & \text{on } H \times H \end{cases}$$

The metric $\widetilde{h}$ admits a compatible almost complex structure $\widetilde{\mathcal{J}}$ given by

$$\widetilde{\mathcal{J}} := \begin{cases} -\mathcal{J} & \text{on } V \\ \mathcal{J} & \text{on } H \end{cases}$$

The next proposition is proved in [Nag02].

Proposition 3.4. *[Nag02] The manifold $(Z, \widetilde{\mathcal{J}}, \widetilde{h})$ is nearly Kähler.*

□

As a corollary, we obtain

Theorem 3. *The twistor space Z of a Riemannian manifold with a parallel even Clifford structure of rank $r \geq 3$ and $\mathrm{Ric} > 0$, admits an almost complex structure $\widetilde{\mathcal{J}}$ and a metric $\widetilde{h}$ such that $(Z, \widetilde{\mathcal{J}}, \widetilde{h})$ is nearly Kähler.*

□

In our case, using the definition of the almost complex structure, one can easily check that this almost complex structure is never integrable.

We conclude by pointing out that even though a classification of parallel even Clifford structures was given in [MS11], and one can try to deal with each of these cases separately, our approach does not rely on this classification (except for dimension 8 in which the curvature condition is not automatically satisfied). Furthermore, the constructions above can be studied in a more general context. One could check integrability conditions of these twistor spaces for manifolds with non parallel even Clifford structures, as in [Fri01]. In fact, in order for the twistor space to be complex, Lemma 3.3 should be satisfied for every S in the twistor space. One nice example is given by $\mathbb{S}^1 \times \mathbb{S}^{15}$, which admits a non parallel Cl_9^0 structure but its twistor space is a complex manifold which cannot be Kähler since its first Betti number is odd.

4. Clifford Structures on Conformal Manifolds

The chapter is organised as follows. Section 4.1 recalls several notions of even Clifford structures, defines Clifford-Weyl structures and introduces the required differential and algebraic objects from differential and conformal geometry. It finishes with a toy problem from Hermitian geometry which inspires the beginning of the proof of Theorem 4 (and provides the proof for the case $n > 4$, $r = 2$). Section 4.2 establishes Theorem 4 for large rank $r \geq 5$ structures and Section 4.3 establishes the theorem in the remaining low rank setting. Section 4.4 considers the generic cases of Theorem 5 and shows that in each of these cases there are examples of Clifford-Weyl structures with non-closed associated Weyl structures.

4.1 Clifford-Weyl Structures

Consider a Riemannian manifold (M^n, g) equipped with an even Clifford structure (E, h, φ). Recall that the Clifford morphism $\varphi : \mathrm{Cl}^0(E, h) \to \mathrm{End}(\mathrm{T}M)$ is required to send $\Lambda^2 E \subset \mathrm{Cl}^0(E, h)$ to the bundle of skew-symmetric endomorphisms $\mathrm{End}^-(\mathrm{T}M)$. As $\mathrm{End}^-(\mathrm{T}M)$ is invariant under a conformal change of the metric g, the notion of an even Clifford structure extends directly to the setting of a conformal manifold (M, c). In the setting of a conformal manifold, where no distinguished connection (on the tangent bundle) is present, the condition of parallelism is transferred by considering Weyl connections giving what we term Clifford-Weyl structures.

Definition 4.1. *A rank $r \geq 2$ Clifford-Weyl structure on a conformal manifold (M^n, c), is a tuple $(E, h, \varphi, \nabla^E, D)$ where*

- *(E, h) is an oriented locally defined rank r Euclidean bundle;*
- *$\varphi : \mathrm{Cl}^0(E, h) \to \mathrm{End}(\mathrm{T}M)$ is an algebra bundle morphism sending $\Lambda^2 E$ to $\mathrm{End}^-(\mathrm{T}M)$;*
- *∇^E is a metric connection on E;*
- *D is a Weyl connection on (M, c),*

such that φ, seen as a section of $\mathrm{Cl}^0(E, h)^* \otimes \mathrm{End}(\mathrm{T}M)$, is parallel with respect to $\nabla^E \otimes D$.

Let $(E, h, \varphi, \nabla^E, D)$ be a rank r Clifford-Weyl structure on a conformal manifold (M^n, c) and let L denote the weight bundle of M (the real line bundle associated with the principal bundle of frames via the representation $|\det|^{1/n}$ of $\mathrm{GL}(n; \mathbb{R})$, cf. [BM11, Section 2]).

Consider a metric g in the conformal class c. Associated with g we have the Levi-Civita connection ∇ as well as the gauge ℓ (a section of L) and Lee form θ defined respectively by

$$c = g \otimes \ell^2, \qquad D\ell = \theta \otimes \ell.$$

This gives the useful formula $Dg = -2\theta \otimes g$. Independent of the choice of metric in the conformal class, we have the Faraday form $F = \mathrm{d}\,\theta$. The Weyl structure D is closed if and only if D is locally the Levi-Civita connection of a metric in the conformal class. This is equivalent to $F = 0$.

Let $\{\xi_i\}_{1\le i\le r}$ be a local oriented orthonormal frame for (E, h). We introduce the collection of connection coefficients and the curvature two-forms

$$\nabla^E \xi_j =: \sum_i \eta_{ij} \otimes \xi_i, \quad \mathrm{R}^E\,\xi_j =: \sum_i \omega_{ij} \otimes \xi_i,$$

(with $\omega_{ij} = \mathrm{d}\,\eta_{ij} + \sum_k \eta_{ik} \wedge \eta_{kj}$), and define endomorphisms $J_{ij} := \varphi(\xi_i \cdot \xi_j)$ where $\cdot$ denotes Clifford multiplication. As in the original setting of even Clifford structures, it is the fact that φ is an algebra bundle morphism mapping $\Lambda^2 E$ into $\mathrm{End}^-(\mathrm{T}M)$, that implies that the endomorphisms J_{ij} are locally defined almost Hermitian structures on M for $i \ne j$. Moreover, for mutually distinct indices i, j, k we have

$$(\xi_i \cdot \xi_j) \cdot (\xi_i \cdot \xi_k) = \xi_j \cdot \xi_k = -(\xi_i \cdot \xi_k) \cdot (\xi_i \cdot \xi_j)$$

and thus J_{ij} anticommutes with J_{ik}. In particular, this shows (similar to the final statement in Lemma 2.6 but without the restriction on $r \ne 4$) that

$$\langle J_{ij}, J_{ik}\rangle = 0, \qquad \forall\ i \ne j \ne k \ne i, \tag{4.1}$$

where $\langle\cdot,\cdot\rangle$ denotes, as usual, minus the trace of the product of two endomorphisms.

Let $\{e_i\}_{1\le i\le n}$ denote a local orthonormal frame for $(\mathrm{T}M, g)$. For each J_{ij} with $i \ne j$, we obtain an associated non-degenerate two-form Ω_{ij}:

$$\Omega_{ij}(\cdot,\cdot) := g(J_{ij}\,\cdot,\cdot).$$

(In the process of establishing (4.3), as well in the the rank $r = 4$ case, the calculations are simplified by summing indiscriminately over subscripts, for this we define $\Omega_{ii} := 0$.) Using the natural scalar product on the bundle of exterior forms induced by g, we obtain the Lefschetz-type operators for $i \ne j$,

$$\mathrm{L}_{ij} := \Omega_{ij}\wedge, \quad \Lambda_{ij} := \mathrm{L}^*_{ij} = \tfrac{1}{2}\sum_{a=1}^{n} J_{ij}(e_a) \lrcorner\, e_a \lrcorner\,.$$

For later use, notice that by the usual identification using the metric g of $\Lambda^2(\mathrm{T}^*M)$ with $\mathrm{End}^-(\mathrm{T}M)$, the Lefschetz operator Λ_{ij} acting on $\Lambda^2(\mathrm{T}^*M)$ (with $i \ne j$) is identified with $\frac{1}{2}\langle J_{ij},\cdot\rangle$ acting on $\mathrm{End}^-(\mathrm{T}M)$.

We finish this section with a toy problem: That when $r = 2$ and $n > 4$, the Weyl connection is closed. This is a standard fact in Hermitian geometry, but the proof below contains, at embryonic state, the main ideas of the proof of Theorem 4 which for convenience we now recall.

Theorem 4. *Suppose a conformal manifold of dimension n carries a rank $r \ge 2$ Clifford-Weyl structure such that (n, r) is different from $(2, 2)$, $(4, 2)$, $(4, 3)$, $(4, 4)$ and $(8, 8)$. Then the associated Weyl connection is closed. The same conclusion holds if $(n, r) = (8, 4)$, provided that the restriction of the Clifford morphism φ to $\Lambda^2 E$ is not injective.*

Proof. Proposition 4.2 for $r = 2$ and $n > 4$. Section 4.2 for $r \ge 5$. Section 4.3 for $r = 3, 4$. □

Proposition 4.2. *Suppose a conformal manifold of dimension $n > 4$ carries a Clifford-Weyl structure of rank $r = 2$. Then the associated Weyl connection is closed.*

Proof. With the notation established in this section, we choose as before a metric g in the conformal class and drop the superfluous subscripts on J_{12} and Ω_{12}. First, the fact that φ is $\nabla^E \otimes D$-parallel implies

$$DJ = D(\varphi(\xi_1 \cdot \xi_2)) = \varphi(\nabla^E(\xi_1 \cdot \xi_2))$$

and as

$$\nabla^E(\xi_1 \cdot \xi_2) = (\eta_{21} \otimes \xi_2) \cdot \xi_2 + \xi_1 \cdot (\eta_{12} \otimes \xi_1) = -(\eta_{12} + \eta_{21}) 1_{\mathrm{Cl}^0(E,h)} = 0,$$

we conclude J is parallel with respect to D. Differentiating Ω with respect to the Weyl connection

$$(D\Omega)(\cdot,\cdot) = (Dg)(J\cdot,\cdot) + g(DJ\cdot,\cdot)$$

and using the formulae $DJ = 0$ and $Dg = -2\theta \otimes g$ gives $D\Omega = -2\theta \otimes \Omega$ which upon extracting the totally antisymmetric part yields

$$\mathrm{d}\,\Omega = -2\theta \wedge \Omega.$$

Differentiating this equation gives

$$0 = \mathrm{d}^2\,\Omega = -2F \wedge \Omega - 4\theta \wedge \theta \wedge \Omega$$

and as Ω is non-degenerate with $n > 4$, the equation $F \wedge \Omega = 0$ forces $F = 0$. □

4.2 Large Rank Clifford-Weyl Structures

For this section suppose that $(E, h, \varphi, \nabla^E, D)$ is a rank r Clifford-Weyl structure on a conformal manifold (M^n, c), with $r \geq 5$. The structure of Cl^0_r forces the dimension of the manifold to be a multiple of 8. Apart from the generic situation $n = 8$, $r = 8$, which will be treated later on, we will show that the Weyl connection is closed.

As in the previous section, we consider an arbitrary Riemannian metric g in the conformal class c. Then (E, h) is an even Clifford structure, and we may build Lefschetz-type operators as well as identify $\Lambda^2(\mathrm{T}^*M)$ with $\mathrm{End}^-(\mathrm{T}M)$.

The connection coefficients give for all $i \neq j$

$$\begin{aligned}\nabla^E(\xi_i \cdot \xi_j) &= \sum_k \eta_{ki} \otimes (\xi_k \cdot \xi_j) + \eta_{kj} \otimes (\xi_i \cdot \xi_k) \\ &= \sum_{k \neq i,j} \eta_{ki} \otimes (\xi_k \cdot \xi_j) - \eta_{kj} \otimes (\xi_k \cdot \xi_i)\end{aligned}$$

and as the even Clifford structure is parallel,

$$DJ_{ij} = \sum_{k \neq i,j} \eta_{ki} \otimes J_{kj} - \eta_{kj} \otimes J_{ki}.$$

Differentiating Ω_{ij} with respect to the Weyl connection

$$(D\Omega_{ij})(\cdot,\cdot) = (Dg)(J_{ij}\cdot,\cdot) + g(DJ_{ij}\cdot,\cdot)$$

and using the previous formula as well as the fundamental formula $Dg = -2\theta \otimes g$ provides

$$D\Omega_{ij} = -2\theta \otimes \Omega_{ij} + \sum_{k\neq i,j} \eta_{ki} \otimes \Omega_{kj} - \eta_{kj} \otimes \Omega_{ki}.$$

Taking the totally antisymmetric part of this equation (and recalling $\Omega_{ii} := 0$) gives

$$\mathrm{d}\,\Omega_{ij} = -2\theta \wedge \Omega_{ij} + \sum_{k} \eta_{ki} \wedge \Omega_{kj} - \eta_{kj} \wedge \Omega_{ki}.$$

Differentiating this equation and replacing appearances of $\mathrm{d}\,\Omega_{ij}$ (as well as $\mathrm{d}\,\Omega_{kj}$ and $\mathrm{d}\,\Omega_{ki}$) using this same equation yields

$$\begin{aligned} 2F \wedge \Omega_{ij} &= \sum_k \Big(2\theta \wedge \eta_{ki} \wedge \Omega_{kj} + \mathrm{d}\,\eta_{ki} \wedge \Omega_{kj} - \eta_{ki} \wedge \mathrm{d}\,\Omega_{kj}\Big) - \{i \leftrightarrow j\} \\ &= \sum_k \Big(\mathrm{d}\,\eta_{ki} \wedge \Omega_{kj} - \eta_{ki} \wedge \sum_\ell \Big(\eta_{\ell k} \wedge \Omega_{\ell j} - \eta_{\ell j} \wedge \Omega_{\ell k}\Big)\Big) - \{i \leftrightarrow j\} \\ &= \sum_k \Big(\mathrm{d}\,\eta_{ki} + \sum_\ell \eta_{k\ell} \wedge \eta_{\ell i}\Big) \wedge \Omega_{kj} - \{i \leftrightarrow j\} \end{aligned}$$

where $\{i \leftrightarrow j\}$ corresponds to the previously displayed term with indices i and j interchanged. The previous equation simplifies upon introducing the curvature two-forms ω_{ij} of ∇^E into

$$2F \wedge \Omega_{ij} = \sum_k \omega_{ki} \wedge \Omega_{kj} - \omega_{kj} \wedge \Omega_{ki}.$$

Writing this using the Lefschetz-type operators establishes, for all $i \neq j$,

$$2\,\mathrm{L}_{ij}\,F = \sum_k \mathrm{L}_{kj}\,\omega_{ki} - \mathrm{L}_{ki}\,\omega_{kj}. \tag{4.2}$$

Assuming $i \neq j$, we apply Λ_{ij} to (4.2). Calculating Λ_{ij} applied to the left hand side of (4.2) is aided by the $\mathfrak{sl}(2)$ structure of the Lefschetz-type operators. Specifically $2[\Lambda_{ij}, \mathrm{L}_{ij}]F = (n-4)F$ and $\mathrm{L}_{ij}(2\,\Lambda_{ij}\,F)$ identifies, as an endomorphism, with $\langle J_{ij}, F\rangle J_{ij}$. In order to calculate Λ_{ij} applied to the right hand side of (4.2), we note that the sum in (4.2) may be taken over $k \neq i, j$ and we write

$$\Lambda_{ij} = \tfrac{1}{2} \sum_a J_{ij}(e_a) \lrcorner\, e_a \lrcorner$$

(recall that $\{e_i\}_{1\leq i\leq n}$ denotes a local orthonormal frame for $(\mathrm{T}M, g)$). This gives, for $k \neq i, j$,

$$2\,\Lambda_{ij}\,\mathrm{L}_{kj}\,\omega_{ki} = \sum_a J_{ij}(e_a) \lrcorner \Big(\Omega_{kj}(e_a) \wedge \omega_{ki} + \Omega_{kj} \wedge \omega_{ki}(e_a)\Big)$$

The summands in the previous display consist of four terms, the first of which vanishes because of

(4.1). Developing the remaining three terms gives

$$2\,\Lambda_{ij}\,\mathrm{L}_{kj}\,\omega_{ki} = \sum_a -\Omega_{kj}(e_a)\wedge(\omega_{ki}\circ J_{ij})(e_a) + (\Omega_{kj}\circ J_{ij})(e_a)\wedge\omega_{ki}(e_a) + \langle J_{ij},\omega_{ki}\rangle\Omega_{kj}.$$

Testing against two tangent vectors shows

$$\sum_a -\Omega_{kj}(e_a)\wedge(\omega_{ki}\circ J_{ij})(e_a) = \omega_{ki}(J_{ik}\cdot,\cdot) + \omega_{ki}(\cdot,J_{ik}\cdot)$$
$$\sum_a (\Omega_{kj}\circ J_{ij})(e_a)\wedge\omega_{ki}(e_a) = \omega_{ki}(J_{ik}\cdot,\cdot) + \omega_{ki}(\cdot,J_{ik}\cdot)$$

which, viewed as endomorphisms via the metric, are each precisely $[J_{ki},\omega_{ki}]$. Therefore, for i,j,k distinct,

$$\Lambda_{ij}\,\mathrm{L}_{kj}\,\omega_{ki} = [J_{ki},\omega_{ki}] + \tfrac{1}{2}\langle J_{ij},\omega_{ki}\rangle\Omega_{kj}$$

which establishes, for $i\neq j$,

$$(n-4)F + \langle J_{ij},F\rangle\Omega_{ij} = \sum_{k\neq i,j}[J_{ki},\omega_{ki}] + [J_{kj},\omega_{kj}] + \tfrac{1}{2}\langle J_{ij},\omega_{ki}\rangle\Omega_{kj} - \tfrac{1}{2}\langle J_{ij},\omega_{kj}\rangle\Omega_{ki}. \tag{4.3}$$

Working with (4.3), we apply $2\,\Lambda_{ij}, 2\,\Lambda_{ia}, 2\,\Lambda_{ab}$ for a,b different from i,j. For $2\,\Lambda_{ij}$ applied to (4.3) we remark that $\langle J_{ij},J_{ij}\rangle = n$ while J_{ij} is orthogonal to J_{kj} and J_{ki} for i,j,k distinct hence

$$\begin{aligned}(2n-4)\langle J_{ij},F\rangle &= \sum_{k\neq i,j}\langle J_{ij},[J_{ki},\omega_{ki}]\rangle + \langle J_{ij},[J_{kj},\omega_{kj}]\rangle\\ &= \sum_{k\neq i,j}\langle [J_{ij},J_{ki}],\omega_{ki}\rangle + \langle [J_{ij},J_{kj}],\omega_{kj}\rangle\end{aligned}$$

establishing

$$(2n-4)\langle J_{ij},F\rangle = 2\sum_{k\neq i,j}\langle J_{kj},\omega_{ki}\rangle - \langle J_{ki},\omega_{kj}\rangle, \tag{4.4}$$

For $2\,\Lambda_{ia}$ applied to (4.3), we remark that $\langle J_{ia},J_{ki}\rangle = -n\delta_{ak}$ for $k\neq i,j$ and importantly, as $r\geq 5$, the terms involving $\langle J_{ia},J_{kj}\rangle$ vanish by Lemma 2.6. Therefore

$$(n-4)\langle J_{ia},F\rangle = \tfrac{n}{2}\langle J_{ij},\omega_{aj}\rangle + \sum_{k\neq i,j}\langle [J_{ia},J_{ki}],\omega_{ki}\rangle + \langle [J_{ia},J_{kj}],\omega_{kj}\rangle$$

where

$$\sum_{k\neq i,j}\langle [J_{ia},J_{ki}],\omega_{ki}\rangle = \sum_{k\neq i,j}\langle J_{ka}-J_{ak},\omega_{ki}\rangle = 2\sum_{k\neq j}\langle J_{ka},\omega_{ki}\rangle$$

and since J_{ia} commutes with J_{kj} for $k\neq i,j$ except when $k=a$,

$$\sum_{k\neq i,j}\langle [J_{ia},J_{kj}],\omega_{kj}\rangle = \langle [J_{ia},J_{aj}],\omega_{aj}\rangle = -2\langle J_{ij},\omega_{aj}\rangle$$

establishing

$$(n-4)\langle J_{ia}, F\rangle = (\tfrac{n}{2}-2)\langle J_{ij}, \omega_{aj}\rangle + 2\sum_{k\neq j}\langle J_{ka}, \omega_{ki}\rangle, \tag{4.5}$$

For $2\,\Lambda_{ab}$ applied to (4.3), we again use the large rank hypothesis $r \geq 5$. Indeed, due to this condition, for $k \neq i, j$, terms involving $\langle J_{ab}, J_{kj}\rangle$ and $\langle J_{ab}, J_{ki}\rangle$ vanish. So

$$\begin{aligned}(n-4)\langle J_{ab}, F\rangle &= \sum_{k\neq i,j} \langle [J_{ab}, J_{ki}], \omega_{ki}\rangle + \langle [J_{ab}, J_{kj}], \omega_{kj}\rangle \\ &= \sum_{k\in\{a,b\}} \langle [J_{ab}, J_{ki}], \omega_{ki}\rangle + \langle [J_{ab}, J_{kj}], \omega_{kj}\rangle\end{aligned}$$

and developing the four terms from the summation in the preceding display establishes

$$(n-4)\langle J_{ab}, F\rangle = 2\langle J_{bi}, \omega_{ai}\rangle - 2\langle J_{ai}, \omega_{bi}\rangle + 2\langle J_{bj}, \omega_{aj}\rangle - 2\langle J_{aj}, \omega_{bj}\rangle. \tag{4.6}$$

Armed with the preceding numbered equations, we may establish the orthogonality between J_{ij} and F. From (4.6), by collecting the first two terms, and collecting the second two terms, we see that $\langle J_{bi}, \omega_{ai}\rangle - \langle J_{ai}, \omega_{bi}\rangle$ is independent of i so

$$(n-4)\langle J_{ab}, F\rangle = 4\langle J_{bi}, \omega_{ai}\rangle - 4\langle J_{ai}, \omega_{bi}\rangle$$

Summing the previous display over $i \neq a, b$ and changing the notation of indices $a, b, i \to i, j, k$ gives

$$(r-2)(n-4)\langle J_{ij}, F\rangle = 4\sum_{k\neq i,j} \langle J_{kj}, \omega_{ki}\rangle - \langle J_{ki}, \omega_{kj}\rangle.$$

Comparing this equation with (4.4) provides the constraint

$$(r-2)(n-4)\langle J_{ij}, F\rangle = 2(2n-4)\langle J_{ij}, F\rangle$$

whence $\langle J_{ij}, F\rangle = 0$ unless $4(2n-4) = 2(r-2)(n-4)$. As $r \geq 5$ and n is a multiple of 8, the only obstructive case is the generic case $n = 8$, $r = 8$, which was excluded. Therefore

$$\langle J_{ij}, F\rangle = 0 \qquad \forall\ i \neq j.$$

Updating (4.5) and (4.6) using this orthogonality, we obtain a pair symmetry from (4.6)

$$\langle J_{ia}, \omega_{ja}\rangle = \langle J_{ja}, \omega_{ia}\rangle, \qquad \forall\ i, j, a \text{ distinct}. \tag{4.7}$$

which, upon switching the variables j, a in (4.5), provides

$$(2-\tfrac{n}{2})\langle J_{ia}, \omega_{ja}\rangle = -2\langle J_{aj}, \omega_{ai}\rangle + 2\sum_{k}\langle J_{kj}, \omega_{ki}\rangle,$$

giving

$$(2-\tfrac{n}{4})\langle J_{ia}, \omega_{ja}\rangle = \sum_{k}\langle J_{ik}, \omega_{jk}\rangle, \qquad \forall\ i, j, a \text{ distinct}. \tag{4.8}$$

Therefore if $n = 8$, the sum on the right hand side vanishes, while if $n \neq 8$, $\langle J_{ia}, \omega_{ja}\rangle$ is independent of $a \neq i, j$ hence

$$(2 - \tfrac{n}{4})\langle J_{ia}, \omega_{ja}\rangle = (r-2)\langle J_{ia}, \omega_{ja}\rangle$$

and so J_{ia} is orthogonal to ω_{ja}. It thus turns out that the sum $\sum_k \langle J_{ik}, \omega_{jk}\rangle$ vanishes no matter what the dimension n is.

As a penultimate result, we remark that upon summation over $j \neq i$ (for i fixed), the final two terms of (4.3) vanish:

$$\sum_{j\neq i}\left(\sum_{k\neq i,j}\langle J_{ij}, \omega_{ki}\rangle J_{kj} - \langle J_{ij}, \omega_{kj}\rangle J_{ki}\right) = 0. \tag{4.9}$$

In fact, the previous display naturally splits into two collections of summations, each collection vanishing independently as we now show. The first collection of summations in (4.9) may be written as the sum over j, k both different from i and from each-other:

$$\sum_{j\neq i}\left(\sum_{k\neq i,j}\langle J_{ij}, \omega_{ki}\rangle J_{kj}\right) = \sum_{\substack{j,k\neq i\\ j\neq k}}\langle J_{ij}, \omega_{ki}\rangle J_{kj}$$

which thus vanishes as $\langle J_{ij}, \omega_{ki}\rangle$ is symmetric in j, k due to (4.7) while J_{jk} is antisymmetric in j, k. Considering the second collection of summations in (4.9), we rearrange the summation,

$$\begin{aligned}\sum_{j\neq i}\left(\sum_{k\neq i,j}\langle J_{ij}, \omega_{kj}\rangle J_{ki}\right) &= \sum_{j\neq i}\left(\sum_{k\neq i}\langle J_{ij}, \omega_{kj}\rangle J_{ki}\right)\\ &= \sum_{k\neq i}\left(\sum_{j\neq i}\langle J_{ij}, \omega_{kj}\rangle\right) J_{ki}\\ &= \sum_{k\neq i,j}\left(\sum_{j}\langle J_{ij}, \omega_{kj}\rangle\right) J_{ki}\end{aligned}$$

and by the remark following (4.8), the preceding display vanishes and provides (4.9).

We may now establish the result. By defining $S_i := \sum_k [J_{ki}, \omega_{ki}]$, (4.3) now reads

$$(n-4)F = S_i + S_j - 2[J_{ij}, \omega_{ij}] + \tfrac{1}{2}\sum_{k\neq i,j}\langle J_{ij}, \omega_{ki}\rangle J_{kj} - \langle J_{ij}, \omega_{kj}\rangle J_{ki}.$$

Keeping i fixed and summing over $j \neq i$, making use of (4.9), we obtain

$$\begin{aligned}(r-1)(n-4)F &= \sum_{j\neq i}(S_i + S_j - 2[J_{ij}, \omega_{ij}])\\ &= (r-4)S_i + \sum_j S_j\end{aligned}$$

which implies (as $r \neq 4$) that S_i is independent of i and proportional to F:

$$(r-1)(n-4)F = 2(r-2)S_i.$$

Equation (4.3) thus develops to

$$\frac{2(n-4)}{r-2}F = 4[J_{ij}, \omega_{ij}] - \sum_{k\neq i,j} \langle J_{ij}, \omega_{ki}\rangle J_{kj} - \langle J_{ij}, \omega_{kj}\rangle J_{ki}.$$

Commuting F with J_{ij} we see that

$$\frac{2(n-4)}{r-2}FJ_{ij} = 4(J_{ij}\omega_{ij}J_{ij} + \omega_{ij}) - \sum_{k\neq i,j} \langle J_{ij}, \omega_{ki}\rangle J_{ki} + \langle J_{ij}, \omega_{kj}\rangle J_{kj},$$
$$\frac{2(n-4)}{r-2}J_{ij}F = -4(\omega_{ij} + J_{ij}\omega_{ij}J_{ij}) + \sum_{k\neq i,j} \langle J_{ij}, \omega_{ki}\rangle J_{ki} + \langle J_{ij}, \omega_{kj}\rangle J_{kj}.$$

Therefore F anticommutes with J_{ij} for every $i \neq j$. By taking some k different from both i and j we get that F commutes with $J_{ik}J_{jk} = J_{ij}$. Hence $F = 0$, thus proving Theorem 4 when the rank of the Clifford-Weyl structure is at least 5.

4.3 Low Rank Clifford-Weyl Structures

We consider now the remaining cases from Theorem 4.

That D is closed for $r = 3$ and $n \geq 8$ is a standard result in quaternion Hermitian Weyl geometry (or locally conformally quaternion Kähler geometry) [Orn01]. We present a proof which can also be adapted to the case $r = 4$.

Define $\Omega := \Omega_{12}^2 + \Omega_{23}^2 + \Omega_{31}^2$ to be the fundamental four-form (or Kraines form) of quaternion Hermitian geometry. By (4.2), which continues to hold for $r = 3$, we obtain

$$\Omega_{12}^2 \wedge F = \tfrac{1}{2}\Omega_{12} \wedge \Omega_{13} \wedge \omega_{32} - \tfrac{1}{2}\Omega_{12} \wedge \Omega_{23} \wedge \omega_{31}$$

and cyclically commuting $(1, 2, 3)$ gives two similar equations. Upon summation, cancellations give $\Omega \wedge F = 0$ and, as the fundamental four-form is well-known to be non-degenerate (and $n \neq 4$), $F = 0$. (Alternatively, if one follows the derivation of (4.2), one obtains similar equations for $\mathrm{d}\,\Omega_{ij}$ in terms of the connection coefficients η_{ij} which result in the equation $\mathrm{d}\,\Omega = -4\theta \wedge \Omega$. Differentiating a second time gives $\Omega \wedge F = 0$.)

Finally, if $(E, h, \nabla^E, \varphi)$ is a rank 4 Clifford-Weyl structure and $n \geq 8$, let us consider $A \in \mathrm{End}(\mathrm{T}M)$ to be the image under φ of the volume element of E. From the properties of φ, A is a symmetric involution, hence the tangent bundle splits into a direct sum $\mathrm{T}M = T^+ \oplus T^-$ of the ± 1 eigenspaces of A. If either T^+ or T^- are of dimension zero, then the rank 4 even Clifford structure is effectively a rank 3 even Clifford structure and the result follows from the previous paragraph. We may thus assume the decomposition of $\mathrm{T}M$ is non-trivial. In particular φ is injective upon restriction to $\Lambda^2 E$, so we only need to consider the case $n \geq 12$.

We construct quaternionic structures on $T^\pm$

$$J_{12}^\pm := \mp\tfrac{1}{2}(J_{14} \pm J_{23}), \quad J_{31}^\pm := \mp\tfrac{1}{2}(J_{13} \mp J_{24}), \quad J_{23}^\pm := \mp\tfrac{1}{2}(J_{12} \pm J_{34})$$

which vanish upon restriction to $T^{\mp}$. We may thus define two four-forms $\Omega_\pm \in \Lambda^4(T^\pm)^*$ as in the case of quaternion Hermitian geometry and set $\Omega = \Omega_+ + \Omega_-$. We decompose the exterior algebra $\Lambda^* M = \oplus(\Lambda^p(T^+)^* \oplus \Lambda^q(T^-)^*)$ and say that elements of $\Lambda^p(T^+)^* \oplus \Lambda^q(T^-)^*$ are of type (p,q). The decomposition of $\Lambda^2 M$ enables us to write $F = F_+ + F_m + F_-$ where F_+, F_m, F_- are respectively of type $(2,0)$, $(1,1)$, $(0,2)$. Using this we calculate $\Omega \wedge F$ (whose 6 pieces are of distinct type). Meanwhile, we remark that

$$\Omega = \tfrac{1}{2}\sum_{i<j} \Omega_{ij}^2 = \tfrac{1}{4}\sum_{i,j} \Omega_{ij}^2$$

and via (4.2), which continues to hold for $r = 4$,

$$\Omega_{ij}^2 \wedge F = \tfrac{1}{2}\sum_k \Omega_{ij} \wedge \Omega_{ik} \wedge \omega_{kj} - \Omega_{ij} \wedge \Omega_{jk} \wedge \omega_{ki}$$

Summing over i,j the second term in the previous sum, $\sum_{i,j,k} \Omega_{ij} \wedge \Omega_{jk} \wedge \omega_{ki}$ may be written, under a permutation $i,j,k \to k,i,j$, as the first term $\sum_{i,j,k} \Omega_{ij} \wedge \Omega_{ik} \wedge \omega_{kj}$ hence

$$\Omega \wedge F = 0.$$

Since they have different types, each of the six terms in the expansion of $\Omega \wedge F$ also individually vanish. As M is at least 12-dimensional with both subbundles $T^\pm$ being non-trivial, we deduce from $\Omega_- \wedge F_+ = 0$ and $\Omega_+ \wedge F_- = 0$ that $F_\pm = 0$. And as one of the subbundles $T^\pm$ has rank larger than 4 (say T^+) then $F_m = 0$ (from $\Omega_+ \wedge F_m = 0$).

This finishes the proof of Theorem 4 when the rank of the Clifford-Weyl structure is 3 or 4.

4.4 Generic Cases

In this final section we prove Theorem 5 and, in the process, show examples of Clifford-Weyl structures with non-closed associated Weyl covariant derivatives.

Theorem 5. *(i) Let D be a Weyl structure on an oriented conformal manifold (M,c) of dimension 2, 4 or 8. Then (M,c) carries a Clifford-Weyl structure of rank $r = 2$ for $n = 2$, $r = 3$ or $r = 4$ for $n = 4$ and $r = 8$ for $n = 8$, whose associated Weyl structure is D.*

(ii) Let D be a Weyl structure on a conformal manifold (M^n, c). Then there exists a Clifford-Weyl structure of rank 2 on (M,c) with associated Weyl structure D if and only if D preserves a complex structure compatible with c. If $n = 4$, every complex structure J compatible with c is preserved by a unique Weyl structure D^J, which is closed if and only if J is locally conformally Kähler.

(iii) Let D be a Weyl structure on a conformal manifold (M^8, c). Then there exists a Clifford-Weyl structure of rank 4 whose Clifford morphism $\varphi : \mathrm{Cl}^0(E,h) \to \mathrm{End}(\mathrm{T}M)$ is injective upon restriction to $\Lambda^2 E$, if and only if D is the adapted Weyl structure of a conformal product structure on (M,c) with 4-dimensional factors (cf. [BM11]).

Proof. (i) If M has dimension 2, we define (E,h) to be the trivial rank 2 Euclidean vector bundle with trivial flat connection ∇^E, and $\varphi : \Lambda^2 E \to \mathrm{End}^-(\mathrm{T}M)$ by $\varphi(\xi_1 \wedge \xi_2) := J$, where ξ_1, ξ_2 is an oriented orthonormal frame of E and J is the rotation in $\mathrm{T}M$ by $\pi/2$ in the positive direction determined by c. Since $DJ = 0$, $(E, h, \varphi, \nabla^E, D)$ is a rank 2 Clifford-Weyl structure.

If M has dimension 4, we define $E := \Lambda^+ M \otimes L^2$ (the bundle of self-dual two-forms of conformal weight 0), ∇^E to be the covariant derivative induced by D on E and h to be the canonical scalar product induced by c on E. Since $\Lambda^2 M \otimes L^2$ is canonically isomorphic to $\mathrm{End}^-(\mathrm{T}M)$, E is in fact a rank 3 sub-bundle of the bundle of skew-symmetric endomorphisms of M. Moreover, since E is oriented, the metric h provides an identification of $\Lambda^2 E$ with E, and thus a map $\varphi : \Lambda^2 E \to \mathrm{End}^-(\mathrm{T}M)$. It is straightforward to check that this map extends to an algebra morphism from $\mathrm{Cl}^0(E,h)$ to $\mathrm{End}(\mathrm{T}M)$ which is tautologically parallel with respect to $\nabla^E \otimes D$, thus defining a rank 3 Clifford-Weyl structure.

Moreover, every rank 3 Clifford-Weyl structure $(E, h, \nabla^E, \varphi, D)$ determines in a tautological way a rank 4 Clifford-Weyl structure $(\tilde{E}, \tilde{h}, \nabla^{\tilde{E}}, \tilde{\varphi}, D)$ where $\tilde{E} = E \oplus \mathbb{R}$ with induced metric $\tilde{h}$ and connection $\nabla^{\tilde{E}}$, and $\tilde{\varphi}$ is defined on $\Lambda^2 \tilde{E} \simeq \Lambda^2 E \oplus E$ by $\tilde{\varphi} = \varphi$ on $\Lambda^2 E$ and $\tilde{\varphi} = \varphi \circ *$ on E, where $*$ denotes the Hodge isomorphism $* : E \to \Lambda^2 E$.

If M has dimension 8, we define $E = \Sigma_0^+ M$ (the bundle of real half-spinors of conformal weight 0, cf. [BHM+15]) and ∇^E and h to be the covariant derivative and the scalar product induced on E by D and c. Of course, if M is not spin, E is only locally defined, but $\Lambda^2 E$ is always globally defined. We consider the map $\varphi : \Lambda^2 E \to \mathrm{End}^-(\mathrm{T}M)$ defined by

$$\varphi(\psi \wedge \phi) := X \mapsto -\sum_{i=1}^{8} h(\ell^{-2} e_i \cdot X \cdot \psi, \phi)\, e_i - h(\psi, \phi)\, X,$$

where ℓ is a local section of L and $\{e_i\}_{1 \le i \le n}$ is a local frame of $\mathrm{T}M$ satisfying $c(e_i, e_j) = \ell^2 \delta_{ij}$. The map φ is tautologically parallel with respect to $\nabla^E \otimes D$. Moreover, φ extends to an algebra morphism from $\mathrm{Cl}^0(E,h)$ to $\mathrm{End}(\mathrm{T}M)$. Indeed, in order to check the universality property for the even Clifford algebra [MS11, Lemma A.1], consider local sections ψ, ϕ and ξ of E such that ψ is orthogonal to ϕ and ξ and $h(\psi, \psi) = 1$. Then $\{\ell^{-1} e_i \cdot \psi\}_{1 \le i \le n}$ is a local orthonormal basis of the zero-weight half-spin bundle $\Sigma_0^- M$ (whose metric is also denoted by h) and thus

$$\begin{aligned}
[\varphi(\psi \wedge \phi) \circ \varphi(\psi \wedge \xi)](X) &= -\sum_{i=1}^{8} h(\ell^{-2} e_i \cdot X \cdot \psi, \xi)\, \varphi(\psi \wedge \phi)(e_i) \\
&= \sum_{i,j=1}^{8} h(\ell^{-2} e_i \cdot X \cdot \psi, \xi) h(\ell^{-2} e_j \cdot e_i \cdot \psi, \phi)\, e_j \\
&= \sum_{i,j=1}^{8} h(\ell^{-2} X \cdot e_i \cdot \psi + 2\ell^{-2} c(e_i, X)\psi, \xi) h(\ell^{-2} e_i \cdot \psi, e_j \cdot \phi)\, e_j \\
&= -\sum_{j=1}^{8} h(\ell^{-2} X \cdot \xi, e_j \cdot \phi)\, e_j \\
&= -\varphi(\xi \wedge \phi)(X) - h(\phi, \xi)\, X \\
&= \varphi(\phi \wedge \xi)(X) - h(\phi, \xi)\, X.
\end{aligned}$$

This shows that $(E, h, \varphi, \nabla^E, D)$ is a rank 8 Clifford-Weyl structure on M.

(*ii*) With any Clifford-Weyl structure of rank 2 on M one can associate the image, J, of the volume form of E through the Clifford morphism φ. Clearly J is an almost complex structure on M compatible with c and D-parallel. On the other hand, every almost complex structure preserved by a torsion-free connection is integrable.

Conversely, if D is a Weyl structure on (M, c) and J is a D-parallel Hermitian structure, we define as before a rank 2 Clifford-Weyl structure on M by taking (E, h) to be the trivial rank 2 Euclidean vector bundle with trivial flat connection ∇^E, and $\varphi : \Lambda^2 E \to \mathrm{End}^-(\mathrm{T}M)$ defined by the fact that it maps the volume form of E onto J.

For the second point, recall that on 4-dimensional conformal manifolds, every complex structure J compatible with the conformal structure is preserved by a unique Weyl covariant derivative D (see e.g. the proof of [BM11, Lemma 5.7]) which is closed if and only if (J, c) is locally conformally Kähler.

(iii) If $(E, h, \varphi, \nabla^E, D)$ is a Clifford-Weyl structure of rank 4 with φ injective on $\Lambda^2 E$ on an 8-dimensional conformal manifold (M, c), then the image of the volume form of E through φ is a D-parallel involution of $\mathrm{T}M$ whose eigenbundles are 4-dimensional D-parallel distributions. By [BM11, Theorem 4.3], (M, c) has a conformal product structure defined by these two distributions.

Conversely, every conformal product structure on (M, c) with 4-dimensional distributions $T^\pm$ defines a unique Weyl connexion D (called the adapted Weyl structure in [BM11, Definition 4.4]) such that the splitting $\mathrm{T}M = T^+ \oplus T^-$ is D-parallel. We obtain in this way a structure group reduction from $\mathrm{CO}(8)$ to $G := \mathrm{CO}(8) \cap (\mathrm{CO}(4) \times \mathrm{CO}(4))$, that is, a G-principal bundle P over M and a connection induced by D on P. Since $\mathrm{CO}(4) = \mathbb{R}^* \times (\mathrm{Spin}(3) \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Spin}(3))$, the projections from $\mathbb{R}^* \times (\mathrm{Spin}(3) \times \mathrm{Spin}(3))$ to the second and third factors respectively define group morphisms i_l and i_r from $\mathrm{CO}(4)$ to $\mathrm{SO}(3)$. Let E denote the (locally defined) rank 4 Euclidean vector bundle over M associated with P via the group morphism $i_l \times i_r$ from G to $\mathrm{SO}(3) \times \mathrm{SO}(3) = \mathrm{PSO}(4)$, and let ∇^E denote the induced covariant derivative. By construction, $\Lambda^2 E$ is globally defined, and isomorphic to the weightless bundle $(\Lambda^+(T^+) \otimes L^{-2}) \oplus (\Lambda^-(T^-) \otimes L^{-2})$. The composition of this isomorphism with the canonical inclusion in $\Lambda^+(\mathrm{T}M) \otimes L^{-2} = \mathrm{End}^-(\mathrm{T}M)$ yields as before a Clifford morphism, which is $\nabla^E \otimes D$-parallel by naturality of the construction.

Examples of conformal products with non-closed adapted Weyl structures can be easily constructed. Take (M_1, g_1) and (M_2, g_2) two 4-dimensional Riemannian manifolds and let $M = M_1 \times M_2$ with conformal class $c = [e^f g_1 + g_2]$ where f is any smooth map on M. Then the adapted Weyl structure of this conformal product structure is closed if and only if there exist functions f_i on M_i such that $f = \pi_1^*(f_1) + \pi_2^*(f_2)$ where $\pi_i : M \to M_i$ are the canonical projections (see [BM11, Section 6.1] for details). □

5. Asymptotically Hyperbolic Manifolds

This chapter is structured as follows. Section 5.1 recalls the model geometry of hyperbolic space as a submanifold in Minkowski space. It is worth mentioning as it provides a clear geometric motivation for the construction of the ambient space as well as the Minkowski and Euclidean scales. Section 5.2 introduces properly asymptotically hyperbolic spaces. Finally, Section 5.3 describes the construction of the two (families of) differential operators $\mathcal{Q}_\lambda$ and $\mathcal{P}_\lambda$ when considering functions. This section does not make reference to any ambient operators however it is useful to have the primary calculations of [Vas13b] recorded in a notation consistent with this thesis.

5.1 Model Geometry

Let $\mathbb{R}^{1,n+1}$ be Minkowski space with the Lorentzian metric

$$\eta := -dx_0^2 + \sum_{i=1}^{n+1} dx_i^2$$

and set M_e to be Minkowski space minus the closure of the backward light cone. The metric gives the Minkowski distance function, denoted η^2, on $\mathbb{R}^{1,n+1}$ from the origin:

$$\eta^2(x) := -x_0^2 + \sum_{i=1}^{n+1} x_i^2.$$

Hyperbolic space $X = \mathbb{H}^{n+1}$ is then identified with the (connected) hypersurface

$$X := \left\{ x \in \mathbb{R}^{1,n+1} \;\middle|\; \eta^2(x) = -1, x_0 > 0 \right\}$$

and is given the metric g induced by the restriction of η. The boundary at infinity of hyperbolic space, i.e. the sphere $Y = \mathbb{S}^n$, is identified with the (connected) submanifold

$$Y := \left\{ x \in \mathbb{R}^{1,n+1} \;\middle|\; \eta^2(x) = 0, x_0 = 1 \right\}$$

which, as an aside, inherits the standard metric, denoted h, by restriction of η. For completeness we introduce de Sitter space dS^{n+1} as the hypersurface

$$dS^{n+1} := \left\{ x \in \mathbb{R}^{1,n+1} \;\middle|\; \eta^2(x) = 1 \right\}.$$

We define the forward light cone

$$M := \left\{ x \in \mathbb{R}^{1,n+1} \;\middle|\; \eta^2(x) < 0, x_0 > 0 \right\}$$

and note the decomposition $M = \mathbb{R}_s^+ \times X$ via the identification

$$\mathbb{R}_s^+ \times X \ni (s, x) \mapsto s \cdot x \in X.$$

In these coordinates, the metric η restricted to M takes the form

$$\eta = -ds \otimes ds + s^2 g$$

and we refer to s as the Minkowski scale. We define X_e to be the subset of the $(n+1)$-sphere contained in M_e

$$X_e := \left\{ x \in \mathbb{R}^{1,n+1} \;\middle|\; \sum_{i=0}^{n+1} x_i^2 = 1, x_0 > \tfrac{-1}{\sqrt{2}} \right\}$$

and note that the ambient space M_e is diffeomorphic to $\mathbb{R}_t^+ \times X_e$ via the identification

$$\mathbb{R}_t^+ \times X_e \ni (t, x) \mapsto t \cdot x \in M_e.$$

We refer to t as the Euclidean scale. The dilations induced by the Euclidean scale allow the following identification

$$X_e \simeq X \sqcup Y \sqcup dS^{n+1}.$$

5.2 Asymptotically Hyperbolic Manifolds

We now properly introduce the geometric structure of even asymptotically hyperbolic spaces. Let (X, g) be a Riemannian manifold of dimension $n+1$ which is even asymptotically hyperbolic [Gui05, Definition 1.2] with boundary at infinity denoted Y. We recall the definition of evenness.

Definition 5.1. *Let (X, g) be an asymptotically hyperbolic manifold. We say that g is even if there exists a boundary defining function ρ and a family of tensors $(h_{2i})_{i\in\mathbb{N}_0}$ on $Y = \partial\overline{X}$ such, for all N, one has the following decomposition of g near Y*

$$\phi^*(\rho^2 g) = dr^2 + \sum_{i=0}^{N} h_{2i} r^{2i} + O(r^{2N+2})$$

where ϕ is the diffeomorphism induced by the flow ϕ_r of the gradient $\mathrm{grad}_{\rho^2 g}(\rho)$:

$$\phi : \begin{cases} [0,1) \times Y & \to \quad \phi([0,1) \times Y) \subset \overline{X} \\ (r, y) & \mapsto \quad \phi_r(y) \end{cases}$$

We define $X^2 := (\overline{X} \sqcup \overline{X})/Y$ to be the topological double of $\overline{X}$. (For a slicker definition, we stray ever so slightly from the model geometry.) From the diffeomorphism ϕ we initially construct a C^∞ atlas on X^2 by noting that $Y \subset X^2$ is contained in an open set $U^2 := (U_- \sqcup U_+)/Y$ with

$U_\pm := \phi([0,1)\times Y)$ and we declare this set to be C^∞ diffeomorphic to $(-1,1)\times Y$ via

$$\begin{array}{rcl} (-1,1)\times Y & \simeq & U^2 \\ (t,y) & \mapsto & \begin{cases} \phi_{-t}(y)\in U_-, & \text{if } t\le 0 \\ \phi_{+t}(y)\in U_+, & \text{if } t\ge 0 \end{cases} \end{array}$$

Charts on the interior of X in $\overline{X}$ complete the atlas on X^2.

We want to consider the boundary defining function ρ as a function from X^2 to $[-1,1]$ such that X may be identified with $\{\rho>0\}$. Using the previous chart for $U^2\simeq(-1,1)\times Y$ we initially set

$$\rho:\begin{cases} (-1,1)\times Y & \to & (-1,1) \\ (r,y) & \mapsto & r \end{cases}$$

and extend ρ to a continuous function on X^2 by demanding that ρ be constant on $X^2\backslash U^2$. In order to ensure smoothness at $\partial\overline{U^2}$ we deform ρ smoothly on the two subsets $(-1,-1+\varepsilon)\times Y$ and $(1-\varepsilon,1)\times Y$ of U^2. This achieves our goal. We now define the function μ on X^2 by declaring

$$\mu : X^2\to[-1,1]:\begin{cases} \mu=-\rho^2, & \text{if } \rho\le 0 \\ \mu=\rho^2 & \text{if } \rho\ge 0 \end{cases}$$

Remark 5.2. Although we have performed a deformation of ρ near $\partial\overline{U^2}$ we will continue to think of ρ and μ as coordinates for the first factor of $U^2=(-1,1)\times Y$ (if we wanted to be correct, in what follows we would replace $(-1,1)$ with $(-1+\varepsilon,1-\varepsilon)$ but this is cumbersome and we prefer to free up the variable ε). Of course, only the coordinates (μ,y) provide a smooth chart for X^2 near Y.

We now weaken the atlas on X^2 near Y. By the previous remark, we may think of μ as coordinates for the first factor of U^2 and we thus demand that the C^∞ atlas is with respect to this coordinate rather than ρ (as was the case for the initial atlas). It is now the case that on X^2, only μ (and not ρ) is a smooth function.

Smooth functions

We define the set $C^\infty_{\text{even}}(\overline{X})$ to be the subset of functions in $C^\infty(X)$ which are extensible to $C^\infty(X^2)$ and whose extension is invariant with respect to the natural involution on X^2. (For example, the restriction of μ to X. However such an invariant extension would of course not give the function μ previously constructed due to a sign discrepency.) We remark that $\dot{C}^\infty(X)$, the subset of functions in $C^\infty(\overline{X})$ which vanish to all orders at Y, injects naturally into $C^\infty(X^2)$ and may be identified with the subset of $C^\infty(X^2)$ whose elements vanish on $\{\rho<0\}$.

Such constructions may also readily be extended to the setting of vector bundles above X by using a local basis near Y of such a vector bundle which smoothly extends across Y.

Definition 5.3. *We denote by X_e the following extension of X*

$$X_e := \{\mu>-1\}\subset X^2,$$

by S the hypersurface $\{\mu=-\frac{1}{2}\}\subset X_e$, and by X_{cs} the open submanifold $\{\mu>-\frac{1}{2}\}\subset X_e$ such that $\partial\overline{X_{cs}}=S$.

We construct two product manifolds $M = \mathbb{R}^+_s \times X$ and $M_e = \mathbb{R}^+_t \times X_e$. We supply M with the Lorentzian cone metric

$$\eta := -ds \otimes ds + s^2 g$$

and explain how this structure may be smoothly extended to M_e.

Using the even neighbourhod at infinity $U := (0,1)_\mu \times Y$ we remark that, on $\mathbb{R}^+_s \times U$, the Lorentzian metric takes the form

$$\eta = -ds \otimes ds + s^2 \left(\frac{d\mu \otimes d\mu}{4\mu^2} + \frac{h}{\mu} \right)$$

where h has a smooth Taylor expansion about $\mu = 0$ by the evenness hypothesis. Upon the change of variables $t = s/\rho$ with $t \in \mathbb{R}^+$, the metric, on $\mathbb{R}^+_t \times U$ takes the form

$$\eta = -\mu dt \otimes dt - \tfrac{1}{2} t (d\mu \otimes dt + dt \otimes d\mu) + t^2 h$$

or, in a slightly more attractive convention,

$$t^{-2}\eta = -\tfrac{\mu}{2}(\tfrac{dt}{t})^2 - \tfrac{1}{2}\tfrac{dt}{t} \cdot d\mu + h \tag{5.1}$$

From this display we see that, by extending h to a family of Riemannian metrics on Y parametrised smoothly by $\mu \in (-1,1)$, we can extend η smoothly onto the chart $\mathbb{R}^+_t \times U^2 \subset M_e$. We do this, thus furnishing M_e with a Lorentzian metric. As in the model geometry we refer to s (which is only defined on M) as the Minkowski scale, and to t (which is defined on M_e) as the Euclidean scale.

From (5.1), the measure associated with $t^{-2}\eta$ on U^2 is $\frac{dt}{t}\, dx$ where $dx = \frac{1}{2} d\mu\, d\mathrm{vol}_h$. On U, we have $dx = \rho^{n+2} d\mathrm{vol}_g$, hence dx extends smoothly to a measure on X_e, also denoted dx, and which agrees with $d\mathrm{vol}_g$ on $X \backslash U$.

A comment on notation in the literature

The idea of considering an ambient space is present in several works. As such, there are several differing notations scattered throughout the literature. We note below two such notations (up to constant factors).

The ambient metric approach of Fefferman and Graham to conformal geometry has notations established in [FG12]. The microlocal approach to resonances of Vasy has notations established in [Vas17]. The in vogue style of using ρ to denote a boundary defining function has lead to the following notational compromise for this thesis:

	Thesis	[FG12]	[Vas17]
Conformal boundary defining function	ρ	r	x
Projective boundary defining function	μ	ρ	μ
Lorentzian scale	s	s	τ
Euclidean scale	t	t	ρ
Riemannian metric	g	g	g
Ambient metric	η	$\widetilde{g}$	η

5.3 Vasy's Construction for Functions

With the notation established from the previous section, denote by Δ the positive Laplacian acting on functions on the asymptotically even hyperbolic manifold (X, g). Define the operator

$$\mathcal{Q}_\lambda := \Delta - \tfrac{n^2}{4} + \lambda^2 \in \mathrm{Diff}^2(X)$$

for $\lambda \in \mathbb{C}$. We restrict our attention to the (non-even) collar neighbourhood near infinity $(0,1)_\rho \times Y$. On this coordinate patch, with local coordinates $\{y^i\}_{1\leq i\leq n}$ for Y, we introduce the endomorphism $\rho^{-2}B \in C^\infty_{\mathrm{even}}(X; \mathrm{End}(\mathrm{T}^*Y))$ by setting $B dy^i := \sum_{jk} \frac{1}{2}(h^{-1})^{ij}(\rho\partial_\rho h_{jk}) dy^k$ so that $\mathrm{tr}_h B = \sum_{ij} \frac{1}{2}(h^{-1})^{ij}(\rho\partial_\rho h_{ij}) \in \rho^2 C^\infty_{\mathrm{even}}(X)$. On this neighbourhood, the operator $\mathcal{Q}_\lambda$ takes the following form

$$\mathcal{Q}_\lambda = -(\rho\partial_\rho)^2 + (n - \mathrm{tr}_h B)\rho\partial_\rho + \rho^2\Delta_h - \tfrac{n^2}{4} + \lambda^2$$

with Δ_h the Laplacian of (Y, h) (dependent on the parameter ρ). Conjugating this operator by $\rho^{\lambda+\frac{n}{2}}$ provides

$$\begin{aligned}
\rho^{-\lambda-\frac{n}{2}}\, \mathcal{Q}_\lambda\, \rho^{\lambda+\frac{n}{2}} &= -(\rho\partial_\rho + \lambda + \tfrac{n}{2})^2 + (n - \mathrm{tr}_h B)(\rho\partial_\rho + \lambda + \tfrac{n}{2}) + \rho^2\Delta_h - \tfrac{n^2}{4} + \lambda^2 \\
&= -(\rho\partial_\rho)^2 - (\mathrm{tr}_h B + 2\lambda)\rho\partial_\rho + \rho^2\Delta_h - \lambda\, \mathrm{tr}_h B.
\end{aligned}$$

Introducing the projective boundary defining function $\mu = \rho^2$ whence $2\mu\partial_\mu = \rho\partial_\rho$, this conjugated operator takes the form

$$\mu^{-\frac{\lambda}{2}-\frac{n}{4}}\, \mathcal{Q}_\lambda\, \mu^{\frac{\lambda}{2}+\frac{n}{4}} = -4(\mu\partial_\mu)^2 - 2(\mathrm{tr}_h B + 2\lambda)\mu\partial_\mu + \mu\Delta_h - \lambda\, \mathrm{tr}_h B$$

and it is from this display that we may divide out (from the left) a factor of μ providing the operator $\mathcal{P}_\lambda$. It takes the form

$$\mathcal{P}_\lambda := \mu^{-\frac{\lambda}{2}-\frac{n}{4}-1}\, \mathcal{Q}_\lambda\, \mu^{\frac{\lambda}{2}+\frac{n}{4}} = -4\mu\partial_\mu^2 - 4\lambda\partial_\mu + \Delta_h + \mathcal{A}_\lambda$$

where the final term is explicitly

$$\mathcal{A}_\lambda := -4\partial_\mu - 2\,\mathrm{tr}_h B\partial_\mu - \lambda\mu^{-1}\,\mathrm{tr}_h B$$

and this final term is (evenly) smooth on $\overline{X}$ because $\mathrm{tr}_h B \in \mu C^\infty_{\mathrm{even}}(X)$.

The even structure of $\mathcal{P}_\lambda$ allows it to be extended to an operator on X_e using the extension of the metric h described in the previous section. The operator is no longer everywhere an elliptic operator. Indeed, on the extension $X_e \backslash \overline{X}$, it becomes a hyperbolic operator. Of course, this is only the beginning of a long story; the necessary PDE analysis for this operator is performed in [Vas13b] while the extension of such analysis pertinent to the setting of this thesis is performed in Section 6.5.

6. Quantum Resonances for Symmetric Tensors

The chapter is structured as follows. Section 6.1 recalls the aspects of symmetric tensors as detailed in Appendix A and establishes several relationships between symmetric tensors when working relative to the Lorentzian and Euclidean scales. Section 6.2 recalls standard notions from microlocal analysis and explicits several notions from the b-calculus framework adapted to vector bundles. Section 6.3 contains the bulk of the calculations of this chapter, relating $\mathbf{Q}$ and $\mathcal{Q}$ with the Lichnerowicz Laplacian. Sections 6.4 and 6.5 introduce the operators $\mathbf{P}$ and $\mathcal{P}$ and provide the desired meromorphic inverse. Section 6.6 establishes the four theorems. Section 6.7 details the particular case of symmetric cotensors of rank $m = 2$. It is useful to gain insight into this problem via this low rank setting, and it is hoped that the presentation of this case will aid the reader particularly during Sections 6.3 and 6.6. Finally, Section 6.8 announces the high energy estimates one would obtain if the microlocal analysis performed in Section 6.5 was performed using semiclassical notions.

6.1 Symmetric Tensors

This section recalls the necessary algebraic aspects of symmetric tensors as detailed in Appendix A and considers also a Lorentzian analogue.

A single fibre

Let E be a vector space of dimension $n + 1$ equipped with an inner product g and let $\{e_i\}_{0\leq i\leq n}$ be an orthonormal basis for E. In this chapter we will distinguish between vectors and covectors; let $\{e^i\}_{0\leq i\leq n}$ be the corresponding duall basis for E^*. We denote by $\mathrm{Sym}^k E^*$ the k-fold symmetric tensor product of E^*. The symmetric product $\cdot$ provides a map from $\mathrm{Sym}^k E^* \times \mathrm{Sym}^{k'} E^*$ to $\mathrm{Sym}^{k+k'} E^*$. The inner product takes the form $g = \frac{1}{2}\sum_{i=0}^{n} e^i \cdot e^i$. The inner product induces an inner product on $\mathrm{Sym}^k E^*$ which in this chapter will be denoted $\langle\cdot,\cdot\rangle$. For $u \in E^*$, the metric adjoint of the linear map $u\cdot : \mathrm{Sym}^k E^* \to \mathrm{Sym}^{k+1} E^*$ is the contraction $u \lrcorner : \mathrm{Sym}^{k+1} E^* \to \mathrm{Sym}^k E^*$. Contraction and multiplication with the metric g define two additional linear maps $\Lambda : \mathrm{Sym}^k E^* \to \mathrm{Sym}^{k-2} E^*$ and $\mathrm{L} : \mathrm{Sym}^k E^* \to \mathrm{Sym}^{k+2} E^*$. which are adjoint to each other. They are referred to as Lefschetz-type operators. Recall the notation $\mathscr{A}^k$ for sequences (whose terms are integers between 0 and n) of length k. We set $e^K := e^{\tilde{k}_1} \cdot \ldots \cdot e^{\tilde{k}_k} \in \otimes^k E^*$ for $K = \tilde{k}_1 \ldots \tilde{k}_k \in \mathscr{A}^k$.

Let F be the vector space $\mathbb{R} \times E$ equipped with the standard Lorentzian inner product $-f \otimes f + g$ where f is the canonical vector in $\mathbb{R}^*$. The previous constructions have obvious counterparts on F which will not be detailed. (For this subsection, we write $\langle\cdot,\cdot\rangle_F$ for the Lorentzian inner product on

$\mathrm{Sym}^m F^*$.) The decomposition of F provides a decomposition of $\mathrm{Sym}^m F^*$:

$$\mathrm{Sym}^m F^* = \bigoplus_{k=0}^{m} a_k\, f^{m-k} \cdot \mathrm{Sym}^k E^*, \qquad a_k := \tfrac{1}{\sqrt{(m-k)!}}$$

and we write

$$u = \sum_{k=0}^{m} a_k\, f^{m-k} \cdot u^{(k)}, \qquad u \in \mathrm{Sym}^m F^*, u^{(k)} \in \mathrm{Sym}^k E^*.$$

The choice of the normalising constant a_k is chosen so that $\langle u, v\rangle_F = \sum_{k=0}^{m}(-1)^{m-k}\langle u^{(k)}, v^{(k)}\rangle$. There is a simple relationship between the terms $u^{(k)}$ in this decomposition of u when u is trace-free.

Lemma 6.1. *Let Λ_F and Λ denote the Lefschetz-type trace operators obtained from the inner products on F and E respectively. For $u \in \mathrm{Sym}^m F^*$ in the kernel of Λ_F, we have*

$$\Lambda\, u^{(k)} = -b_{k-2} b_{k-1} u^{(k-2)}$$

where $u = \sum_{k=0}^{m} a_k f^{m-k} \cdot u^{(k)}$ for $u^{(k)} \in \mathrm{Sym}^k E^*$ and constants $b_k := \sqrt{m-k}$.

Proof. Beginning with $\Lambda_F\, f^{m-k} = (m-k)(m-k-1) f^{m-k-2}$ we obtain

$$\Lambda_F \left(a_k\, f^{m-k} \cdot u^{(k)} \right) = a_{k+2} \sqrt{(m-k)(m-k-1)} f^{m-k-2} \cdot u^{(k)} + a_k\, f^{m-k} \cdot \Lambda\, u^{(k)}.$$

Therefore, as $u \in \ker \Lambda_F$, equating powers of f in the resulting formula for

$$\Lambda_F \left(\sum_{k=0}^{m} a_k\, f^{m-k} \cdot u^{(k)} \right)$$

gives

$$a_k f^{m-k}\, \Lambda\, u^{(k)} + a_k \sqrt{(m-k+2)(m-k+1)} f^{m-k} u^{(k-2)} = 0.$$ □

Vector bundles

These constructions are naturally extended to vector bundles above manifolds. We include this subsection in order to announce our notations and conventions. Consider M and X (with similar constructions for M_e and X_e). We denote

$$\mathcal{F} := \mathrm{Sym}^m \mathrm{T}^* M, \qquad \mathcal{E}^{(k)} := \mathrm{Sym}^k \mathrm{T}^* X, \qquad \mathcal{E} := \oplus_{k=0}^{m} \mathcal{E}^{(k)}.$$

If we want to make precise that $\mathcal{F}$ consists of rank m symmetric cotensors, we will write $\mathcal{F}^{(m)}$. The Minkowski scale gives the decomposition $M = \mathbb{R}_s^+ \times X$ and we denote by π the projection onto the second factor $\pi : M \to X$. (Remark that on M this gives the same map as the projection $\pi : M_e \to X_e$ using the Euclidean scale $M_e = \mathbb{R}_t^+ \times X_e$.) This enables $\mathcal{E}^{(k)}$ to be pulled back to a bundle over M which we will also denote by $\mathcal{E}^{(k)}$.

Given $u \in C^\infty(M; \mathcal{F})$, we decompose u in the following way

$$u = \sum_{k=0}^{m} a_k \left(\tfrac{ds}{s}\right)^{m-k} \cdot u^{(k)}, \qquad u^{(k)} \in C^\infty(M; \mathcal{E}^{(k)}) \tag{6.1}$$

where a_k is the previously introduced constant $((m-k)!)^{-1/2}$. We say that such a decomposition is relative to the Minkowski scale.

For a fixed value of s, say s_0, there is an identification of the corresponding subset of M with X via the map $\pi_{|s=s_0}$. We will thus reuse π for the following map

$$\pi_{s=s_0} : \begin{cases} C^\infty(M;\mathcal{F}) & \to & C^\infty(X;\mathcal{E}) \\ u = \sum_{k=0}^m a_k\,(\frac{ds}{s})^{m-k}\cdot u^{(k)} & \mapsto & \sum_{k=0}^m \pi_{|s=s_0} u^{(k)} \end{cases}$$

and in order to map from $C^\infty(X;\mathcal{E})$ to $C^\infty(M;\mathcal{F})$, taking into account the Minkowski scale, we introduce

$$\pi_s^* : \begin{cases} C^\infty(X;\mathcal{E}) & \to & C^\infty(M;\mathcal{F}) \\ u = \sum_{k=0}^m u^{(k)} & \mapsto & \sum_{k=0}^m a_k\,(\frac{ds}{s})^{m-k}\cdot \pi^* u^{(k)} \end{cases}$$

On M we have two useful metrics. First, $s^{-2}\eta$ which takes the model form of the metric on F introduced in the previous subsection

$$s^{-2}\eta = -\tfrac{1}{2}(\tfrac{ds}{s})^2 + g.$$

Second, we have the metric η which is geometrically advantageous as it gives the Lorentzian cone metric on M. Notationally we will distinguish the two constructions by decorating the Lefschetz-type operators with a subscript of the particular metric used. A similar decoration will be used for the two inner products on $\mathcal{F}$. There are two useful relationships. First,

$$\Lambda_{s^{-2}\eta} u = s^4\,\Lambda_\eta\, u, \qquad u \in \mathcal{F} \tag{6.2}$$

and second,

$$\langle u, v\rangle_{s^{-2}\eta} = s^{2m}\langle u, v\rangle_\eta, \qquad u, v \in \mathcal{F} \tag{6.3}$$

On X, when the metric g is used, no such decoration will be added. We can however make use of the metric $s^{-2}\eta$ by appealing to π_s^*. We introduce $\langle\cdot,\cdot\rangle_s$ on $C^\infty(X;\mathcal{E})$ by declaring

$$\langle u, v\rangle_s := \langle \pi_s^* u, \pi_s^* v\rangle_{s^{-2}\eta}, \qquad u, v \in C^\infty(X;\mathcal{E}).$$

Note that such a definition does not depend on the value of $s \in \mathbb{R}^+$ at which point the inner product on $\mathcal{F}$ is applied. With this inner product given, and the measure $d\mathrm{vol}_g$ previously introduced, we obtain the notion of L^2 sections and define

$$L^2_s(X;\mathcal{E}) := L^2(X, d\mathrm{vol}_g; \mathcal{E}, \langle\cdot,\cdot\rangle_s)$$

whose inner product is provided by

$$(u, v)_s := \int_X \langle u, v\rangle_s\, d\mathrm{vol}_g, \qquad u, v \in C_c^\infty(X;\mathcal{E}).$$

On X_e, we define L^2 sections with respect to the measure dx,

$$L^2_t(X_e;\mathcal{E}) := L^2(X_e, dx; \mathcal{E}, \langle\cdot,\cdot\rangle_t).$$

On X, the necessary correspondences between the constructions using the Lorentzian and Euclidean scales are given in the following lemma.

Lemma 6.2. *There exists* $J \in C^\infty(X; \operatorname{End} \mathcal{E})$ *such that*

$$\pi_s^* u = \pi_t^* J u, \qquad u \in C^\infty(X; \mathcal{E})$$

whose entries are homogeneous polynomials of degree at most m *in* $\frac{d\rho}{\rho}$*, upper triangular in the sense that* $J(\mathcal{E}^{(k_0)}) \subset \oplus_{k=k_0}^m \mathcal{E}^{(k)}$*, and whose diagonal entries are the identity. Moreover,*

$$\langle u, v \rangle_s = \rho^{2m} \langle Ju, Jv \rangle_t, \qquad u, v \in C^\infty(X; \mathcal{E}).$$

Finally,

$$L_s^2(X; \mathcal{E}) = \rho^{\frac{n}{2}-m+1} J^{-1} L_t^2(X; \mathcal{E}).$$

Proof. As $t = s/\rho$, the differentials are related by

$$\tfrac{ds}{s} = \tfrac{dt}{t} + \tfrac{d\rho}{\rho}$$

hence by the binomial expansion

$$a_k (\tfrac{ds}{s})^{m-k} \cdot \pi^* u^{(k)} = \sum_{j=0}^{m-k} a_{k+j} (\tfrac{dt}{t})^{m-k-j} \cdot \tbinom{m-k}{j} \tfrac{a_k}{a_{k+j}} (\tfrac{d\rho}{\rho})^j \cdot \pi^* u^{(k)}.$$

where $u^{(k)} \in C^\infty(X; \mathcal{E}^{(k)})$. This defines the endomorphism J by declaring

$$J u^{(k)} = \sum_{j=0}^{m-k} \tbinom{m-k}{j} \tfrac{a_k}{a_{k+j}} (\tfrac{d\rho}{\rho})^j \cdot u^{(k)}.$$

The second claim is direct from $s^{-2}\eta = \rho^{-2} t^{-2} \eta$, hence on $\mathcal{F}$, where the inner product requires m applications of the inverse metric, $\langle \cdot, \cdot \rangle_{s^{-2}\eta} = \rho^{2m} \langle \cdot, \cdot \rangle_{t^{-2}\eta}$. The final claim follows from the second claim and the previously remarked correspondence, $dx = \rho^{n+2} d\mathrm{vol}_g$. ☐

Smooth tensors which are even

We explicit the notion of even sections $C^\infty_{\text{even}}(\overline{X}; \mathcal{E}^{(m)})$. Such sections may be characterised as smooth sections of $\mathrm{Sym}^m \mathrm{T}^* X_e$ (over X_e) restricted to tensors over X (using the smooth structure defined by μ.) Near the boundary at infinity, we recall the sets $U = (0,1)_\mu \times Y \subset X$ and $U^2 = (-1,1)_\mu \times Y \subset X_e$. A smooth base for $\mathrm{T}^* X_e$ on U^2 is given by $\{d\mu, dy^i\}_{1 \leq i \leq n}$ where $\{y^i\}_{1 \leq i \leq n}$ are local coordinates on Y. Now $d\mu = 2\rho d\rho$ so a given $u \in C^\infty_{\text{even}}(\overline{X}; \mathcal{E}^{(m)})$ may be written, near $\partial \overline{X}$, as

$$u = \sum_{k=0}^m \sum_{K \in \mathscr{A}^k} u_{k,K} (\rho d\rho)^{m-k} \cdot dy^K, \qquad u_{k,K} \in C^\infty_{\text{even}}(\overline{U}).$$

6.2 b-Calculus and Microlocal Analysis

This section introduces the necessary b-calculus formalism on symmetric cotensors. The standard reference is [Mel93], in particular we make much use of Chapters 2 and 5. We also recall some now standard ideas from microlocal analysis.

b-calculus

For convenience we will only work on $M = \mathbb{R}_s^+ \times X$ rather than on both M and M_e. We define $\overline{M}$ to be the closure of M seen as a submanifold of $\mathbb{R}_s \times X$ with its usual topology. Then

$$\overline{M} = M \sqcup X$$

where X is naturally identified with the boundary $\partial\overline{M} = \{s = 0\}$'.

We let $\{e_i\}_{0 \leq i \leq n}$ denote a (local) holonomic frame for $\mathrm{T}X$ and $\{e^i\}_{0 \leq i \leq n}$ its dual frame for T^*X. The Lie algebra of b-vector fields consists of smooth vector fields on $\overline{M}$ tangent to the boundary X. It is thus generated by $\{s\partial_s, e_i\}$. This provides the smooth vector bundle ${}^{\mathrm{b}}\mathrm{T}\overline{M}$. The dual bundle, ${}^{\mathrm{b}}\mathrm{T}^*\overline{M}$, has basis $\{\frac{ds}{s}, e^i\}$. This dual bundle is used to construct the b-symmetric bundle of m-cotensors, denoted ${}^{\mathrm{b}}\mathcal{F}$. On the interior of $\overline{M}$, this bundle is canonically isomorphic to $\mathcal{F}$.

An operator $\mathbf{Q}$ belongs to $\mathrm{Diff}^p_{\mathrm{b}}(\overline{M}; \mathrm{End}\, {}^{\mathrm{b}}\mathcal{F})$ if, relative to a frame generated by $\{\frac{ds}{s}, e^i\}$ the operator $\mathbf{Q}$ may be written as a matrix

$$\mathbf{Q} = [\mathbf{Q}_{i,j}]$$

whose coefficients $\mathbf{Q}_{i,j}$ belong to $\mathrm{Diff}^p_{\mathrm{b}}(\overline{M})$. That is, if $\mathbf{Q}_{i,j}$ may be written

$$\mathbf{Q}_{i,j} = \sum_{k,|\alpha| \leq p} q_{i,j,k,\alpha} (s\partial_s)^k \partial_x^\alpha$$

for smooth functions $q_{i,j,k,\alpha} \in C^\infty(\overline{M})$.

Operators in $\mathrm{Diff}^p_{\mathrm{b}}(\overline{M}; \mathrm{End}\, {}^{\mathrm{b}}\mathcal{F})$ provide indicial families of operators belonging to $\mathrm{Diff}^p(X; \mathrm{End}\, \mathcal{E})$. In order to define this mapping we recall the operator $\pi_{s=s_0}$ defined in the previous section for $s_0 \in \mathbb{R}^+$. This family of maps clearly has an extension to $\overline{M}$ giving

$$\pi_{s=s_0} : C^\infty(\overline{M}; {}^{\mathrm{b}}\mathcal{F}) \to C^\infty(X; \mathcal{E})$$

where $s_0 \in [0, \infty)$. The indicial family mapping (with respect to the Minkowski scale s)

$$\mathrm{I}_s : \mathrm{Diff}^p_{\mathrm{b}}(\overline{M}; \mathrm{End}\, {}^{\mathrm{b}}\mathcal{F}) \to \mathcal{O}(\mathbb{C}; \mathrm{Diff}^p(X; \mathrm{End}\, \mathcal{E})).$$

is defined by

$$\mathrm{I}_s(\mathbf{Q}, \lambda)\,(u) = \pi_{s=0}\left(s^\lambda\, \mathbf{Q}\, s^{-\lambda}\,(\pi_s^* u)\right), \qquad u \in C^\infty(X; \mathcal{E}).$$

When the scale s is understood, we will use the convention of removing the bold font from such an operator and write

$$\mathcal{Q} = \mathrm{I}_s(\mathbf{Q}, \cdot), \qquad \mathcal{Q}_\lambda = \mathrm{I}_s(\mathbf{Q}, \lambda).$$

Remark 6.3. This definition effectively does three things. First, if $\mathbf{Q}$ is written as a matrix, relative to the decomposition established by the Minkowski scale (6.1), then $\mathcal{Q}$ will take the same form but without the appearances of $a_k(\frac{ds}{s})^{m-k}\cdot$. Next, the functions $q_{i,j,k,\alpha}$ are frozen to their values at $s = 0$. (These two results are due to the appearance of $\pi_{s=0}$.) Finally, due to the conjugation by s^λ, all appearances of $s\partial_s$ in $\mathbf{Q}$ are replaced by the complex parameter $-\lambda$.

Remark 6.4. The choice to conjugate by $s^{-\lambda}$ is to ensure that the subsequent operators (in particular $\mathcal{P}$) acting on L^2 sections, have physical domains corresponding to $\operatorname{Re}\lambda \gg 1$. If one is convinced that the convention ought to be conjugation by s^λ rather than $s^{-\lambda}$ one can kill two birds with one stone: Considering the model geometry, which motivates the viewpoint of hyperbolic space "at infinity" inside the forward light cone of compactified Minkowski space, it would be somewhat more natural to introduce the coordinate $\tilde{s} = s^{-1}$ on M, then construct the closure of M as a submanifold of $\mathbb{R}_{\tilde{s}} \times X$. The indicial family would then by constructed via a conjugation of $\tilde{s}^\lambda$ and appearances of $\tilde{s}\partial_{\tilde{s}} = -s\partial_s$ would be replaced by λ. For this chapter, the aesthetics of such a choice are outweighed by the superfluous introduction of two dual variables, one for each of s and t.

The b-operators we consider are somewhat simpler than the previous definition in that the coefficients $q_{i,j,k,\alpha}$ do not depend on s (in the correct basis).

Definition 6.5. *A b-operator $\mathbf{Q} \in \operatorname{Diff}^p_{\mathrm{b}}(\overline{M}; {}^{\mathrm{b}}\mathcal{F})$ is b-trivial if, for all $s_0 \in \mathbb{R}^+$,*

$$\mathrm{I}_s(\mathbf{Q}, \lambda)\,(u) = \pi_{s=s_0}\left(s^\lambda\,\mathbf{Q}\,s^{-\lambda}\,(\pi_s^* u)\right), \qquad u \in C^\infty(X; \mathcal{E}).$$

One advantage of this property is that self-adjointness of $\mathbf{Q}$ easily implies self-adjointness of $\mathcal{Q}_\lambda$ for $\lambda \in i\mathbb{R}$.

Lemma 6.6. *Suppose $\mathbf{Q}$ is b-trivial and formally self-adjoint relative to the inner product*

$$(u, v)_{s^{-2}\eta} = \int_M \langle u, v\rangle_{s^{-2}\eta}\, \tfrac{ds}{s} \mathrm{dvol}_g, \qquad u, v \in C_c^\infty(M; \mathcal{F}).$$

Then, the indicial family $\mathcal{Q}$ is, upon restriction to $\lambda \in i\mathbb{R}$, formally self-adjoint relative to the inner product

$$(u, v)_s = \int_X \langle u, v\rangle_s\, \mathrm{dvol}_g, \qquad u, v \in C_c^\infty(X; \mathcal{E}).$$

Moreover, for all λ, $\mathcal{Q}_\lambda^* = \mathcal{Q}_{-\bar{\lambda}}$.

Proof. We prove only the first claim. That $\mathcal{Q}_\lambda^* = \mathcal{Q}_{-\bar{\lambda}}$ for all λ follows by the same reasoning making the obvious changes in the second display provided below. Let ψ be a smooth function on $\mathbb{R}_s^+$ with compact support (away from $s = 0$) and with unit mass $\int_{\mathbb{R}^+} \psi\, \frac{ds}{s} = 1$. Let $u, v \in C_c^\infty(X; \mathcal{E})$. The b-triviality provides

$$\begin{aligned}(\mathcal{Q}_\lambda\, u, v)_s &= \int_{\mathbb{R}^+} (\mathcal{Q}_\lambda\, u, v)_s\, \psi\, \tfrac{ds}{s} \\ &= (s^\lambda\, \mathbf{Q}\, s^{-\lambda} \pi_s^* u, \psi \pi_s^* v)_{s^{-2}\eta}\end{aligned}$$

For $\lambda \in i\mathbb{R}$ this develops as

$$\begin{aligned}(\mathcal{Q}_\lambda\, u, v)_s &= (\pi_s^* u, s^\lambda\, \mathbf{Q}\, s^{-\lambda} \psi \pi_s^* v)_{s^{-2}\eta} \\ &= (\pi_s^* u, \psi s^\lambda\, \mathbf{Q}\, s^{-\lambda} \pi_s^* v)_{s^{-2}\eta} + (\pi_s^* u, [s^\lambda\, \mathbf{Q}\, s^{-\lambda}, \psi] \pi_s^* v)_{s^{-2}\eta} \\ &= (u, \mathcal{Q}_\lambda\, v)_s + (\pi_s^* u, [s^\lambda\, \mathbf{Q}\, s^{-\lambda}, \psi] \pi_s^* v)_{s^{-2}\eta}\end{aligned}$$

where the last line has again used the b-triviality. Thus we require

$$(\pi_s^* u, [s^\lambda \mathbf{Q}\, s^{-\lambda}, \psi]\pi_s^* v)_{s^{-2}\eta} = 0 \tag{6.4}$$

Consider $\mathbf{Q}$ as a matrix $\mathbf{Q} = [\mathbf{Q}_{i,j}]$ with respect to a basis in which

$$\mathbf{Q}_{i,j} = \sum_{k,|\alpha|\leq p} q_{i,j,k,\alpha}(s\partial_s)^k \partial_x^\alpha$$

for $q_{i,j,k,\alpha} \in C^\infty(X)$. The key is to note that we may write

$$[s^\lambda \mathbf{Q}_{i,j}\, s^{-\lambda}, \psi] = \sum_{k,|\alpha|\leq p-1} \kappa_{i,j,k,\alpha}(s\partial_s)^k \partial_x^\alpha \tag{6.5}$$

for smooth functions (which depend on λ) $\kappa_{i,j,k,\alpha} \in C^\infty(X)$ such that every term in each $\kappa_{i,j,k,\alpha}$ is smoothly divisible by some non-zero integer $s\partial_s$-derivative of ψ. Factoring out these appearances and integrating over $\mathbb{R}^+$ in (6.4) causes, by the fundamental theorem of calculus, the problematic term to vanish. The factorisation claim involving the functions $\kappa_{i,j,k,\alpha}$ follows directly from the following calculation. First

$$\begin{aligned}[s^\lambda \mathbf{Q}_{i,j}\, s^{-\lambda}, \psi] &= \sum_{k,|\alpha|\leq p} q_{i,j,k,\alpha}[(s\partial_s - \lambda)^k \partial_x^\alpha, \psi] \\ &= \sum_{\substack{k,|\alpha|\leq p \\ k\geq 1}} q_{i,j,k,\alpha}[(s\partial_s - \lambda)^k, \psi]\partial_x^\alpha\end{aligned}$$

and for $k > 1$,

$$\begin{aligned}[(s\partial_s - \lambda)^k, \psi] &= \sum_{\ell=1}^{k} \tbinom{k}{\ell}(-\lambda)^{k-\ell}[(s\partial_s)^\ell, \psi] \\ &= \sum_{\ell=1}^{k}\sum_{m=1}^{\ell} \tbinom{k}{\ell}(-\lambda)^{k-\ell}\tbinom{\ell}{m}((s\partial_s)^m\psi)(s\partial_s)^{\ell-m}\end{aligned}$$

which, due to the appearance of $(s\partial_s)^m\psi$ gives (6.5) with the desired structure. □

Remark 6.7. The use of $d\mathrm{vol}_g$ is unimportant, the result holds for any measure on X given such a measure also appears as $d\mathrm{vol}_g$ does in the inner product on M.

We finish this subsection by remarking the effect that the scale (Minkowski or Euclidean) has on the indicial family.

Lemma 6.8. *For $\mathbf{Q} \in \mathrm{Diff}_{\mathrm{b}}^p(\overline{M}; {}^{\mathrm{b}}\mathcal{F})$, the indicial families obtained using the scales s and t are related by*

$$\mathrm{I}_s(\mathbf{Q}, \lambda) = \rho^\lambda J^{-1}\, \mathrm{I}_t(\mathbf{Q}, \lambda) J \rho^{-\lambda}$$

with J presented in Lemma 6.2.

Proof. Lemma 6.2 provides $\pi_s^* = \pi_t^* \circ J$. Dual to this equation, $\pi_{s=0} = J^{-1} \circ \pi_{t=0}$. Combining these observations gives the result

$$\begin{aligned}\mathrm{I}_s(\mathbf{Q},\lambda)\,(u) &= \pi_{s=0}\left(s^{\lambda}\,\mathbf{Q}\,s^{-\lambda}\,(\pi_s^* u)\right)\\ &= J^{-1}\pi_{t=0}\left(\rho^{\lambda}t^{\lambda}\,\mathbf{Q}\,t^{-\lambda}\rho^{-\lambda}\,(\pi_t^* J u)\right)\\ &= \rho^{\lambda}J^{-1}\,\mathrm{I}_t(\mathbf{Q},\lambda)(J\rho^{-\lambda}u).\end{aligned}$$

□

Microlocal analysis

We recall standard objects in microlocal analysis (the necessary information is given in [Zwo16] for pseudodifferential operators acting on the trivial bundle, here we merely indicate the small changes that occur when acting on a vector bundle). Using the open manifold X_{cs}, we will assume that $L^2_t(X_e;\mathcal{E})$ provides a notion of sections above X_{cs} with Sobolev regularity s, denoted $H^s(X_{cs};\mathcal{E})$, with norm $\|\cdot\|_{H^s}$ (see Subsection 6.5 for subtleties arising due to the boundary S) . Let ζ denote the coefficients of a covector relative to some local base for T^*X_{cs} such that we may define the Japanese bracket $\langle\zeta\rangle$. We denote by

$$\Psi^p_{\mathrm{scal}}(X_{cs};\operatorname{End}\mathcal{E}) \subset \Psi^p(X_{cs};\operatorname{End}\mathcal{E})$$

the space of properly supported pseudo-differential operators of order p acting on $\mathcal{E}$ and which have scalar principal symbol. For $A \in \Psi^a_{\mathrm{scal}}(X_{cs};\operatorname{End}\mathcal{E})$ such a symbol is written

$$\sigma(A) \in S^a(\mathrm{T}^*X_{cs}\backslash 0)/S^{a-1}(\mathrm{T}^*X_{cs}\backslash 0;\operatorname{End}\mathcal{E}).$$

For such operators, it continues to hold that, for $B \in \Psi^b_{\mathrm{scal}}(X_{cs};\operatorname{End}\mathcal{E})$,

$$\sigma(AB) = \sigma(A)\sigma(B) \in S^{a+b}(\mathrm{T}^*X_{cs}\backslash 0)/S^{a+b-1}(\mathrm{T}^*X_{cs}\backslash 0;\operatorname{End}\mathcal{E})$$

however now, as lower order terms are not required to be diagonal,

$$\sigma([A,B]) \in S^{a+b-1}(\mathrm{T}^*X_{cs}\backslash 0;\operatorname{End}\mathcal{E})/S^{a+b-2}(\mathrm{T}^*X_{cs}\backslash 0;\operatorname{End}\mathcal{E})$$

In the case that $A \in \Psi^a(X_{cs}) \subset \Psi^a_{\mathrm{scal}}(X_{cs};\operatorname{End}\mathcal{E})$ we get $\sigma(\frac{1}{2i}[A,B]) = \frac{1}{2}H_{\sigma(B)}(\sigma(A))$ where $H_{\sigma(B)}$ is the Hamiltonian vector field associated with $\sigma(B)$. Exactly as in the case that $\mathcal{E}$ is the trivial bundle, associated with the operator A are the notions of the wave front set $\mathrm{WF}(A)$ and the characteristic variety $\mathrm{Char}(A)$.

There are two radial estimates used in the analysis of $\mathcal{P}$ (the family of operators introduced in Section 6.4) in order to prove Proposition 6.35. The analysis is performed in [Vas13a, Section 2.4] for functions with an alternative description given in [DZ16, E.5.2]. We will follow the second approach and translate the results into a (non semiclassical) setting adapted to vector bundles. For this, and to follow closely the referenced works, we introduce [DZ16, Subsection E.1.2] the radially compactified cotangent bundle $\overline{\mathrm{T}}^*X_{cs}$ and projection map $\kappa : \mathrm{T}^*X_{cs}\backslash 0 \to \partial\overline{\mathrm{T}}^*X_{cs}$. Consider $P \in \Psi^p_{\mathrm{scal}}(X_{cs};\operatorname{End}\mathcal{E})$ with real principal symbol $\sigma(P)$ and Hamiltonian vector field $H_{\sigma(P)}$. Write P in the following way

$$P = \operatorname{Re}P + i\operatorname{Im}P$$

for

$$\operatorname{Re}P = \frac{P+P^*}{2} \in \Psi^p_{\mathrm{scal}}(X_{cs};\operatorname{End}\mathcal{E}), \qquad \operatorname{Im}P = \frac{P-P^*}{2i} \in \Psi^{p-1}(X_{cs};\operatorname{End}\mathcal{E}).$$

In the sense of [DZ16, Definition E.52], let Γ_+ and Γ_- be a source and a sink of $\sigma(P)$ respectively. Suppose that $\langle\zeta\rangle^{1-p}H_{\sigma(P)}$ vanishes on $\Gamma_\pm$. Then

Lemma 6.9. *Let s satisfy the following threshold condition on Γ_+ that*

$$\langle\zeta\rangle^{1-p}(\sigma(\operatorname{Im} P) + (s + \tfrac{1-p}{2})H_{\sigma(P)}\log\langle\zeta\rangle) \quad \text{is negative definite.}$$

Then for all $B_1 \in \Psi^0(X_{cs})$ with $\mathrm{WF}(I - B_1) \cap \Gamma_+ = \varnothing$, there exists $A \in \Psi^0(X_{cs})$ with $\mathrm{Char}(A) \cap \Gamma_+ = \varnothing$ such that for any $u \in C_c^\infty(X_{cs};\mathcal{E})$ (and any N large enough)

$$\|Au\|_{H^s} \leq C(\|B_1Pu\|_{H^{s-p+1}} + \|u\|_{H^{-N}}).$$

Lemma 6.10. *Let s satisfy the following threshold condition on Γ_-*

$$\langle\zeta\rangle^{1-p}(\sigma(\operatorname{Im} P) + (s + \tfrac{1-p}{2})H_{\sigma(P)}\log\langle\zeta\rangle) \quad \text{is negative definite.}$$

Then for all $B_1 \in \Psi^0(X_{cs})$ with $\mathrm{WF}(I - B_1) \cap \Gamma_- = \varnothing$, there exists $A, B \in \Psi^0(X_{cs})$ with $\mathrm{Char}(A) \cap \Gamma_- = \varnothing$ and $\mathrm{WF}(B) \cap \Gamma_- = \varnothing$ such that for any $u \in C_c^\infty(X_{cs};\mathcal{E})$ (and any N large enough)

$$\|Au\|_{H^s} \leq C(\|Bu\|_{H^s} + \|B_1Pu\|_{H^{s-p+1}} + \|u\|_{H^{-N}}).$$

Remark 6.11. There are two trivial but important points to make. First, a source for P is a sink for $-P$ (and similarly a sink for P is a source for $-P$). Second, we have assumed P has real principal symbol therefore, when considering its adjoint P^*, we have $H_{\sigma(P^*)} = H_{\sigma(P)}$. Less trivially, by approximation [DZ16, Lemma E.47], these results do not need to assume $u \in C_c^\infty(X_{cs};\mathcal{E})$. In Lemma 6.9, if $s > \tilde{s}$ with $\tilde{s}$ satisfying the threshold condition and $u \in H^{\tilde{s}}(X_{cs};\mathcal{E})$ then the inequality holds (on the condition that the right hand side is finite). Similarly in Lemma 6.10, if u is a distribution such that the right hand side of the inequality is well defined, then so too is the left hand side, and the inequality holds.

6.3 The Laplacian, the d'Alembertian and the Operator Q

This section shows the relationship between the Laplacian on (X, g) and the d'Alembertian on (M, η).

We first recall several differential operators as detailed in Appendix A using the Levi-Civita connection ∇ of g extended to all associated vector bundles associated with the principal orthonormal frame bundle. The symmetrisation of the covariant derivative is the symmetric differential

$$\mathrm{d} : C^\infty(X;\mathcal{E}^{(k)}) \to C^\infty(X;\mathcal{E}^{(k+1)})$$

and its formal adjoint is the divergence

$$\delta : C^\infty(X;\mathcal{E}^{(k)}) \to C^\infty(X;\mathcal{E}^{(k-1)}).$$

The rough Laplacian on this space is

$$\nabla^*\nabla : C^\infty(X;\mathcal{E}^{(k)}) \to C^\infty(X;\mathcal{E}^{(k)})$$

where ∇^* is the formal adjoint of $\nabla : C^\infty(X;\mathcal{E}^{(k)}) \to C^\infty(X;\mathrm{T}^*X \otimes \mathcal{E}^{(k)})$. Recall $\nabla^*\nabla = -\operatorname{tr} \circ \nabla \circ \nabla$ where tr is a trace operator on $\mathrm{T}^*X \otimes \mathrm{T}^*X$ obtained from g and extended to $\mathrm{T}^*X \otimes \mathrm{T}^*X \otimes \mathcal{E}^{(k)}$. The

Lichnerowicz Laplacian uses the Riemann curvature tensor R giving the curvature endomorphism

$$q(\mathrm{R}) : \mathcal{E}^{(k)} \to \mathcal{E}^{(k)}$$

The Lichnerowicz Laplacian, hereafter simply referred to as the Laplacian, is

$$\Delta : \begin{cases} C^\infty(X;\mathcal{E}^{(k)}) & \to & C^\infty(X;\mathcal{E}^{(k)}) \\ u & \mapsto & (\nabla^*\nabla + q(\mathrm{R}))u \end{cases}$$

We decompose symmetric k-cotensors using the symmetrised basis elements:

$$u = \sum_{K\in\mathscr{A}^k} u_K e^K, \qquad u \in C^\infty(X;\mathcal{E}^{(k)}), u_K \in C^\infty(X).$$

Useful formulae for the preceding operators thus far introduced are given in

Lemma 6.12. *Let $u \in C^\infty(X;\mathcal{E}^{(k)})$. At a point in X about which $\{e_i\}_{0\le i\le n}$ are normal coordinates and which give dual coordinates $\{e^i\}_{0\le i\le n}$, the trace is*

$$\Lambda\, u = \sum_{K\in\mathscr{A}^k}\sum_{k_r\in K}\sum_{k_p\in\{k_r\to\}K} g^{k_r k_p} u_K e^{\{k_p\to,k_r\to\}K},$$

the symmetric differential is

$$\mathrm{d}\, u = \sum_{K\in\mathscr{A}^k}\sum_{i=0}^{n} e_i u_K e^{\{\to i\}K},$$

the divergence is

$$\delta\, u = -\sum_{K\in\mathscr{A}^k}\sum_{k_r\in K}\sum_{i=0}^{n} g^{ik_r} e_i u_K e^{\{k_r\to\}K},$$

the rough Laplacian is

$$\nabla^*\nabla\, u = \sum_{K\in\mathscr{A}^k}\sum_{i,j=0}^{n}\left(-g^{ij}e_ie_ju_Ke^K + \sum_{k_r\in K}\sum_{\ell=0}^{n} g^{i\ell}u_K(\nabla_{e_\ell}\Gamma^{k_r}_{ij})e^{\{k_r\to j\}K}\right)$$

where the connection coefficients are given locally by $\nabla_{e_i}e^k = -\sum_{j=0}^n \Gamma^k_{ij}e^j$. Finally, (at a point using normal coordinates), the Riemann curvature takes the form

$$\mathrm{R}_{e_i,e_j}\, e^\ell = -\sum_{k=0}^{n}\mathrm{R}_{ij}{}^{\ell}{}_{k}e^k, \quad \mathrm{R}_{ij}{}^{\ell}{}_{k} = \nabla_{e_i}\Gamma^\ell_{jk} - \nabla_{e_j}\Gamma^\ell_{ik}.$$

Similar in vein to the commutations relations of (A.1) we have the following two useful results, the second of which originates from [Lic61, Section 10].

Lemma 6.13. *Let $u \in C^\infty(X,\mathcal{E}^{(k)})$ The Laplacian commutes with the Lefschetz-type trace operator*

$$[\Lambda,\Delta]u = 0$$

and commutes with the divergence under the following conditions

$$[\delta, \Delta]u = 0 \text{ if } \begin{cases} k = 0, 1, \\ k = 2 \text{ and } X \text{ is Ricci parallel,} \\ k \geq 3 \text{ and } X \text{ is locally isomorphic to } \mathbb{H}^{n+1}. \end{cases}$$

Proof. The first result is very standard. As the metric is parallel, the Riemann curvature tensor (acting as a derivation on $\mathcal{E}^{(k)}$) commutes with L hence

$$[\mathrm{L}, q(\mathrm{R})]u = \sum_{i,j=0}^{n} (\mathrm{L}\, e^j \lrcorner\, e^i \lrcorner - e^j \lrcorner\, e^i \lrcorner\, \mathrm{L})\ \mathrm{R}_{e_i, e_j}\, u$$

and developing the second term with the aid of the commutation formula $[e^i \lrcorner, \mathrm{L}] = 2e^i \cdot$ provides

$$\begin{aligned} [\mathrm{L}, q(\mathrm{R})]u &= \sum_{i,j=0}^{n} -2(e^j \cdot e^i \lrcorner + e^j \lrcorner\, e^i \cdot)\ \mathrm{R}_{e_i, e_j}\, u \\ &= \sum_{i,j=0}^{n} -2(e^j \cdot e^i \lrcorner + \delta^{ij} + e^i \cdot e^j \lrcorner)\ \mathrm{R}_{e_i, e_j}\, u \end{aligned}$$

which vanishes due to the skew-symmetry of the Riemann curvature tensor. By duality, $[\Lambda, q(\mathrm{R})] = 0$. Now using the commutation relations (A.1) and the following characterisation of the Laplacian [HMS16, Proposition 6.2]

$$\Delta = \delta\, \mathrm{d} - \mathrm{d}\, \delta + 2q(\mathrm{R})$$

provides the commutation of Λ with Δ.

The second result is more involved as a demonstration via a direct calculation (however as these statements are well known, we only sketch said calculations). For $k = 0, 1$ the Laplacian and divergence agree with Hodge Laplacian and the adjoint of the exterior derivative. We will thus assume X is Ricci parallel (and $k \geq 2$). We break the calculation into two parts studying $[\delta, \nabla^*\nabla]$ and $[\delta, q(\mathrm{R})]$. As usual, we use a frame $\{e_i\}_{0 \leq i \leq n}$ for $\mathrm{T}X$ with dual frame $\{e^i\}_{0 \leq i \leq n}$ and calculate at a point about which the connection coefficients vanish. We act on $u = u_K e^K \in C^\infty(X; \mathcal{E}^{(k)})$. That the Ricci tensor is parallel implies, by the (second) Bianchi identity, $\sum_\ell \nabla_{e_\ell} \mathrm{R}_{ij}{}^\ell{}_k = 0$. This observation is repeatedly used. Also, the Ricci endomorphism may be written $\sum_{i,j} \mathrm{Ric}_i^j\, e^i \otimes e_j$ with $\mathrm{Ric}_i^j = \sum_{k,\ell} g^{k\ell}(\nabla_{e_i} \Gamma_{k\ell}^j - \nabla_{e_k} \Gamma_{\ell i}^j)$.

Consider $[\delta, \nabla^*\nabla]$. Calculating simply $\delta\, \nabla^*\nabla$ gives

$$\begin{aligned} \delta\, \nabla^*\nabla &= -\textstyle\sum_k e^k \lrcorner\, \nabla_{e_k}(-\operatorname{tr} \sum_{i,j} e^i \otimes \nabla_{e_i}(e^j \otimes \nabla_{e_j})) \\ &= \textstyle\sum_{i,j,k} g^{ij} e^k \lrcorner\, \nabla_{e_k} \nabla_{e_i} \nabla_{e_j} - \sum_{i,j,k,\ell} g^{i\ell}(\nabla_{e_k} \Gamma_{i\ell}^j)\, e^k \lrcorner\, \nabla_{e_j} \end{aligned}$$

with a similar calculation for $\nabla^*\nabla\, \delta$. Combining these results and commuting ∇_{e_k} with $\nabla_{e_i} \nabla_{e_j}$ gives

$$\begin{aligned} [\delta, \nabla^*\nabla] &= \textstyle\sum_{i,j,k} g^{ij} e^k \lrcorner [\nabla_{e_k}, \nabla_{e_i} \nabla_{e_j}] - \sum_i (\mathrm{Ric}\, e^i) \lrcorner\, \nabla_{e_i} \\ &= -\textstyle\sum_{i,j,k} g^{ij} e^k \lrcorner \{\nabla_{e_i}, \mathrm{R}_{e_j, e_k}\} - \sum_i (\mathrm{Ric}\, e^i) \lrcorner\, \nabla_{e_i} \end{aligned}$$

where $\{\cdot,\cdot\}$ is the anticommutator. After a tedious calculation, we obtain

$$[\delta,\nabla^*\nabla]u = \sum_i (\mathrm{Ric}\, e^i) \lrcorner \nabla_{e_i} u + 2(\mathrm{R},\nabla,u) \tag{6.6}$$

where (R,∇,u) is shorthand for the unwieldy term

$$(\mathrm{R},\nabla,u) = \sum_{i,j} \sum_{k_r\in K} \sum_{k_p\in\{k_r\to\}K} \mathrm{R}^{ik_rk_p}{}_j (\nabla_{e_i} u_K) e^{\{k_p\to j,k_r\to\}K}.$$

For completeness we outline this calculation

$$\begin{aligned} -\textstyle\sum_{i,j,\ell} g^{ij} e^\ell \lrcorner \{\nabla_{e_i}, \mathrm{R}_{e_j,e_\ell}\}u &= -\textstyle\sum_{i,j,\ell}\sum_{k_r\in K}(\{\nabla_{e_i}, \mathrm{R}_\ell{}^{ik_r}{}_j\}u_K) e^\ell \lrcorner e^{\{k_r\to j\}K} \\ &= -2\textstyle\sum_{i,j,\ell}\sum_{k_r\in K} \mathrm{R}_\ell{}^{ik_r}{}_j (\nabla_{e_i} u_K) e^\ell \lrcorner e^{\{k_r\to j\}K} \end{aligned}$$

where the anticommutator has been removed using $\sum_\ell \nabla_{e_\ell} \mathrm{R}_{ij}{}^\ell{}_k = 0$. Developing the final term in the preceding display gives

$$e^\ell \lrcorner e^{\{k_r\to j\}K} = g^{j\ell} e^{\{k_r\to\}K} + \textstyle\sum_{k_p\in\{k_r\to\}K} g^{k_p\ell} e^{\{k_p\to j,k_r\to\}K}$$

which after a little rearrangement of dummy indices and using the algebraic symmetries of the Riemann curvature tensor gives

$$-\textstyle\sum_{i,j,\ell} g^{ij} e^\ell \lrcorner \{\nabla_{e_i}, \mathrm{R}_{e_j,e_\ell}\}u = 2\textstyle\sum_i (\mathrm{Ric}\, e^i)\lrcorner \nabla_{e_i} u + 2(\mathrm{R},\nabla,u)$$

upon subtraction of $\sum_i (\mathrm{Ric}\, e^i)\lrcorner \nabla_{e_i} u$, this provides (6.6).

Consider $[\delta, q(\mathrm{R})]$. Similar to the previous calculations we obtain

$$\begin{aligned} [\delta, q(\mathrm{R})] &= \textstyle\sum_{i,j,k} -e^k \lrcorner e^j \cdot e^i \lrcorner \nabla_{e_k} \mathrm{R}_{e_i,e_j} + e^j \cdot e^i \lrcorner \mathrm{R}_{e_i,e_j} (e^k \lrcorner \nabla_{e_k}) \\ &= \textstyle\sum_{i,j,k} e^j \cdot e^i \lrcorner e^k \lrcorner [\mathrm{R}_{e_i,e_j}, \nabla_{e_k}] - g^{jk} e^i \lrcorner \nabla_{e_k} \mathrm{R}_{e_i,e_j} + e^j \cdot e^i \lrcorner (\mathrm{R}_{e_i,e_j} e^k) \lrcorner \nabla_{e_k} \end{aligned}$$

After an even more tedious calculation treating each of the three terms in the previous display, we obtain

$$[\delta, q(\mathrm{R})]u = -[\delta,\nabla^*\nabla]u - (\nabla,\mathrm{R},u) \tag{6.7}$$

where (∇,R,u) represents the even more unwieldy term

$$(\nabla,\mathrm{R},u) = \sum_{i,j,\ell} \sum_{\substack{k_r\in K\\ k_p\in\{k_r\to\}K \\ k_s\in\{k_p\to,k_r\to\}K}} g^{\ell k_s} (\nabla_{e_\ell} \mathrm{R}_i{}^{k_rk_p}{}_j) u_K e^{\{k_s\to i,k_p\to j,k_r\to\}K}.$$

Again, we sketch the calculation. One of the three terms is easy to calculate directly giving

$$\textstyle\sum_{i,j,k} e^j \cdot e^i \lrcorner (\mathrm{R}_{e_i,e_j} e^k) \lrcorner \nabla_{e_k} u = -(\mathrm{R},\nabla,u).$$

Another term is also relatively easy, again using the trick that $\sum_\ell \nabla_{e_\ell} \mathrm{R}_{ij}{}^\ell{}_k = 0$,

$$-\textstyle\sum_{i,j,k} g^{jk} e^i \lrcorner \nabla_{e_k} \mathrm{R}_{e_i,e_j} u = -\textstyle\sum_i (\mathrm{Ric}\, e^i) \lrcorner \nabla_{e_i} u - (\mathrm{R},\nabla,u)$$

The involved step is treating $\sum_{i,j,k} e^j \cdot e^i \lrcorner e^k \lrcorner [\mathrm{R}_{e_i,e_j}, \nabla_{e_k}]$. We first obtain

$$\textstyle\sum_{i,j,\ell} e^j \cdot e^i \lrcorner e^\ell \lrcorner [\mathrm{R}_{e_i,e_j}, \nabla_{e_\ell}]u = \sum_{i,k,\ell,m} \sum_{k_r \in K} ([\mathrm{R}_{ji}{}^{k_r}{}_m, \nabla_{e_\ell}]u_K)\, e^j \cdot e^i \lrcorner e^\ell \lrcorner e^{\{k_r \to m\}K}$$

and it is important to realise that whenever the index ℓ contracts with m (or i or j), the resulting sum vanishes (as $\sum_\ell \nabla_{e_\ell} \mathrm{R}_{ij}{}^\ell{}_k = 0$). Similarly, if i and m are contracted then, as Ricci is parallel, the resulting sum vanishes. Expanding the final part of the previous display (and letting terms$(g^{\ell m}, g^{im})$ denote any terms involving $g^{\ell m}$ or g^{im}) gives

$$\begin{aligned} e^j \cdot e^i \lrcorner e^\ell \lrcorner e^{\{k_r \to m\}K} &= \sum_{k_p \in \{k_r \to\}K} g^{\ell k_p} e^j \cdot e^i \lrcorner e^{\{k_p \to m, k_r \to\}K} + \mathrm{terms}(g^{\ell m}) \\ &= \sum_{\substack{k_p \in \{k_r \to\}K \\ k_s \in \{k_p \to, k_r \to\}K}} g^{\ell k_p} g^{ik_s} e^{\{k_s \to j, k_p \to m, k_r \to\}K} + \mathrm{terms}(g^{\ell m}, g^{im}) \end{aligned}$$

and after a little rearrangement of dummy indices, this gives

$$\textstyle\sum_{i,j,\ell} e^j \cdot e^i \lrcorner e^\ell \lrcorner [\mathrm{R}_{e_i,e_j}, \nabla_{e_\ell}]u = -(\nabla, \mathrm{R}, u)$$

whence (6.7) is obtained.

Combining (6.6) with (6.7) gives $[\delta, \Delta]u = -(\nabla, \mathrm{R}, u)$. For symmetric tensors of rank two, such a summation (over k_r, k_p, k_s) does not arrive so such a term instantly vanishes and the result follows. For tensors of higher rank, one needs the Riemann curvature to be parallel. This is assured in the constant curvature setting of $\mathbb{H}^{n+1}$. □

The objects thus far introduced in this section all have natural analogues in the Lorentzian setting on (M, η). We denote by ${}^M\nabla$ the Levi-Civita connection of η extended to all associated vector bundles and ${}^M\mathrm{R}$ the Riemann curvature tensor of η. We let d_η and δ_η denote the symmetric differential and the divergence with respect to η. Finally we let ${}^M\nabla^{*M}\nabla$ denote the rough d'Alembertian and $\Box$ the (Lichnerowicz) d'Alembertian both constructed with respect to the metric η.

Minkowski scale and the operator Q

We define the first of our two main operators.

Definition 6.14. *The second-order differential operator* $\mathbf{Q} \in \mathrm{Diff}^2(M; \mathrm{End}\,\mathcal{F})$ *is the following conjugation of the d'Alembertian:*

$$\mathbf{Q} : \begin{cases} C^\infty(M; \mathcal{F}) & \to \quad C^\infty(M; \mathcal{F}) \\ \qquad u & \mapsto \quad s^{\frac{n}{2} - m + 2} \Box\, s^{-\frac{n}{2} + m}\, u \end{cases}$$

Lemma 6.15. *The differential operator* $\mathbf{Q}$ *is formally self-adjoint with respect to the inner product*

$$(u, v)_{s^{-2}\eta} = \int_M \langle u, v \rangle_{s^{-2}\eta} \tfrac{ds}{s} \mathrm{dvol}_g, \qquad u, v \in C_c^\infty(M; \mathcal{F}).$$

Proof. The d'Alembertian is self-adjoint with respect to the following inner product

$$(u, v)_\eta = \int_M \langle u, v \rangle_\eta \, \mathrm{dvol}_\eta, \qquad u, v \in C_c^\infty(M; \mathcal{F}).$$

The two inner products on $\mathcal{F}$ are related via (6.3). Tracking the effects of the conjugations by powers of s on $\Box$ as well as the multiplication by s^2 in order to obtain $\mathbf{Q}$ implies self-adjointness when using the inner product $\langle \cdot, \cdot \rangle_{s^{-2}\eta}$ with the measure $s^{-(n+2)} d\mathrm{vol}_\eta$ which gives the result as $d\mathrm{vol}_\eta = s^{n+2} \frac{ds}{s} d\mathrm{vol}_g$. □

Lemma 6.16. *The operator* $\mathbf{Q}$ *commutes with the Lefschetz-type trace operator* $s^{-2}\Lambda_{s^{-2}\eta}$.

$$[s^{-2}\Lambda_{s^{-2}\eta}, \mathbf{Q}]u = 0, \qquad u \in C^\infty(M; \mathcal{F}).$$

Proof. The Lorentzian analogue of Lemma 6.13 is that the d'Alembertian commutes with Λ_η

$$[\Lambda_\eta, \Box] = 0.$$

This operator is related to our standard Lefschetz-type operator $\Lambda_{s^{-2}\eta}$ via (6.2). The result is now a direct calculation. For clarity we denote differential operators with a superscript (m) to indicate that they act on symmetric cotensors of rank m. In particular, on $C^\infty(M; \mathcal{F})$ we have

$$\begin{aligned} s^{-2}\Lambda_{s^{-2}\eta}\, \mathbf{Q}^{(m)} &= s^2\, \Lambda_\eta\, s^{\frac{n}{2}-m+2}\, \Box^{(m)}\, s^{-\frac{n}{2}+m} \\ &= s^2 s^{\frac{n}{2}-m+2}\, \Box^{(m-2)}\, s^{-\frac{n}{2}+m}\, \Lambda_\eta \\ &= s^{\frac{n}{2}-(m-2)+2}\, \Box^{(m-2)}\, s^{-\frac{n}{2}+(m-2)} s^2\, \Lambda_\eta \\ &= \mathbf{Q}^{(m-2)}\, s^{-2}\Lambda_{s^{-2}\eta}. \end{aligned}$$ □

The rest of this subsection is dedicated to proving

Proposition 6.17. *For* $u \in C^\infty(M; \mathcal{F})$ *decomposed relative to the Minkowski scale* (6.1)*, the conjugated d'Alembertian* $\mathbf{Q}$ *is given by*

$$\begin{aligned} \mathbf{Q}\, a_k \left(\tfrac{ds}{s}\right)^{m-k} \cdot u^{(k)} = \;& a_{k+2} \left(\tfrac{ds}{s}\right)^{m-k-2} \cdot \big(- b_k b_{k+1}\, \mathrm{L} \big) u^{(k)} + \\ & a_{k+1} \left(\tfrac{ds}{s}\right)^{m-k-1} \cdot \big(2b_k\, \mathrm{d} \big) u^{(k)} + \\ & a_k \quad \left(\tfrac{ds}{s}\right)^{m-k} \quad \cdot \big(\Delta + (s\partial_s)^2 - c_k - \mathrm{L}\,\Lambda \big) u^{(k)} + \\ & a_{k-1} \left(\tfrac{ds}{s}\right)^{m-k+1} \cdot \big(- 2b_{k-1}\, \delta \big) u^{(k)} + \\ & a_{k-2} \left(\tfrac{ds}{s}\right)^{m-k+2} \cdot \big(- b_{k-2} b_{k-1}\, \Lambda \big) u^{(k)} \end{aligned}$$

with constants

$$\begin{aligned} a_k &= ((m-k)!)^{-1/2}, \\ b_k &= \sqrt{m-k}, \\ c_k &= \tfrac{n^2}{4} + m(n+2k+1) - k(2n+3k-1). \end{aligned}$$

Consequently, relative to this scale, there exist $\mathbf{D} \in \mathrm{Diff}^1(M; \mathrm{End}\,\mathcal{F})$ *and* $\mathbf{G} \in C^\infty(M; \mathrm{End}\,\mathcal{F})$ *independent of* s *such that*

$$\mathbf{Q} = \nabla^*\nabla + (s\partial_s)^2 + \mathbf{D} + \mathbf{G}$$

Proof. The result will follow from Lemmas 6.19 and 6.20. The conjugation by $s^{-\frac{n}{2}+m}$ is chosen so that the term $(s\partial_s + \frac{n}{2} - m)^2$ in Lemma 6.19 becomes simply $(s\partial_s)^2$. □

Proposition 6.17 is a direct calculation which we present in the rest of this subsection. To begin we announce the following lemma whose proof need not be detailed.

Lemma 6.18. *In the Minkowski scale, with* $\{e_i\}_{0\le i\le n}$ *a local holonomic frame on* (X, g) *with dual frame* $\{e^i\}_{0\le i\le n}$ *such that* $g = \sum_{i,j} g_{ij} e^i \otimes e^j$, *the connection* ${}^M\nabla$ *acts in the following manner:*

$$\begin{aligned} {}^M\nabla_{s\partial_s}\tfrac{ds}{s} &= -\tfrac{ds}{s}, & {}^M\nabla_{e_i}\tfrac{ds}{s} &= -\textstyle\sum_{j=0}^n g_{ij}e^j,\\ {}^M\nabla_{s\partial_s}e^i &= -e^i, & {}^M\nabla_{e_i}e^j &= \delta_i^j\tfrac{ds}{s} + \nabla_{e_i}e^j. \end{aligned}$$

This lemma provides the following two important formulae for the symmetrised basis

$$ {}^M\nabla_{s\partial_s}(\tfrac{ds}{s})^{m-k}\cdot e^K = -m(\tfrac{ds}{s})^{m-k}\cdot e^K \tag{6.8}$$

and

$$ {}^M\nabla_{e_i}(\tfrac{ds}{s})^{m-k}\cdot e^K = \begin{array}{l} (\tfrac{ds}{s})^{m-k-1}\cdot\Big(-(m-k)g_{ij}e^{\{\to j\}K}\Big) + \\ (\tfrac{ds}{s})^{m-k}\quad\cdot\Big(-\sum_{k_r\in K}\Gamma_{ij}^{k_r}e^{\{k_r\to j\}K}\Big) + \\ (\tfrac{ds}{s})^{m-k+1}\cdot\Big(\sum_{k_r\in K}\delta_i^{k_r}e^{\{k_r\to\}K}\Big) \end{array} \tag{6.9}$$

where the second result is a consequence of

$$ {}^M\nabla_{e_i}e^K = \sum_{k_r\in K}\delta_i^{k_r}\tfrac{ds}{s}\cdot e^{\{k_r\to\}K} + \nabla_{e_j}e^K $$

and we recall that the connection coefficients were introduced in Lemma 6.12. We split the calculation of the d'Alembertian into two calculations, treating the rough d'Alembertian separately from the curvature endomorphism.

Lemma 6.19. *For* $u \in C^\infty(M;\mathcal{F})$ *decomposed relative to the Minkowski scale* (6.1), *the rough d'Alembertian is given by*

$$ s^2\,{}^M\nabla^*{}^M\nabla\, a_k\,(\tfrac{ds}{s})^{m-k}\cdot u^{(k)} = \begin{array}{l} a_{k+2}\,(\tfrac{ds}{s})^{m-k-2}\cdot\big(-b_kb_{k+1}\,\mathrm{L}\big)u^{(k)} + \\ a_{k+1}\,(\tfrac{ds}{s})^{m-k-1}\cdot\big(2b_k\,\mathrm{d}\big)u^{(k)} + \\ a_k\quad(\tfrac{ds}{s})^{m-k}\quad\cdot\big(\nabla^*\nabla + (s\partial_s + \tfrac{n}{2} - m)^2 - \tilde{c}_k\big)u^{(k)} + \\ a_{k-1}\,(\tfrac{ds}{s})^{m-k+1}\cdot\big(-2b_{k-1}\,\delta\big)u^{(k)} + \\ a_{k-2}\,(\tfrac{ds}{s})^{m-k+2}\cdot\big(-b_{k-2}b_{k-1}\,\Lambda\big)u^{(k)} \end{array}$$

with modified constants

$$\tilde{c}_k = \tfrac{n^2}{4} + m(n+2k+1) - k(n+2k).$$

Proof. It suffices to consider a single term $u_K(\frac{ds}{s})^{m-k}\cdot e^K$ and we will ignore the normalising constants a_k until the final step. Upon a first application of ${}^M\nabla$ we obtain a section of $\mathrm{T}^*M \otimes \mathcal{F}$

$$\begin{aligned}{}^M\nabla u_K(\tfrac{ds}{s})^{m-k}\cdot e^K = {} & s\partial_s u_K \tfrac{ds}{s} \otimes (\tfrac{ds}{s})^{m-k}\cdot e^K \\ & + u_K \tfrac{ds}{s} \otimes {}^M\nabla_{s\partial_s}\left((\tfrac{ds}{s})^{m-k}\cdot e^K\right) \\ & + \textstyle\sum_i e_i u_K e^i \otimes (\tfrac{ds}{s})^{m-k}\cdot e^K \\ & + \textstyle\sum_i u_K e^i \otimes {}^M\nabla_{e_i}\left((\tfrac{ds}{s})^{m-k}\cdot e^K\right).\end{aligned}$$

Using (6.8) and (6.9) to develop the terms involving ${}^M\nabla_{s\partial_s}$ and ${}^M\nabla_{e_i}$ we group the result in terms of symmetric powers of $\frac{ds}{s}$. In order to handle the equations we write

$$ {}^M\nabla u_K(\tfrac{ds}{s})^{m-k}\cdot e^K = \boxed{1} + \boxed{2} + \boxed{3} + \boxed{4} \tag{6.10}$$

where

$$\begin{aligned}\boxed{1} &= -(m-k)\textstyle\sum_{i,j} u_K g_{ij} e^i \otimes (\tfrac{ds}{s})^{m-k-1}\cdot e^{\{\to j\}K},\\ \boxed{2} &= (s\partial_s - m)u_K \tfrac{ds}{s} \otimes (\tfrac{ds}{s})^{m-k}\cdot e^K,\\ \boxed{3} &= \textstyle\sum_i e_i u_K e^i \otimes (\tfrac{ds}{s})^{m-k}\cdot e^K - \sum_{i,j} u_K e^i \otimes (\tfrac{ds}{s})^{m-k}\cdot \sum_{k_r\in K}\Gamma_{ij}^{k_r} e^{\{k_r\to j\}K},\\ \boxed{4} &= -\textstyle\sum_i u_K e^i \otimes (\tfrac{ds}{s})^{m-k+1}\cdot \sum_{k_r\in K}\delta_i^{k_r} e^{\{k_r\to\}K}.\end{aligned}$$

Taking the second derivative, we calculate at a point about which $\{e_i\}_{0\le i\le n}$ are normal coordinates. Of course, we only need to keep track of terms which are not subsequently killed upon applying the trace tr_η (which, as the notation suggests, is the trace map on $\mathrm{T}^*M\otimes \mathrm{T}^*M$ built using the metric η).

$\boxed{1}$. Considering the first term in (6.10), applying ${}^M\nabla_{s\partial_s}$ provides only terms in the kernel of tr_η and applying ${}^M\nabla_{e_i}$ gives

$$\begin{aligned}\textstyle\sum_\ell e^\ell \otimes {}^M\nabla_{e_\ell}\boxed{1} = {} & -(m-k)\textstyle\sum_{i,j,\ell} e_\ell u_K g_{ij} e^\ell \otimes e^i \otimes (\tfrac{ds}{s})^{m-k-1}\cdot e^{\{\to j\}K}\\ & -(m-k)\textstyle\sum_{i,j,\ell} u_K g_{ij} e^\ell \otimes e^i \otimes {}^M\nabla_{e_\ell}\left((\tfrac{ds}{s})^{m-k-1}\cdot e^{\{\to j\}K}\right) + \ker \mathrm{tr}_\eta\end{aligned}$$

and we immediately apply tr_η to get

$$\begin{aligned}-s^2(\mathrm{tr}_\eta \circ {}^M\nabla)\boxed{1} = {} & (m-k)(\tfrac{ds}{s})^{m-k-1}\cdot\left(\textstyle\sum_i e_i u_K e^{\{\to i\}K}\right)\\ & + (m-k)u_K \textstyle\sum_j {}^M\nabla_{e_j}\left((\tfrac{ds}{s})^{m-k-1}\cdot e^{\{\to j\}K}\right).\end{aligned}$$

The first term of the preceding display reduces to the symmetric differential $(m-k)(\frac{ds}{s})^{m-k-1}\cdot \mathrm{d}(u_K e^K)$ by Lemma 6.12. The second term of the preceding display is calculated with the aid of (6.9) and remembering that the connection coefficients cancel at the point of interest. Specifically

$$\begin{aligned}{}^M\nabla_{e_j}\left((\tfrac{ds}{s})^{m-k-1}\cdot e^{\{\to j\}K}\right) = {} & (\tfrac{ds}{s})^{m-k-2}\cdot\left(-\textstyle\sum_i (m-k-1) g_{ij} e^{\{\to i,\to j\}K}\right) + \\ & (\tfrac{ds}{s})^{m-k} \cdot\left(\delta_j^j e^K + \textstyle\sum_{k_r\in K}\delta_j^{k_r} e^{\{\to j,k_r\to\}K}\right).\end{aligned}$$

Observe that $\sum_j \sum_{k_r\in K}\delta_j^{k_r} e^{\{\to j,k_r\to\}K} = k e^K$. Using Lemma 6.12 again this time to recover L, the result is

$$-s^2(\mathrm{tr}_\eta \circ {}^M\nabla)\boxed{1} = \begin{aligned}&(\tfrac{ds}{s})^{m-k-2} \cdot (-(m-k)(m-k-1)\,\mathrm{L})\, u_K e^K + \\ &(\tfrac{ds}{s})^{m-k-1} \cdot ((m-k)\,\mathrm{d})\, u_K e^K + \\ &(\tfrac{ds}{s})^{m-k} \quad \cdot (-(m-k)(n+1+k))\, u_K e^K.\end{aligned}$$

$\boxed{2}$. Considering the second term in (6.10) is much simpler. A second application of ${}^M\nabla$ provides

$$\begin{aligned}{}^M\nabla\boxed{2} &= (s\partial_s - m - 1)(s\partial_s - m))u_K \tfrac{ds}{s} \otimes \tfrac{ds}{s} \otimes (\tfrac{ds}{s})^{m-k} \cdot e^K \\ &\quad - (s\partial_s - m)u_K \sum_{i,j} g_{ij} e^i \otimes e^j \otimes (\tfrac{ds}{s})^{m-k} \cdot e^K + \ker \mathrm{tr}_\eta .\end{aligned}$$

and the desired result is

$$-s^2(\mathrm{tr}_\eta \circ {}^M\nabla)\boxed{2} \;=\; (\tfrac{ds}{s})^{m-k} \cdot ((s\partial_s - m + n)(s\partial_s - m))\, u_K e^K,$$

$\boxed{3}$. Considering the third term in (6.10) is somewhat similar to the first term in that ${}^M\nabla_{s\partial_s}$ provides only terms in the kernel of tr_η. Remembering that at the point of interest, the connection coefficients vanish, applying ${}^M\nabla_{e_j}$ gives

$$\begin{aligned}\textstyle\sum_j e^j \otimes {}^M\nabla_{e_j}\boxed{3} &= \textstyle\sum_{i,j} e_j e_i u_K e^j \otimes e^i \otimes (\tfrac{ds}{s})^{m-k} \cdot e^K \\ &\quad + \textstyle\sum_{i,j} e_i u_K e^j \otimes e^i \otimes {}^M\nabla_{e_j} \left((\tfrac{ds}{s})^{m-k} \cdot e^K\right) \\ &\quad - \textstyle\sum_{i,j,\ell} u_K e^\ell \otimes e^i \otimes (\tfrac{ds}{s})^{m-k} \cdot \sum_{k_r \in K} \left(\nabla_{e_\ell} \Gamma_{ij}^{k_r}\right) e^{\{k_r \to j\}K} + \ker \mathrm{tr}_\eta\end{aligned}$$

and we immediately apply tr_η to recover the rough Laplacian from the first and third terms in the previous display

$$\begin{aligned}-s^2(\mathrm{tr}_\eta \circ {}^M\nabla)\boxed{3} &= (\tfrac{ds}{s})^{m-k} \cdot \nabla^*\nabla(u_K e^K) \\ &\quad - \textstyle\sum_{i,j} g^{ij} e_i u_K {}^M\nabla_{e_j} \left((\tfrac{ds}{s})^{m-k} \cdot e^K\right)\end{aligned}$$

while the second term in the previous display is first treated using (6.9) and then Lemma 6.12 to recover the symmetric differential and the divergence

$$\begin{aligned}-\textstyle\sum_{i,j} g^{ij} e_i u_K {}^M\nabla_{e_j} \left((\tfrac{ds}{s})^{m-k} \cdot e^K\right) &= \textstyle\sum_{i,j,\ell} g^{ij} e_i u_K (m-k)(\tfrac{ds}{s})^{m-k-1} \cdot g_{\ell j} e^{\{\to \ell\}K} \\ &\quad + \textstyle\sum_{i,j,\ell} g^{ij} e_i u_K (\tfrac{ds}{s})^{m-k+1} \cdot \sum_{k_r \in K} \delta_j^{k_r} e^{\{k_r \to\}K} \\ &= (m-k)(\tfrac{ds}{s})^{m-k-1} \cdot \mathrm{d}(u_K e^K) \\ &\quad - (\tfrac{ds}{s})^{m-k+1} \cdot \delta(u_K e^K).\end{aligned}$$

The result is

$$-s^2(\mathrm{tr}_\eta \circ {}^M\nabla)\boxed{3} = \begin{aligned}&(\tfrac{ds}{s})^{m-k-1} \cdot ((m-k)\,\mathrm{d})\, u_K e^K + \\ &(\tfrac{ds}{s})^{m-k} \quad \cdot (\nabla^*\nabla)\, u_K e^K + \\ &(\tfrac{ds}{s})^{m-k+1} \cdot (-\,\delta)\, u_K e^K.\end{aligned}$$

$\boxed{4}$. Considering finally the fourth term in (6.10) we immediately remove the sum over i using the

Kronecker delta. Again ${}^M\nabla_{s\partial_s}$ provides only terms in the kernel of tr_η and applying ${}^M\nabla_{e_i}$ gives

$$\begin{aligned}\textstyle\sum_i e^i \otimes {}^M\nabla_{e_i}\boxed{4} = &-\textstyle\sum_i \sum_{k_r\in K} e_i u_K e^i \otimes e^{k_r} \otimes (\frac{ds}{s})^{m-k+1} \cdot e^{\{k_r\to\}K} \\ &-\textstyle\sum_i \sum_{k_r\in K} u_K e^i \otimes e^{k_r} \otimes {}^M\nabla_{e_i}\big((\frac{ds}{s})^{m-k+1}\cdot e^{\{k_r\to\}K}\big) + \ker\mathrm{tr}_\eta\,.\end{aligned}$$

and we immediately apply tr_η to get

$$\begin{aligned}-s^2(\mathrm{tr}_\eta \circ {}^M\nabla)\boxed{4} = &(\tfrac{ds}{s})^{m-k+1}\cdot\big(\textstyle\sum_i\sum_{k_r\in K} g^{ik_r} e_i u_K e^{\{k_r\to\}K}\big) \\ &+\textstyle\sum_i\sum_{k_r\in K} g^{ik_r} u_K {}^M\nabla_{e_i}\big((\frac{ds}{s})^{m-k+1}\cdot e^{\{k_r\to\}K}\big)\end{aligned}$$

The first term provides the divergence $-(\frac{ds}{s})^{m-k+1}\cdot\delta(u_K e^K)$ while the second term is treated using (6.9) and then Lemma 6.12 to recover a multiple of $u_K e^K$ and a term involving Λ

$$\begin{aligned}\textstyle\sum_i\sum_{k_r\in K} & g^{ik_r} u_K {}^M\nabla_{e_i}\Big((\tfrac{ds}{s})^{m-k+1}\cdot e^{\{k_r\to\}K}\Big) \\ &= -(m-k+1)(\tfrac{ds}{s})^{m-k}\cdot\textstyle\sum_{i,j}\sum_{k_r\in K} g^{ik_r} g_{ij} e^{\{\to j,k_r\to\}K} \\ &\quad -(\tfrac{ds}{s})^{m-k+2}\cdot\textstyle\sum_{k_r\in K}\sum_{k_p\in\{k_r\to\}K} g^{k_rk_p} e^{\{k_p\to,k_r\to\}K} \\ &= -k(m-k+1)(\tfrac{ds}{s})^{m-k}\cdot u_K e^K \\ &\quad -(\tfrac{ds}{s})^{m-k+2}\cdot\Lambda(u_K e^K).\end{aligned}$$

The result is

$$\begin{aligned}-s^2(\mathrm{tr}_\eta\circ{}^M\nabla)\boxed{4} = \; &(\tfrac{ds}{s})^{m-k}\cdot(-k(m-k+1))\,u_K e^K + \\ &(\tfrac{ds}{s})^{m-k+1}\cdot(-\,\delta)\,u_K e^K + \\ &(\tfrac{ds}{s})^{m-k+2}\cdot(-\,\Lambda)\,u_K e^K.\end{aligned}$$

Upon summation of these four terms coming from (6.10) we obtain

$$\begin{aligned} s^2\,{}^M\nabla^*{}^M\nabla(\tfrac{ds}{s})^{m-k}\cdot u^{(k)} = \; &(\tfrac{ds}{s})^{m-k-2}\cdot(-(m-k)(m-k-1)\,\mathrm{L})\,u^{(k)} + \\ &(\tfrac{ds}{s})^{m-k-1}\cdot(2(m-k)\,\mathrm{d})\,u^{(k)} + \\ &(\tfrac{ds}{s})^{m-k}\cdot\big(\nabla^*\nabla + (s\partial_s + \tfrac{n}{2} - m)^2 - \tilde{c}_k\big)\,u^{(k)} + \\ &(\tfrac{ds}{s})^{m-k+1}\cdot(-2k\,\delta)\,u^{(k)} + \\ &(\tfrac{ds}{s})^{m-k+2}\cdot(-k(k-1)\,\Lambda)\,u^{(k)}\end{aligned}$$

with constant $\tilde{c}_k$ as announced in the proposition. The final step is to reintroduce the normalisation constants a_k. Treating, for example, the term containing $(\frac{ds}{s})^{m-k-1}$ amounts to observing

$$a_{k+1}^{-1}(m-k)a_k = \sqrt{(m-k)}$$

This completes the demonstration. □

Lemma 6.20. *For $u \in C^\infty(M;\mathcal{F})$ decomposed relative to the Minkowski scale* (6.1)*, the curvature endomorphism acts diagonally with respect to the Minkowski scale and is given by*

$$s^2 q({}^M\mathrm{R})u^{(k)} = (q(\mathrm{R}) + k(n+k-1) - \mathrm{L}\,\Lambda)\,u^{(k)}.$$

Proof. We need only concern ourselves with the effect of $q({}^M\mathrm{R})$ on $(\frac{ds}{s})^{m-k}\cdot e^K$. It is easy to see from

Lemma 6.18 that ${}^{M}\mathrm{R}_{s\partial_s,e_i}$ is the zero endomorphism, that ${}^{M}\mathrm{R}_{e_i,e_j}\frac{ds}{s}=0$, and that $\eta({}^{M}\mathrm{R}_{e_i,e_j}e_k^*,\frac{ds}{s})=0$. Therefore we need only calculate the effect of $q({}^{M}\mathrm{R})$ on e^K. The non-trivial information of ${}^{M}\mathrm{R}$ is encoded in the following equation:

$${}^{M}\mathrm{R}_{ij}{}^{k}{}_{\ell}=g_{j\ell}\delta_i^k-g_{i\ell}\delta_j^k+\mathrm{R}_{ij}{}^{k}{}_{\ell}\,.$$

We extend ${}^{M}\mathrm{R}_{e_i,e_j}$ to $\mathcal{E}^{(k)}$ giving

$${}^{M}\mathrm{R}_{e_i,e_j}e^K=\mathrm{R}_{e_i,e_j}\,e^K+\sum_{k_r\in K}\left(\delta_j^{k_r}g_{i\ell}-\delta_i^{k_r}g_{j\ell}\right)e^{\{k_r\to\ell\}K}$$

Calculating the interior product requires the metric, in particular

$$s^2e^i\lrcorner_\eta\,{}^{M}\mathrm{R}_{e_i,e_j}=e^i\lrcorner\,{}^{M}\mathrm{R}_{e_i,e_j}$$

where $\lrcorner_\eta$ uses the metric η to identify $\mathrm{T}M$ with T^*M. Consequently calculating

$$\sum_i\left(s^2e^i\lrcorner_\eta\,{}^{M}\mathrm{R}_{e_i,e_j}e^K-e^i\lrcorner\,\mathrm{R}_{e_i,e_j}\,e^K\right)$$

gives

$$\sum_i\sum_{k_r\in K}\left(\delta_j^{k_r}g_{i\ell}-\delta_i^{k_r}g_{j\ell}\right)\left(g^{i\ell}e^{\{k_r\to\}K}+\sum_{k_p\in\{k_r\to\}K}g^{ik_p}e^{\{k_p\to,k_r\to\ell\}K}\right).$$

Applying $\sum_j e^j\cdot$ to the preceding display provides $s^2q({}^{M}\mathrm{R})-q(\mathrm{R})$. Splitting the calculation into four terms, the results are

$$\begin{aligned}
\textstyle\sum_{i,j}\sum_{k_r\in K}e^j\cdot\delta_j^{k_r}g_{i\ell}g^{i\ell}e^{\{k_r\to\}K}&=k(n+1)e^K,\\
-\textstyle\sum_{i,j}\sum_{k_r\in K}e^j\cdot\delta_i^{k_r}g_{j\ell}g^{i\ell}e^{\{k_r\to\}K}&=-ke^K,\\
\textstyle\sum_{i,j}\sum_{k_r\in K}\sum_{k_p\in\{k_r\to\}K}e^j\cdot\delta_j^{k_r}g_{i\ell}g^{ik_p}e^{\{k_p\to,k_r\to\ell\}K}&=k(k-1)e^K,\\
-\textstyle\sum_{i,j}\sum_{k_r\in K}\sum_{k_p\in\{k_r\to\}K}e^j\cdot\delta_i^{k_r}g_{j\ell}g^{ik_p}e^{\{k_p\to,k_r\to\ell\}K}&=-\mathrm{L}\,\Lambda\,e^K.
\end{aligned}$$

Upon summation of these four terms, the proof is complete. □

Proposition 6.21. *Suppose $u\in C^\infty(M;\mathcal{F})$, decomposed relative to the Minkowski scale (6.1), is trace-free with respect to the trace operator $\Lambda_{s^{-2}\eta}$. Then the conjugated d'Alembertian* **Q** *is given by*

$$\mathbf{Q}\,a_k\,(\tfrac{ds}{s})^{m-k}\cdot u^{(k)}\;=\;\begin{array}{l}a_{k+1}\,(\tfrac{ds}{s})^{m-k-1}\cdot\big(2b_k\,\mathrm{d}\,\big)u^{(k)}+\\ a_k\quad(\tfrac{ds}{s})^{m-k}\quad\cdot\big(\Delta+(s\partial_s)^2-c_k'\,\big)u^{(k)}+\\ a_{k-1}\,(\tfrac{ds}{s})^{m-k+1}\cdot\big(-2b_{k-1}\,\delta\,\big)u^{(k)}\end{array}$$

with constants a_k, b_k announced in Proposition 6.17 and the modified constants

$$c_k'=c_k-(m-k)(m-k-1).$$

Proof. This follows directly from the structure of **Q** given in Proposition 6.17 and the condition that $\Lambda\,u^{(k)}=-b_{k-2}b_{k-1}u^{(k-2)}$ coming from Lemma 6.1. □

The indicial family of Q

Definition 6.22. *Denote by $\mathcal{Q}$ the indicial family of the operator $\mathbf{Q} \in \mathrm{Diff}^2_{\mathrm{b}}(\overline{M};\mathcal{F})$ relative to the Minkowski scale s.*

$$\mathcal{Q} = \mathrm{I}_s(\mathbf{Q};\lambda) \in \mathrm{Diff}^2(X;\mathcal{E}).$$

The previous section introduced $\mathbf{Q}$ as a differential operator on $\mathcal{F}$ above M however, from the structure of $\mathbf{Q}$ given as announced in Proposition 6.17, it is clear that the operator extends to $\overline{M}$. Moreover by the same proposition we immediately get the structure of $\mathcal{Q}$.

Proposition 6.23. *For $u = \sum_{k=0}^m u^{(k)} \in C^\infty(X;\mathcal{E})$, the operator $\mathcal{Q}$ is given by*

$$\begin{aligned} & \big(-b_k b_{k+1}\,\mathrm{L}\,\big)u^{(k)}+ \\ & \big(2b_k\,\mathrm{d}\,\big)u^{(k)}+ \\ \mathcal{Q}_\lambda\, u^{(k)} = {} & \big(\Delta + \lambda^2 - c_k - \mathrm{L}\,\Lambda\,\big)u^{(k)}+ \\ & \big(-2b_{k-1}\,\delta\,\big)u^{(k)}+ \\ & \big(-b_{k-2}b_{k-1}\,\Lambda\,\big)u^{(k)} \end{aligned}$$

with constants

$$b_k = \sqrt{m-k}, \qquad c_k = \tfrac{n^2}{4} + m(n+2k+1) - k(2n+3k-1).$$

Consequently, there exist $\mathcal{D} \in \mathrm{Diff}^1(X;\mathrm{End}\,\mathcal{E})$ and $\mathcal{G} \in C^\infty(X;\mathrm{End}\,\mathcal{E})$ independent of λ such that

$$\mathcal{Q}_\lambda = \nabla^*\nabla + \lambda^2 + \mathcal{D} + \mathcal{G}$$

Proposition 6.24. *The family of differential operators $\mathcal{Q}$ is, upon restriction to $\lambda \in i\mathbb{R}$, a family of formally self-adjoint operators with respect to the inner product*

$$(u,v)_s = \int_X \sum_{k=0}^m (-1)^{m-k} \langle u^{(k)}, v^{(k)}\rangle \,\mathrm{dvol}_g$$

where $u = \sum_{k=0}^m u^{(k)}$, $v = \sum_{k=0}^m v^{(k)}$ for $u^{(k)}, v^{(k)} \in C_c^\infty(X;\mathcal{E}^{(k)})$. Moreover, for all λ, $\mathcal{Q}_\lambda^* = \mathcal{Q}_{-\bar\lambda}$.

Proof. Lemmas 6.6 and 6.15. □

The operator $\mathbf{Q}$ preserves the subbundle $\mathcal{F}\cap\ker\Lambda_{s^{-2}\eta}$ by Lemma 6.16. As π_s^* is algebraic, we may consider it as a map from $\mathcal{E}$ over X to $\mathcal{F}$ over M. We thus obtain the subbundle $\mathcal{E}\cap\ker(\Lambda_{s^{-2}\eta}\circ\pi_s^*)$ over X; symmetric tensors above X which are trace-free with respect to the ambient trace operator $\Lambda_{s^{-2}\eta}$. It thus follows that $\mathcal{Q}$ may also be considered a family of differential operators on this subbundle and we obtain

Proposition 6.25. *For $u = \sum_{k=0}^m u^{(k)} \in C^\infty(X;\mathcal{E}) \cap \ker(\Lambda_{s^{-2}\eta} \circ \pi_s^*)$, the operator $\mathcal{Q}$ is given by*

$$\begin{aligned} & \big(2b_k\,\mathrm{d}\,\big)u^{(k)}+ \\ \mathcal{Q}_\lambda\, u^{(k)} = {} & \big(\Delta + \lambda^2 - c'_k\big)u^{(k)}+ \\ & \big(-2b_{k-1}\,\delta\,\big)u^{(k)}. \end{aligned}$$

6.4 The Operator P and its Indicial Family

This section introduces the operator $\mathbf{P}$ on M_e and its indicial family $\mathcal{P}$ on X_e and similar results to those presented for $\mathbf{Q}$ and $\mathcal{Q}$ are given. The relationship between these two constructions is also detailed.

Euclidean scale

The manifold $M_e = \mathbb{R}^+_t \times X_e$ has been equipped with the Lorentzian metric η which agrees with the Lorentzian cone metric put on M. Recalling the smooth chart $U = (0,1)_\mu \times Y \subset X \subset X_e$ the metric, on $\mathbb{R}^+_t \times U$ takes the form of (5.1) and we may assume that this is the form of η on the larger chart $\mathbb{R}^+_t \times U^2$ where $U^2 = (-1,1)_\mu \times Y$. For later use we record the behaviour of ${}^{M_e}\nabla$.

Lemma 6.26. *On the chart $\mathbb{R}^+_t \times (-1,1)_\mu \times Y$ with $\{e_i\}_{1\le i\le n}$ a local holonomic frame on Y with dual frame $\{e^i\}_{1\le i\le n}$ such that $h = \sum_{i,j} h_{ij} e^i \otimes e^j$, the connection ${}^{M_e}\nabla$ acts in the following manner:*

$$\begin{aligned}
{}^{M_e}\nabla_{t\partial_t}\tfrac{dt}{t} &= 0, & {}^{M_e}\nabla_{\partial_\mu}\tfrac{dt}{t} &= 0,\\
{}^{M_e}\nabla_{t\partial_t} d\mu &= -d\mu, & {}^{M_e}\nabla_{\partial_\mu} d\mu &= -\tfrac{dt}{t},\\
{}^{M_e}\nabla_{t\partial_t} e^i &= -e^i, & {}^{M_e}\nabla_{\partial_\mu} e^i &= -\tfrac{1}{2}h^{ij}(\partial_\mu h_{jk})e^k,
\end{aligned}$$

and

$$\begin{aligned}
{}^{M_e}\nabla_{e_i}\tfrac{dt}{t} &= -(\partial_\mu h_{ij})e^j,\\
{}^{M_e}\nabla_{e_i} d\mu &= -2((1-\mu\partial_\mu)h_{ij})e^j,\\
{}^{M_e}\nabla_{e_i} e^j &= -\delta_i^j\tfrac{dt}{t} - \tfrac{1}{2}h^{jk}(\partial_\mu h_{ik})d\mu + {}^{Y}\nabla_{e_i} e^j.
\end{aligned}$$

Motivated by the structure of $\mathbf{Q}$ from the previous section we define the second of a our two main operators

Definition 6.27. *The second-order differential operator $\mathbf{P} \in \mathrm{Diff}^2(M_e;\mathcal{F})$ is the following conjugation of the d'Alembertian:*

$$\mathbf{P} : \begin{cases} C^\infty(M;\mathcal{F}) & \to \quad C^\infty(M_e;\mathcal{F}) \\ \qquad u & \mapsto \quad t^{\frac{n}{2}-m+2}\,\Box\, t^{-\frac{n}{2}+m}\,u \end{cases}$$

Note that on $M \subset M_e$ there is a trivial correspondence between $\mathbf{P}$ and $\mathbf{Q}$,

$$\mathbf{P} = \rho^{-\frac{n}{2}+m-2}\,\mathbf{Q}\,\rho^{\frac{n}{2}-m}$$

and that, since $\rho = 1$ on $X\backslash U$, we have equality $\mathbf{P} = \mathbf{Q}$ on $M\backslash(\mathbb{R}^+ \times U)$.

Lemma 6.28. *The operator $\mathbf{P} \in \mathrm{Diff}^2(M_e;\mathcal{F})$ naturally extends to an operator $\mathbf{P} \in \mathrm{Diff}^2_{\mathrm{b}}(\overline{M_e};{}^{\mathrm{b}}\mathcal{F})$ and is b-trivial.*

Proof. The important point is to verify that at $\mu = 0$, $\mathbf{P}$ fits into the b-calculus framework. This is reasonably clear from Lemma 6.26. Indeed, the Lie algebra of b-vector fields is generated by $\{t\partial_t, \partial_\mu, e_i\}$ where $\{e_i\}_{1\le i\le n}$ is a local holonomic frame on Y, while the b-cotangent bundle has basis $\{\frac{dt}{t}, d\mu, e^i\}$ with $\{e^i\}_{1\le i\le n}$ the dual frame on T^*Y. Lemma 6.26 thus shows that ${}^{M_e}\nabla$ is a b-connection. Taking the trace using η and then multiplying by t^2 is equivalent to taking the trace with $t^{-2}\eta$ whose structure (5.1) indicates it is a b-metric. Therefore $t^2\,{}^{M_e}\nabla^{*\,M_e}\nabla$ is a b-differential operator. That $t^2\,{}^{M_e}\nabla^{*\,M_e}\nabla$ is b-trivial is also immediate from Lemma 6.26 and the structure of $t^{-2}\eta$. A similar line of reasoning for $q({}^{M_e}\mathrm{R})$ (which uses one application of the inverse of the metric η) shows that $t^2\,\Box$ is also a b-differential operator. The final conjugation by powers of t preserves the b-structure (and its b-triviality) as it merely conjugates appearances of $t\partial_t$. This implies the result. □

Lemma 6.29. *The differential operator* $\mathbf{P}$ *is formally self-adjoint with respect to the inner product*

$$(u,v)_{t^{-2}\eta} = \int_{M_e} \langle u,v\rangle_{t^{-2}\eta}\,\tfrac{dt}{t}dx, \qquad u,v\in C_c^\infty(M_e;\mathcal{F}).$$

Proof. By the correspondence between $\mathbf{P}$ and $\mathbf{Q}$ on $M\backslash(\mathbb{R}^+\times U)$ and Lemma 6.15, it suffices to verify this claim when u,v are supported on $\mathbb{R}_t^+\times U^2$. The d'Alembertian is self-adjoint with respect to the following inner product

$$(u,v)_\eta = \int_{M_e} \langle u,v\rangle_\eta\,d\mathrm{vol}_\eta, \qquad u,v\in C_c^\infty(\mathbb{R}_t^+\times U^2;\mathcal{F}).$$

The two inner products on the fibres of $\mathcal{F}$ are related via the Euclidean scale analog of (6.3). Tracking the effects of the conjugations by powers of t on $\Box$ as well as the multiplication by t^2 in order to obtain $\mathbf{P}$ implies self-adjointness when using the inner product $\langle\cdot,\cdot\rangle_{t^{-2}\eta}$ with the measure $t^{-n-2}d\mathrm{vol}_\eta$. As $\det\eta = -\frac{1}{4}t^{2n+2}\det h$ we have

$$t^{-n-2}d\mathrm{vol}_\eta = \tfrac{1}{2}\tfrac{dt}{t}d\mu\,d\mathrm{vol}_h. \qquad \Box$$

The indicial family of P

Definition 6.30. *Denote by* $\mathcal{P}$ *the indicial family of the operator* $\mathbf{P}\in\mathrm{Diff}_{\mathrm{b}}^2(\overline{M_e};{}^{\mathrm{b}}\mathcal{F})$ *relative to the Euclidean scale* t*.*

$$\mathcal{P}_\lambda = \mathrm{I}_t(\mathbf{P};\lambda)\in\mathrm{Diff}^2(X_e;\mathcal{E}).$$

Lemma 6.8 gives the following proposition (whose final statement follows as ρ is constant on $X\backslash U$).

Proposition 6.31. *On* $X\subset X_e$ *the indicial family operators* $\mathcal{P}$ *and* $\mathcal{Q}$ *are related by*

$$\mathcal{P}_\lambda = \rho^{-\lambda-\frac{n}{2}+m-2} J\,\mathcal{Q}_\lambda\,J^{-1}\rho^{\lambda+\frac{n}{2}-m}$$

with J *presented in Lemma 6.2. Moreover, on* $X\backslash U$*, we have equality* $\mathcal{P}=\mathcal{Q}$*.*

Proposition 6.32. *The family of differential operators* $\mathcal{P}$ *is, upon restriction to* $\lambda\in i\mathbb{R}$*, a family of formally self-adjoint operators with respect to the inner product*

$$(u,v)_t = \int_{X_e}\langle u,v\rangle_t\,dx, \qquad u,v\in C_c^\infty(X_e;\mathcal{E}).$$

Moreover, for all λ*,* $\mathcal{P}_\lambda^* = \mathcal{P}_{-\bar\lambda}$*.*

6.5 Microlocal Analysis

This section constructs an inverse to the family $\mathcal{P}$ introduced in the preceding section. This is done by first showing that the family is a family of Fredholm operators and then by considering a Cauchy problem which provides an inverse for $\operatorname{Re}\lambda \gg 1$. In [Vas13b, Zwo16], the procedure is described for functions, rather than symmetric tensors. We are required to alter only minor details in order to apply the technique to symmetric tensors.

Function spaces

From Subsection 6.1, we have the space of L^2 sections $L^2_t(X_e;\mathcal{E})$. This defines $H^s_{\text{loc}}(X_e;\mathcal{E})$, the space of (locally) H^s sections for $s \in \mathbb{R}$. For all notions of Sobolev regularity, we will only use the Euclidean scale; we thus need not decorate these spaces with a subscript t.

As is standard, we denote by $\dot{C}^\infty(X_{cs};\mathcal{E})$ the set of smooth sections which are extensible to smooth sections over X_e and whose support is contained in $\overline{X_{cs}}$. And by $C^\infty(\overline{X_{cs}};\mathcal{E})$ all smooth sections which are smoothly extensible to X_e.

Following [Hör07, Appendix B.2] we obtain, for $s \in \mathbb{R}$, the Sobolev spaces

$$\dot{H}^s(\overline{X_{cs}};\mathcal{E}) \quad \text{and} \quad \overline{H}^s(X_{cs};\mathcal{E})$$

which are, respectively, the set of elements in $H^s_{\text{loc}}(X_e;\mathcal{E})$ supported by $\overline{X_{cs}}$ and the space of restrictions to X_{cs} of $H^s_{\text{loc}}(X_e;\mathcal{E})$. Then $\dot{H}^s(\overline{X_{cs}};\mathcal{E})$ gets its norm directly from that of $H^s_{\text{loc}}(X_e;\mathcal{E})$ while the norm of an element in $\overline{H}^s(X_{cs};\mathcal{E})$ is that obtained by taking the infimum of the norms of all permissible extensions of the element which have compact support in X_e. (Such norms will be denoted, for simplicity, by $\|\cdot\|_{\dot{H}^s}$ and $\|\cdot\|_{\overline{H}^s}$. Furthermore, if an object is supported away from S, these norms correspond and we may simply write $\|\cdot\|_{H^s}$.)

The inner product $\langle\cdot,\cdot\rangle_t$ gives the L^2 pairing

$$(\cdot,\cdot)_t : \dot{C}^\infty(X_{cs};\mathcal{E}) \times C^\infty(\overline{X_{cs}};\mathcal{E}) \to \mathbb{C}$$

which extends by density [Hör07, Theorem B.2.1] to a pairing between the spaces $\dot{H}^{-s}(\overline{X_{cs}};\mathcal{E})$ and $\overline{H}^s(X_{cs};\mathcal{E})$ providing the identification of dual spaces

$$(\overline{H}^s(X_{cs};\mathcal{E}))^* \simeq \dot{H}^{-s}(\overline{X_{cs}};\mathcal{E}), \qquad s \in \mathbb{R}. \tag{6.11}$$

Definition 6.33. *For $s \in \mathbb{R}$, let $\mathcal{X}^s$ and $\mathcal{Y}^s$ be the following two spaces*

$$\mathcal{Y}^s = \overline{H}^s(X_{cs};\mathcal{E}),$$
$$\mathcal{X}^s = \{u : u \in \mathcal{Y}^s, \mathcal{P}\, u \in \mathcal{Y}^{s-1}\}$$

These spaces come with the standard norms, in particular,

$$\|u\|_{\mathcal{X}^s} = \|u\|_{\mathcal{Y}^s} + \|\mathcal{P}\, u\|_{\mathcal{Y}^{s-1}}, \qquad u \in \mathcal{X}^s.$$

Remark 6.34. It will be seen that λ does not appear in the principal symbol of $\mathcal{P}$, it is thus unimportant to explicit with respect to what value of λ the preceding norm is taken as all such norms are equivalent.

When restricting to $U^2 \subset X_e$ we will let $\{e_i\}_{1\le i\le n}$ denote an orthonormal frame for (Y,h) (which depends on $\mu \in (-1,1)$) and by $\{e^i\}_{1\le i\le n}$ its dual frame. The frames are completed to frames for TU^2 and T^*U^2 by including ∂_μ and $d\mu$ respectively. A dual vector will take the notation

$$\xi d\mu + \sum_{i=0}^{n} \eta_i e^i \in \mathrm{T}^*U^2. \tag{6.12}$$

The following subsection proves the following two propositions.

Proposition 6.35. *For fixed s, the family of operators*

$$\mathcal{P} : \mathcal{X}^s \to \mathcal{Y}^{s-1}$$

is Fredholm for $\operatorname{Re}\lambda > \frac{1}{2} - s$.

Proof. Lemmas 6.38 and 6.39. □

Proposition 6.36. *For fixed s, the Fredholm operator $\mathcal{P}_\lambda : \mathcal{X}^s \to \mathcal{Y}^{s-1}$ are Fredholm of index 0 for* $\operatorname{Re}\lambda > m + \frac{1}{2} - s$ *and it has a meromorphic inverse*

$$\mathcal{P}^{-1} : \mathcal{Y}^{s-1} \to \mathcal{X}^s$$

with poles of finite rank.

Proof. Lemmas 6.40 and 6.41. □

Proofs of Propositions 6.35 and 6.36

On $\mathbb{R}_t^+ \times U^2$, the inverse of the metric η takes the form

$$t^2\eta^{-1} = -2t\partial_t \cdot \partial_\mu + 2\mu\partial_\mu \cdot \partial_\mu + h^{-1}$$

which implies to highest order for $t^2\,{}^{M_e}\nabla^{*\,M_e}\nabla$, that

$$t^2\,{}^{M_e}\nabla^{*\,M_e}\nabla = -4\mu\partial_\mu^2 + 4t\partial_t\partial_\mu + \Delta_h + \mathrm{Diff}^1(\mathbb{R}_t^+ \times U^2; \operatorname{End}\mathcal{F})$$

where Δ_h may be considered the rough Laplacian on (Y,h). Considering $\mathcal{P}$, conjugation by $t^{-\frac{n}{2}+m}$ replaces $t\partial_t$ by $(t\partial_t - \frac{n}{2} + m)$ and we can absorb the newly created term $4(-\frac{n}{2}+m)\partial_\mu$ into $\mathrm{Diff}^1(\mathbb{R}_t^+ \times U^2; \operatorname{End}\mathcal{F})$. Also, the curvature term is of order zero so

$$\mathbf{P} = -4\mu\partial_\mu^2 + 4t\partial_t\partial_\mu + \Delta_h + \mathbf{A}$$

for some $\mathbf{A} \in \mathrm{Diff}^1(\mathbb{R}_t^+ \times U^2; \operatorname{End}\mathcal{F})$. This structure of $\mathbf{P}$ immediately gives the structure of $\mathcal{P}$ to highest order. Keeping track of the term $4t\partial_t\partial_\mu$ for the moment, we write

$$\mathcal{P}_\lambda = -4\mu\partial_\mu^2 - 4\lambda\partial_\mu + \Delta_h + \mathcal{A}_\lambda. \tag{6.13}$$

where $\mathcal{A}_\lambda \in \mathrm{Diff}^1(U^2; \operatorname{End}\mathcal{E})$ is the indicial family of $\mathbf{A}$. The most obvious conclusion we draw from such a presentation of $\mathcal{P}$ is that $\mathcal{P}$ is a family of elliptic operators on $U^2 \cap \{\mu > 0\}$ and a family of strictly hyperbolic operators for $\{\mu < 0\}$ (with respect to the level sets $\{\mu = \text{constant}\}$). Of course the ellipticity extends to all of X. The principal symbol on U^2 is also immediately recognisable as

$$\sigma(\mathcal{P}) = 4\mu\xi^2 + |\eta|^2$$

using the notation from (6.12) and $|\eta|^2 = \sum_{i=1}^n \eta_i^2$. And on U^2, the Hamiltonian vector field associated with $\sigma(\mathcal{P})$ is

$$H_{\sigma(\mathcal{P})} = 8\mu\xi\partial_\mu - 4\xi^2\partial_\xi + H_{|\eta|^2}.$$

The strategy to obtain a Fredholm problem is to combine standard results for elliptic and hyperbolic operators with some analysis performed at the junction $Y = \{\mu = 0\}$. The analysis was first presented in [Vas13a, Section 4.4]. It turns out the dynamics of interest are those of radial sources and sinks [DZ16, Definition E.52]. The original radial estimates of Melrose [Mel94] on asymptotically Euclidean spaces have been adapted to functions on asymptotically hyperbolic spaces by Vasy [Vas13a]. Indeed, to see that such dynamics are relevant for $\mathcal{P}$, consider $\sigma(\mathcal{P})$ and $H_{\sigma(\mathcal{P})}$ given in the preceding displays. Define the characteristic variety $\Sigma \subset \mathrm{T}^*X_{cs}\backslash 0$ which is contained in T^*U. As $(\mu, y, 0, \eta) \notin \Sigma$, we may split $\Sigma = \Sigma_+ \sqcup \Sigma_-$ given by $\Sigma_\pm = \Sigma \cap \{\pm\xi > 0\}$. At Y remark that

$$\Sigma \cap \mathrm{T}^*_Y U = \{(0, y, \xi, 0) : \xi \neq 0\} \subset N^*Y$$

and recalling the projection $\kappa : \mathrm{T}^*U\backslash 0 \to \partial\overline{\mathrm{T}}^*U$ define

$$\Gamma_+ = \kappa(\Sigma_+ \cap Y), \quad \Gamma_- = \kappa(\Sigma_- \cap Y).$$

In [Vas13b, Section 3.2], it is shown that $\Gamma_\pm$ are respectively a source and a sink for $\sigma(\mathcal{P})$. In order to apply Lemmas 6.9 and 6.10, we introduce the principal symbol of the imaginary part of $\mathcal{P}$. By Remark 6.11, $H_{\sigma(\mathcal{P})} = H_{\sigma(\mathcal{P}^*)}$ and by Proposition 6.32, $\mathcal{P}^*_\lambda = \mathcal{P}_{-\bar{\lambda}}$ hence $\sigma(\operatorname{Im}\mathcal{P}) = -\sigma(\operatorname{Im}\mathcal{P}^*)$. Also, by a direct calculation using the structure of $H_{\sigma(\mathcal{P})}$,

$$\langle\xi + \eta\rangle^{-1} H_{\sigma(\mathcal{P})} \log\langle\xi + \eta\rangle = \mp 4, \quad \text{on } \Gamma_\pm. \tag{6.14}$$

In fact Proposition 6.32 along with (6.13) gives more precisely

$$\operatorname{Im}\mathcal{P}_\lambda = \frac{\mathcal{P}_\lambda - \mathcal{P}^*_\lambda}{2i} = 4i(\operatorname{Re}\lambda)\partial_\mu + \frac{\mathcal{A}_\lambda - \mathcal{A}_{-\bar{\lambda}}}{2i}$$

however as $\mathbf{A}$ is first order, $\mathcal{A}_\lambda$ may be written as the sum of a first order operator independent of λ and a zeroth order operator (which may depend on λ). Therefore

$$\sigma(\operatorname{Im}\mathcal{P}_\lambda) = -4\operatorname{Re}\lambda\,\xi. \tag{6.15}$$

Bringing this altogether in preparation for the proof of Proposition 6.35 we have

Lemma 6.37. *For $\mathcal{P}$, Γ_+ is a source, while Γ_- is a source for $-\mathcal{P}$. In both situations, the threshold condition, when working on $H^s(X_{cs}; \mathcal{E})$, is satisfied if*

$$s > -\operatorname{Re}\lambda + \tfrac{1}{2}.$$

For $\mathcal{P}^*$, Γ_- is a sink, while Γ_+ is a sink for $-\mathcal{P}^*$. In both situations, the threshold condition, when *working on $H^{\tilde{s}}(X_{cs};\mathcal{E})$, is satisfied if*

$$\tilde{s} < \operatorname{Re}\lambda + \tfrac{1}{2}.$$

Proof. We explain the first result, all others are similar after taking into account Remark 6.11. On Γ_+, by (6.14) and (6.15),

$$\langle \xi+\eta\rangle^{-1}(\sigma(\operatorname{Im}\mathcal{P}) + (s-\tfrac{1}{2})H_{\sigma(\mathcal{P})}\log\langle\xi+\eta\rangle) = -4(\operatorname{Re}\lambda + s - \tfrac{1}{2}).$$

For this to be negative definite requires precisely that $s > -\operatorname{Re}\lambda + \frac{1}{2}$. □

Lemma 6.38. *Restricting to $s > -\operatorname{Re}\lambda + \frac{1}{2}$, the operators $\mathcal{P}_\lambda : \mathcal{X}^s \to \mathcal{Y}^{s-1}$ have finite dimensional kernels.*

Proof. It suffices to obtain an estimate, for $u \in \mathcal{X}^s$, of the form

$$\|u\|_{\overline{H}^s} \le C\left(\|\mathcal{P}_\lambda u\|_{\overline{H}^{s-1}} + \|\psi u\|_{H^{-N}}\right).$$

for some ψ supported on $\{\mu > -\frac{1}{2}\}$ and such that $\psi = 1$ near $\{\mu > -\frac{1}{2}+\varepsilon\}$. This is done by writing $u = (\psi_- + \psi_0 + \psi_+)u$ with the supports of ψ_-, ψ_0, ψ_+ respectively contained in $\{\mu < -\varepsilon\}, \{|\mu| < 2\varepsilon\}, \{\mu > \varepsilon\}$. The estimate for $\psi_+ u$ is due to ellipticity of $\mathcal{P}$. The estimate for $\psi_- u$ is due to hyperbolicity which allows us to reduce to the estimate for $\psi_0 u$:

$$\|\psi_- u\|_{\overline{H}^s} \le C\left(\|\mathcal{P}_\lambda u\|_{\overline{H}^{s-1}} + \|\psi_0 u\|_{H^s}\right).$$

The estimate for $\psi_0 u$ is obtained by microlocalising. Away from Σ, ellipticity gives the result, while near Σ, propagation of singularities implies that the norms can be controlled by $\Gamma_\pm$. The high regularity results for Γ_+ and Γ_- from Lemma 6.9 are applicable as these are sources for $\mathcal{P}$ and $-\mathcal{P}$ respectively. Lemma 6.37 ensures that the threshold conditions are satisfied (by hypothesis of this proposition). The desired estimate is obtained. □

Lemma 6.39. *Restricting to $s > -\operatorname{Re}\lambda + \frac{1}{2}$, the operators $\mathcal{P}_\lambda : \mathcal{X}^s \to \mathcal{Y}^{s-1}$ have finite dimensional cokernels.*

Proof. To show that the range is of finite codimension we study the adjoint operator $\mathcal{P}^*$. By (6.11) the dual space of $\overline{H}^{s-1}(X_{cs};\mathcal{E})$ is $\dot{H}^{1-s}(\overline{X_{cs}};\mathcal{E})$ and the dimension of the kernel of $\mathcal{P}^*$ equals the dimension of the cokernel of $\mathcal{P}$. It suffices to obtain an estimate of the form

$$v \in \dot{H}^{1-s}(\overline{X_{cs}};\mathcal{E}) \cap \ker\mathcal{P}^* \quad\Longrightarrow\quad \|v\|_{\dot{H}^{1-s}} \le C\|\psi v\|_{H^{-N}}$$

with ψ as defined in the previous proof. Again, we use the partition $v = (\psi_- + \psi_0 + \psi_+)v$. Again, the estimate for $\psi_+ v$ is due to ellipticity of $\mathcal{P}^*$. This time, the estimates for $\psi_- v$ are immediate due to hyperbolicity and the requirement at S that v vanish to all orders which implies that $v = 0$ on $\{\mu < 0\}$. The estimate for $\psi_0 v$ is obtained by microlocalising. (Away from $\operatorname{Char}(P)$, the result is obtained by ellipticity.) The low regularity results for Γ_- and Γ_+ from Lemma 6.10 are applicable as these are sinks for $\mathcal{P}^*$ and $-\mathcal{P}^*$ respectively. Lemma 6.37 ensures that the threshold conditions are satisfied. Therefore there exists $A, B \in \Psi^0(X_{cs})$ with $\operatorname{Char}(A) \cap \Gamma_\pm = \varnothing$ and $\operatorname{WF}(B) \cap \Gamma_\pm = \varnothing$ such that $\|A\psi_0 v\|_{H^{1-s}} \le C(\|B\psi_0 v\|_{H^{1-s}} + \|\psi v\|_{H^{-N}})$. As $v = 0$ on $\{\mu < 0\}$ and is smooth (by ellipticity of $\mathcal{P}^*$) on $\{\mu > 0\}$, we have $\operatorname{WF}(B\psi_0 v) \cap \operatorname{Char}(\mathcal{P}^*) = \varnothing$ so microellipticty gives $\|B\psi_0 v\|_{H^{1-s}} \le C\|\psi v\|_{H^{-N}}$. The desired estimate is obtained. □

Lemma 6.40. *For $\mathcal{P}_\lambda$ with $\lambda \in \mathbb{R}$ acting on $\overline{H}^s(X_{cs};\mathcal{E})$, the kernel of $\mathcal{P}_\lambda$ is trivial for $\lambda \gg 1$.*

Proof. Consider $u \in \ker \mathcal{P}_\lambda$. By the estimate obtained in Lemma 6.38, $u \in C^\infty(\overline{X_{cs}};\mathcal{E})$. Restricting our attention to $\{\mu > 0\}$, Proposition 6.31 gives

$$\rho^{-\lambda-\frac{n}{2}+m-2} J\, \mathcal{Q}_\lambda\, J^{-1} \rho^{\lambda+\frac{n}{2}-m} u = 0$$

so defining $\tilde{u} = J^{-1}\rho^{\lambda+\frac{n}{2}-m}u$ we get $\mathcal{Q}_\lambda\, \tilde{u} = 0$. Or by Proposition 6.23,

$$(\nabla^*\nabla + \lambda^2 + \mathcal{D} + \mathcal{G})\tilde{u} = 0.$$

Now $\mathcal{D}$ may be bounded (up to a constant) by ∇ (and $\mathcal{G}$ by a constant as the curvature is bounded on X) so we can find C independent of λ such that

$$|(\mathcal{Q}_\lambda\, \tilde{u}, \tilde{u})_s| \geq C^{-1}\|\nabla\tilde{u}\|_s^2 + (\lambda^2 - C)\|\tilde{u}\|_s^2$$

and taking $\lambda \gg \sqrt{C}$ shows $\tilde{u} = 0$ on $\{\rho > 0\}$. By smoothness, u vanishes on $\{\mu \geq 0\}$ (and so too do all its derivates on Y). Standard hyperbolic estimates give the desired result $u = 0$ if we can show a type of unique continuation result that $u = 0$ on $\{\mu > -\varepsilon\}$.

To this end we work on U^2 and consider $\mathbf{P}$ written in the following form

$$\mathcal{P}_\lambda = -\mu\partial_\mu^2 + \Delta_h + \mathcal{B}_\lambda$$

for $\mathcal{B}_\lambda = -4\lambda\partial_\mu + \mathcal{A}_\lambda \in \mathrm{Diff}^1(U^2;\mathrm{End}\,\mathcal{E})$. Let $\langle\cdot,\cdot\rangle_{h,t}$ on $\mathrm{T}^*Y \otimes \mathcal{E}$ denote the coupling of the metrics h on T^*Y with $\langle\cdot,\cdot\rangle_t$ on $\mathcal{E}$. For ease of presentation, we will assume throughout this demonstration that all objects are real-valued. Consider $u, v \in C_c^\infty(U^2,\mathcal{E})$ (and we may assume $\operatorname{supp} u \subset (-1,0] \times Y$) then we have the following formula

$$\langle {}^Y\nabla u, {}^Y\nabla v\rangle_{h,t} = \langle \Delta_h u, v\rangle_t + \mathrm{div}$$

where div denotes any term which is of divergence nature on Y, hence vanishes upon integrating over Y (using $d\mathrm{vol}_h$). Indeed such an equation is obtained by considering $f \in C^\infty(Y)$ and calculating, at some value μ,

$$\begin{aligned}\int_Y \langle {}^Y\nabla u, {}^Y\nabla v\rangle_{h,t} f d\mathrm{vol}_h &= \int_Y \langle {}^Y\nabla u, {}^Y\nabla(fv)\rangle_{h,t} - \langle {}^Y\nabla u, {}^Y\nabla f \otimes v\rangle_{h,t}\, d\mathrm{vol}_h \\ &= \int_Y (\langle \Delta_h u, v\rangle_t + \mathrm{div}) f\, d\mathrm{vol}_h\end{aligned}$$

where the second term was dealt with in the following way:

$$\begin{aligned}\int_Y \langle {}^Y\nabla u, {}^Y\nabla f \otimes v\rangle_{h,t}\, d\mathrm{vol}_h &= \int_Y \sum_i \langle {}^Y\nabla_{e_i} u, v\rangle_t \,\mathrm{tr}_h(e^i \otimes {}^Y\nabla f)\, d\mathrm{vol}_h \\ &= \int_Y {}^Y\nabla^*(\sum_i \langle {}^Y\nabla_{e_i} u, v\rangle_t e^i) f\, d\mathrm{vol}_h.\end{aligned}$$

With this formula established we define, for given u,

$$\mathcal{H}(\mu) = |\mu|\langle \partial_\mu u, \partial_\mu u\rangle_t + \langle {}^Y\nabla u, {}^Y\nabla u\rangle_{h,t} + \langle u, u\rangle_t$$

and on $\{\mu < 0\}$ (using $v = \partial_\mu u$ in the previously established formula)

$$-\partial_\mu \mathcal{H} = -2\langle \mathcal{P}\, u, \partial_\mu u\rangle_t + \langle (2\mathcal{B}_\lambda - \partial_\mu)u, \partial_\mu u\rangle_t + \operatorname{div} - \tilde{\mathcal{H}}.$$

where $\tilde{\mathcal{H}}$ has the same structure as $\mathcal{H}$ but with appearances of h (used to construct the various inner products) replaced by its Lie derivative, $\mathcal{L}_{\partial_\mu} h$. Recall that $\operatorname{supp} u \subset (-1, 0] \times Y$ and u is smooth, hence $\partial_\mu^N u = 0$ at $\{\mu = 0\}$ for all N. Continuing to work on $\{\mu < 0\}$,

$$\begin{aligned}-\partial_\mu(|\mu|^{-N}\mathcal{H}) &+ |\mu|^{-N}\operatorname{div}\\ &= -N|\mu|^{-N-1}\mathcal{H} - 2|\mu|^{-N}\operatorname{Re}\langle \mathcal{P}_\lambda\, u, \partial_\mu u\rangle_t + |\mu|^{-N}\langle (2\mathcal{B}_\lambda - \partial_\lambda)u, \partial_\mu u\rangle_t - |\mu|^{-N}\tilde{\mathcal{H}}.\end{aligned}$$

Now suppose that $u \in \ker \mathcal{P}_\lambda$. Fix $\delta > 0$ small and let $0 < \varepsilon < \delta$. We take the previous display and insert it into the operator $\int_{-\delta}^{-\varepsilon}\int_Y \dots\, d\mu\, d\mathrm{vol}_h$. The first term on the left hand side of the previous display is treated with the fundamental theorem of calculus, the second term vanishes due to the appearance of $\int_Y \operatorname{div} d\mathrm{vol}_h$. We claim the right hand side is negative for large N. Indeed the second term vanishes as u is assumed in the kernel of $\mathcal{P}_\lambda$. Considering the third term, $\langle (2\mathcal{B}_\lambda - \partial_\lambda)u, \partial_\mu u\rangle_t$ is quadratic in u, ${}^Y\nabla u$, and $\partial_\mu u$ hence for N large enough, it may be bounded by $N|\mu|^{-1}\mathcal{H}$, thus the third term's potential positivity may be absorbed by the negativity of the first term. The fourth term may be treated in a similar manner upon consideration of the Taylor expansion of h at Y. We obtain

$$\delta^{-N}\int_Y \mathcal{H}(-\delta)\, \mathrm{dvol}_h \le \varepsilon^{-N}\int_Y \mathcal{H}(-\varepsilon)\, \mathrm{dvol}_h.$$

As u is smooth and vanishes to all orders at $\mu = 0$, we may bound $\int_Y \mathcal{H}(-\varepsilon)\, \mathrm{dvol}_h$ by $C|\mu|^K$ on $[-\varepsilon, 0]$ for arbitrarily large K. We can obtain a similar bound for $\int_Y \mathcal{H}(-\varepsilon)\, \mathrm{dvol}_h$. In particular, for $K > N$. This produces

$$\delta^{-N}\int_Y \mathcal{H}(-\delta)\, \mathrm{dvol}_h \le C\varepsilon^{-N+K}$$

and letting $\varepsilon \to 0^+$ shows $\int_Y \mathcal{H}(-\delta)\, \mathrm{dvol}_h = 0$ hence $\mathcal{H}(-\delta) = 0$. Doing this for all δ less than the original δ gives $\mathcal{H} = 0$ near 0. Hence $\partial_\mu u$ and $\nabla^Y u$ vanish and $u = 0$ near 0. This suffices to conclude the proof. □

Lemma 6.41. *For $\mathcal{P}_\lambda^*$ with $\lambda \in \mathbb{R}$ acting on $\dot{H}^{1-s}(\overline{X_{cs}}; \mathcal{E})$, the kernel of $\mathcal{P}_\lambda^*$ is trivial for $\lambda \gg 1$.*

Proof. Take λ satisfying the threshold condition and consider $v \in \ker \mathcal{P}_\lambda^*$. Hyperbolicity, as used in Lemma 6.39, implies $v = 0$ on $\{\mu \le 0\}$, and that v is smooth on X due to ellipticity. The strategy given in Lemma 6.39 implies $v \in \dot{H}^{\tilde{s}}(\overline{X_{cs}}; \mathcal{E})$ for all $\tilde{s} < \lambda + \frac{1}{2}$ which with $\lambda \gg n$ implies v is continuous. By the same logic, again by taking λ sufficiently large, we may assume v is regular enough to conclude $\partial_\mu^N v_{|Y} = 0$ for $N \le \frac{1}{2}\lambda$. Equivalently, $v_{|X} \in \rho^{2N} C^\infty_{\mathrm{even}}(\overline{X}; \mathcal{E})$. Meanwhile, direct calculations on $C^\infty(X; \mathcal{E})$ give

$$\begin{aligned}\rho^N \nabla^*\nabla \rho^{-N} &= \nabla^*\nabla - N^2 - N(\Delta \log \rho) + 2N\nabla_{\rho\partial_\rho},\\ \rho^N \,\mathrm{d}\, \rho^{-N} &= d - N\tfrac{d\rho}{\rho}\,\cdot,\\ \rho^N\, \delta\, \rho^{-N} &= \delta + N\tfrac{d\rho}{\rho}\,\lrcorner\end{aligned}$$

where $\Delta \log \rho = n - (\frac{1}{2}\sum_{ij} h^{ij}\rho\partial_\rho h_{ij}) \in n - \rho^2 C^\infty_{\mathrm{even}}(\overline{X};\mathcal{E})$. Also for $\tilde{u} \in C^\infty_c(X;\mathcal{E})$ we have

$$|(2N\nabla_{\rho\partial_\rho}\tilde{u},\tilde{u})_s| = |N\int_X \|u\|_s^2 \partial_\rho \left(\tfrac{d\rho\, d\mathrm{vol}_h}{\rho^n}\right)| \le CN\|u\|_s^2.$$

So consider the difference operator $(\mathcal{Q}_\lambda - N^2 + 2N\nabla_{\rho\partial_\rho}) - \rho^N\,\mathcal{Q}_\lambda\,\rho^{-N}$ acting on $\tilde{u} \in C^\infty_c(X;\mathcal{E})$. All terms are of order N and of differential order 0. Similar to the previous proof (and using the preceding remark in order to treat the term involving $N\nabla_{\rho\partial_\rho}$) we may obtain

$$|(\rho^N\,\mathcal{Q}_\lambda\,\rho^{-N}\tilde{u},\tilde{u})_s| \ge C^{-1}\|\nabla\tilde{u}\|_s^2 + (\lambda^2 - N^2 - CN)\|\tilde{u}\|_s^2$$

and provided $N \gg C$, the final term in the preceding display may be written with coefficient $\lambda^2 - 2N^2$. Set $N = \lfloor \frac{1}{2}\lambda \rfloor$ with $\lambda \gg 2C$. So that

$$|(\rho^N\,\mathcal{Q}_\lambda\,\rho^{-N}\tilde{u},\tilde{u})_s| \ge C^{-1}\|\nabla\tilde{u}\|_s^2 + \tfrac{1}{2}\lambda^2\|\tilde{u}\|_s^2.$$

Considering the Hilbert space $\{w \in L^2_s(X;\mathcal{E}) : B(w,w) < \infty\}$ with $B(w,w) = \|\rho^N\,\mathcal{Q}_\lambda\,\rho^{-N}w\|_s^2 < \infty$, the previous inequality shows that $w \mapsto (w,\tilde{f})_s$ is a linear functional for $\tilde{f} \in L^2_s(X;\mathcal{E})$ so by the Riesz representation theorem, there exists $\tilde{u} \in L^2_s(X;\mathcal{E})$ with $(\rho^N\,\mathcal{Q}_\lambda\,\rho^{-N}w,\tilde{u})_s = (w,\tilde{f})_s$ for all w. To show v vanishes on X, it suffices to show $(f,v)_t = 0$ for all $f \in C^\infty_c(X;\mathcal{E})$. Let $f \in C^\infty_c(X;\mathcal{E})$ and

$$\tilde{f} = \rho^{\lambda+\frac{n}{2}-m+2}J^{-1}\rho^{-N}f \in C^\infty_c(X;\mathcal{E})$$

Then the preceding argument gives $\tilde{u} \in L^2_s(X;\mathcal{E})$ such that $\rho^{-N}\,\mathcal{Q}_\lambda\,\rho^N\tilde{u} = \tilde{f}$ hence $\mathcal{P}_\lambda\,u = f$ where

$$u = J\rho^{-\lambda-\frac{n}{2}+m}\rho^N\tilde{u} \in \rho^{-\frac{1}{2}\lambda+1}L^2_t(X;\mathcal{E})$$

(the inclusion is a consequence of Lemma 6.2). This gives u enough regularity to perform the following pairing which provides the desired result

$$(f,v)_t = (\mathcal{P}_\lambda\,u,v)_t = (u,\mathcal{P}^*_\lambda\,v)_t = (u,0)_t = 0. \qquad \square$$

6.6 Proofs of Theorems 6, 7, 8, and 9

Theorem 6. *Let (X^{n+1},g) be even asymptotically hyperbolic. Then the inverse of*

$$\mathcal{Q}_\lambda \;\; \textit{acting on} \;\; L^2_s(X;\mathcal{E})$$

written $\mathcal{Q}_\lambda^{-1}$ has a meromorphic continuation from $\operatorname{Re}\lambda \gg 1$ *to* $\mathbb{C}$,

$$\mathcal{Q}_\lambda^{-1} : C^\infty_c(X;\mathcal{E}) \to \rho^{\lambda+\frac{n}{2}-m}\bigoplus_{k=0}^{m}\rho^{-2k}C^\infty_{\mathrm{even}}(\overline{X};\mathcal{E}^{(k)})$$

with finite rank poles.

Proof. Proposition 6.31 gives

$$\mathcal{Q}_\lambda = J^{-1}\rho^{\lambda+\frac{n}{2}-m+2}\,\mathcal{P}_\lambda\,\rho^{-\lambda-\frac{n}{2}+m}J$$

By Propositions 6.35 and 6.36, there is a meromorphic family $\mathcal{P}^{-1}$ on $\mathbb{C}$ mapping $\dot{C}^\infty(X;\mathcal{E})$ to

$C^\infty(X_{cs}; \mathcal{E})$. Hence an extension of $\mathcal{Q}^{-1}$ from $\operatorname{Re}\lambda \gg 1$ to all of $\mathbb{C}$ as a meromorphic family is given by

$$\mathcal{Q}_\lambda^{-1} = J^{-1} \rho^{\lambda + \frac{n}{2} - m} r_X \, \mathcal{P}_\lambda^{-1} \, \rho^{-\lambda - \frac{n}{2} + m - 2} J$$

where r_X is the restriction of sections above X_{cs} to sections above X. The previous display implies

$$\mathcal{Q}_\lambda^{-1} : \dot{C}^\infty(X; \mathcal{E}) \to \rho^{\lambda + \frac{n}{2} - m} J^{-1} C^\infty_{\text{even}}(\overline{X}; \mathcal{E})$$

and for $f \in \dot{C}^\infty(X; \mathcal{E})$, we may write near $\partial\overline{X}$

$$\mathcal{Q}_\lambda^{-1} f_{|U} = \mu^{\frac{\lambda}{2} + \frac{n}{4} - \frac{m}{2}} J^{-1} \sum_{k=0}^{m} \sum_{\ell=0}^{k} (d\mu)^{k-\ell} \cdot \tilde{u}^{(\ell)}, \qquad \tilde{u}^{(\ell)} \in C^\infty_{\text{even}}([0,1) \times Y; \operatorname{Sym}^\ell \mathrm{T}^* Y)$$

The proof of Lemma 6.2 shows that the part of J (or J^{-1}) which sends $\mathcal{E}^{(k)}$ to $\mathcal{E}^{(k+p)}$ for $0 \leq p \leq m-k$ is, up to a constant, $(\frac{d\mu}{\mu})^p$. Therefore,

$$\mathcal{Q}_\lambda^{-1} f_{|U} \in \mu^{\frac{\lambda}{2} + \frac{n}{4} - \frac{m}{2}} \bigoplus_{k=0}^{m} \bigoplus_{p=0}^{m-k} (\tfrac{d\mu}{\mu})^p \cdot \bigoplus_{\ell=0}^{k} (d\mu)^{k-\ell} \cdot C^\infty_{\text{even}}([0,1) \times Y; \operatorname{Sym}^\ell \mathrm{T}^* Y)$$

hence on X,

$$\mathcal{Q}_\lambda^{-1} f \in \rho^{\lambda + \frac{n}{2} - m} \bigoplus_{k=0}^{m} \bigoplus_{p=0}^{m-k} \rho^{-2p} C^\infty_{\text{even}}(\overline{X}; \mathcal{E}^{(k+p)})$$

which is contained in $\rho^{\lambda + \frac{n}{2} - m} \bigoplus_{k=0}^{m} \rho^{-2k} C^\infty_{\text{even}}(\overline{X}; \mathcal{E}^{(k)})$. □

Remark 6.42. Suppose that, for $f \in \dot{C}^\infty(X; \mathcal{E})$, it were possible to write in the preceeding proof that near $\partial\overline{X}$

$$\mathcal{Q}_\lambda^{-1} f_{|U} = \rho^{\lambda + \frac{n}{2} - m} J^{-1} \tilde{u}^{(m)}, \qquad \tilde{u}^{(m)} \in C^\infty_{\text{even}}(\overline{U}; \mathcal{E}^{(m)})$$

then as J^{-1} acts as the identity upon restriction to $\mathcal{E}^{(m)}$, we would obtain

$$\mathcal{Q}_\lambda^{-1} f \in \rho^{\lambda + \frac{n}{2} - m} C^\infty_{\text{even}}(\overline{X}; \mathcal{E}^{(m)})$$

This will be useful for the asymptotics given in Theorems 8 and 9.

Theorem 7. *Let (X^{n+1}, g) be even asymptotically hyperbolic. Then the inverse of*

$$\mathcal{Q}_\lambda \text{ acting on } L^2_s(X; \mathcal{E}) \cap \ker(\Lambda_\eta \circ \pi_s^*)$$

written $\mathcal{Q}_\lambda^{-1}$ has a meromorphic continuation from $\operatorname{Re}\lambda \gg 1$ to $\mathbb{C}$,

$$\mathcal{Q}_\lambda^{-1} : C^\infty_c(X; \mathcal{E}) \cap \ker(\Lambda_\eta \circ \pi_s^*) \to \rho^{\lambda + \frac{n}{2} - m} \left(\bigoplus_{k=0}^{m} \rho^{-2k} C^\infty_{\text{even}}(\overline{X}; \mathcal{E}^{(k)}) \right) \cap \ker(\Lambda_\eta \circ \pi_s^*)$$

with finite rank poles.

Proof. The meromorphic inverse of $\mathcal{Q}_\lambda$ is precisely that given in the preceding proof

$$\mathcal{Q}_\lambda^{-1} = J^{-1}\rho^{\lambda+\frac{n}{2}-m} r_X\, \mathcal{P}_\lambda^{-1}\, \rho^{-\lambda-\frac{n}{2}+m-2} J.$$

All we must check is, given $f \in \dot{C}^\infty(X;\mathcal{E}) \cap \ker(\Lambda_{s^{-2}\eta} \circ \pi_s^*)$, that the resulting section $u = \mathcal{Q}_\lambda^{-1} f$ is indeed trace-free with respect to the ambient trace operator. To this end, we first lift the equation $\mathcal{Q}_\lambda u = f$ to an equation on M involving $\mathbf{Q}$ giving

$$s^\lambda\, \mathbf{Q}\, s^{-\lambda}\, (\pi_s^* u) = \pi_s^* f.$$

We apply $\Lambda_{s^{-2}\eta}$ to obtain an equation on $\mathcal{F}^{(m-2)}$. Using the hypothesis $\Lambda_{s^{-2}\eta}\pi_s^* f = 0$ and Lemma 6.16 to commute $s^{-2}\Lambda_{s^{-2}\eta}$ with $\mathbf{Q}$ gives

$$s^2 s^\lambda\, \mathbf{Q}\, s^{-\lambda} s^{-2}\Lambda_{s^{-2}\eta}\, (\pi_s^* u) = 0.$$

Freezing this differential equation at $s = 0$ with $\pi_{s=0}$ to obtain the indicial family of $\mathbf{Q}$ provides the equation

$$\mathrm{I}_s(\mathbf{Q}, \lambda + 2)\pi_{s=0}\Lambda_{s^{-2}\eta}\, (\pi_s^* u) = 0.$$

Section 6.5 ensures that for $\operatorname{Re}\lambda \gg 1$, this operator has trival kernel hence

$$\pi_{s=0}\Lambda_{s^{-2}\eta}\, (\pi_s^* u) = 0$$

and $u \in \ker(\Lambda_{s^{-2}\eta} \circ \pi_s^*)$ as required. □

We are finally in a position to consider the original problem of proving Theorems 8 and 9. We prove these theorems simulaneously.

Theorem 8. *Let (X^{n+1}, g) be even asymptotically hyperbolic and Einstein. Then the inverse of*

$$\Delta - \frac{n(n-8)}{4} + \lambda^2 \text{ acting on } L^2(X;\mathcal{E}^{(2)}) \cap \ker\Lambda \cap \ker\delta$$

written $\mathcal{R}_\lambda$ has a meromorphic continuation from $\operatorname{Re}\lambda \gg 1$ to $\mathbb{C}$,

$$\mathcal{R}_\lambda : C_c^\infty(X;\mathcal{E}^{(2)}) \cap \ker\Lambda \cap \ker\delta \to \rho^{\lambda+\frac{n}{2}-2} C^\infty_{\mathrm{even}}(\overline{X};\mathcal{E}^{(2)}) \cap \ker\Lambda \cap \ker\delta$$

with finite rank poles.

Theorem 9. *Let (X^{n+1}, g) be a convex cocompact quotient of $\mathbb{H}^{n+1}$. Then the inverse of*

$$\Delta - \frac{n^2 - 4m(n+m-2)}{4} + \lambda^2 \text{ acting on } L^2(X;\mathcal{E}^{(m)}) \cap \ker\Lambda \cap \ker\delta$$

written $\mathcal{R}_\lambda$ has a meromorphic continuation from $\operatorname{Re}\lambda \gg 1$ to $\mathbb{C}$,

$$\mathcal{R}_\lambda : C_c^\infty(X;\mathcal{E}^{(m)}) \cap \ker\Lambda \cap \ker\delta \to \rho^{\lambda+\frac{n}{2}-m} C^\infty_{\mathrm{even}}(\overline{X};\mathcal{E}^{(m)}) \cap \ker\Lambda \cap \ker\delta$$

with finite rank poles.

Proof. Let

$$f \in \dot{C}^\infty(X; \mathcal{E}^{(m)}) \cap \ker \Lambda \cap \ker \delta$$

and define, using Theorem 6,

$$u = \sum_{k=0}^{m} u^{(k)} = \mathcal{Q}_\lambda^{-1} f, \qquad u^{(k)} \in \rho^{\lambda + \frac{n}{2} - m - 2k} C^\infty_{\text{even}}(\overline{X}; \mathcal{E}^{(k)}).$$

Note that the growth near $\partial\overline{X}$ of $u^{(k)}$ and $\delta\, u^{(k)}$ may be controlled by the size of $\operatorname{Re}\lambda$ hence for $\operatorname{Re}\lambda \gg 1$ we may assume that they are sections of $L^2_s(X; \mathcal{E}^{(k)})$ and $L^2_s(X; \mathcal{E}^{(k-1)})$ respectively. We claim, for $\operatorname{Re}\lambda \gg 1$ and $|\operatorname{Im}\lambda| \ll 1$, that

$$u = u^{(m)} \in \rho^{\lambda + \frac{n}{2} - m} C^\infty_{\text{even}}(\overline{X}; \mathcal{E}^{(k)}) \cap \ker \Lambda \cap \ker \delta$$

at which point the equation $\mathcal{Q}_\lambda\, u = f$ decouples giving

$$(\Delta + \lambda^2 - c_m) u = f$$

and by uniqueness of the L^2 inverse of the Laplacian, we have the formula, for $\operatorname{Re}\lambda \gg 1$ and $|\operatorname{Im}\lambda| \ll 1$,

$$(\Delta + \lambda^2 - c_m)^{-1} = J^{-1} \rho^{\lambda + \frac{n}{2} - m} r_X\, \mathcal{P}_\lambda^{-1}\, \rho^{-\lambda - \frac{n}{2} + m - 2} J.$$

with the right hand side giving the meromorphic extension of the resolvent announced in the theorems.

To this end take $\operatorname{Re}\lambda \gg 1$ and $|\operatorname{Im}\lambda| \ll 1$. By Theorem 7, we deduce u is trace-free with respect to the ambient trace operator thus $\mathcal{Q}_\lambda$ takes the form detailed in Proposition 6.25. We begin by remarking, that while working on $L^2_s(X; \mathcal{E}^{(k)})$ if $\mathcal{R}_\lambda^{(k)}$ is any operator of the form $(\Delta + \lambda^2 + O(1))^{-1}$ (which has order $O(|\lambda|^{-2})$, then the operator $\mathrm{d}\, \mathcal{R}_\lambda^{(k)}\, \delta$ has norm of order $O(1)$. We define $\mathcal{R}_\lambda^{(0)} = (\Delta + \lambda^2 - c_0')^{-1}$ and for $0 < k < m$,

$$\mathcal{R}_\lambda^{(k)} = \Big(\Delta + \lambda^2 - c_k' + 4(m - k + 1)\, \mathrm{d}\, \mathcal{R}_\lambda^{(k-1)}\, \delta\Big)^{-1}.$$

The component of $\mathcal{Q}_\lambda\, u = f$ in $\mathcal{E}^{(0)}$ reads,

$$(\Delta + \lambda^2 - c_0') u^{(0)} = 2\sqrt{m}\, \delta\, u^{(1)}$$

hence $u^{(0)} = 2\sqrt{m}\, \mathcal{R}_\lambda^{(0)}\, \delta\, u^{(1)}$. The component of $\mathcal{Q}_\lambda\, u = f$ in $\mathcal{E}^{(1)}$ now reads,

$$(\Delta + \lambda^2 - c_1' + 4m\, \mathrm{d}\, \mathcal{R}_\lambda^{(0)}\, \delta) u^{(1)} = 2\sqrt{m-1}\, \delta\, u^{(2)}$$

hence $u^{(1)} = 2\sqrt{m-1}\, \mathcal{R}_\lambda^{(1)}\, \delta\, u^{(2)}$. Continuing, we obtain on $\mathcal{E}^{(m)}$,

$$(\Delta + \lambda^2 - c_m + 4\, \mathrm{d}\, \mathcal{R}_\lambda^{(m-1)}\, \delta) u^{(m)} = f.$$

Applying the divergence, we recall Lemma 6.13. For this, we must assume that if $m = 2$ then X has parallel Ricci curvature, and if $m \geq 3$ then X is locally isomorphic to $\mathbb{H}^{n+1}$. We obtain,

$$(\Delta + \lambda^2 - c_m + 4\,\delta\,\mathrm{d}\,\mathcal{R}_\lambda^{(m-1)})\,\delta\, u^{(m)} = 0.$$

Again, $\delta\,\mathrm{d}\,\mathcal{R}_\lambda^{(m-1)}$ has norm of order $O(1)$ so we may invert this equation and deduce that $\delta\, u^{(m)} = 0$. This implies, for all $k < m$,

$$u^{(k)} = 2\sqrt{m-k}\,\mathcal{R}_\lambda^{(k)}\,\delta\, u^{k+1} = 0.$$

Therefore $u = u^{(m)}$. By Remark 6.42, $u \in \rho^{\lambda+\frac{n}{2}-m} C^\infty_{\mathrm{even}}(\overline{X}; \mathcal{E}^{(m)})$. By Theorem 7, $u \in \ker \Lambda$. And as previously mentioned $u \in \ker \delta$. This completes the proof. □

6.7 Symmetric Cotensors of Rank 2

This section details the results announced in Sections 6.3 and 6.6 for rank 2 symmetric cotensors. In this low rank, writing the action of the d'Alembertian, or its conjugation $\mathbf{Q}$, on $\mathcal{F} = \mathrm{Sym}^2\mathrm{T}^*M$ is tractable.

The operator Q for 2-cotensors

Using the decomposition given by the Minkowksi scale, we write

$$u = \begin{bmatrix} 1 & \frac{ds}{s}\cdot & \frac{1}{\sqrt{2}}(\frac{ds}{s})^2 \end{bmatrix} \begin{bmatrix} u^{(2)} \\ u^{(1)} \\ u^{(0)} \end{bmatrix}, \qquad u \in C^\infty(M;\mathcal{F}), u^{(k)} \in C^\infty(M; \mathcal{E}^{(k)})$$

The change of basis matrix J takes the form

$$J = \begin{bmatrix} 1 & \frac{d\rho}{\rho}\cdot & \frac{1}{\sqrt{2}}(\frac{d\rho}{\rho})^2 \\ 0 & 1 & \sqrt{2}\frac{d\rho}{\rho}\cdot \\ 0 & 0 & 1 \end{bmatrix}.$$

Propositions 6.17 and 6.21 become

Proposition 6.43. *For $u \in C^\infty(M;\mathcal{F})$ decomposed relative to the Minkowski scale* (6.1)*, the conjugated d'Alembertian $\mathbf{Q}$ is given by*

$$\mathbf{Q}\, u = \begin{bmatrix} 1 & \frac{ds}{s}\cdot & \frac{1}{\sqrt{2}}(\frac{ds}{s})^2 \end{bmatrix} \begin{bmatrix} \Delta + (s\partial_s)^2 - c_2 - \mathrm{L}\,\Lambda & 2\,\mathrm{d} & -\sqrt{2}\,\mathrm{L} \\ -2\,\delta & \Delta + (s\partial_s)^2 - c_1 & 2\sqrt{2}\,\mathrm{d} \\ -\sqrt{2}\,\Lambda & -2\sqrt{2}\,\delta & \Delta + (s\partial_s)^2 - c_0 \end{bmatrix} \begin{bmatrix} u^{(2)} \\ u^{(1)} \\ u^{(0)} \end{bmatrix}$$

with constants

$$c_2 = \tfrac{1}{4}n(n-8), \qquad c_1 = \tfrac{1}{4}(n^2+16), \qquad c_0 = \tfrac{1}{4}(n^2+8n+8).$$

If, furthermore, u is trace-free with respect to the trace operator $\Lambda_{s^{-2}\eta}$, then $\Lambda\, u^{(2)} = -\sqrt{2}u^{(0)}$, and

$$\mathbf{Q}\, u = \begin{bmatrix} 1 & \frac{ds}{s}\cdot & \frac{1}{\sqrt{2}}(\frac{ds}{s})^2 \end{bmatrix} \begin{bmatrix} \Delta + (s\partial_s)^2 - c_2' & 2\,\mathrm{d} & 0 \\ -2\,\delta & \Delta + (s\partial_s)^2 - c_1' & 2\sqrt{2}\,\mathrm{d} \\ 0 & -2\sqrt{2}\,\delta & \Delta + (s\partial_s)^2 - c_0' \end{bmatrix} \begin{bmatrix} u^{(2)} \\ u^{(1)} \\ u^{(0)} \end{bmatrix}$$

with modified constants

$$c_2' = c_2, \qquad c_1' = c_1, \qquad c_0' = \tfrac{1}{4}(n^2 + 8n).$$

The indicial family of Q for 2-cotensors

Propositions 6.23 and 6.25 become

Proposition 6.44. *For $u = \sum_{k=0}^{2} u^{(k)} \in C^\infty(X;\mathcal{E})$ the operator $\mathcal{Q}$ is given by*

$$\mathcal{Q}_\lambda\, u = \begin{bmatrix} \Delta + \lambda^2 - c_2 - \mathrm{L}\,\Lambda & 2\,\mathrm{d} & -\sqrt{2}\,\mathrm{L} \\ -2\,\delta & \Delta + \lambda^2 - c_1 & 2\sqrt{2}\,\mathrm{d} \\ -\sqrt{2}\,\Lambda & -2\sqrt{2}\,\delta & \Delta + \lambda^2 - c_0 \end{bmatrix} \begin{bmatrix} u^{(2)} \\ u^{(1)} \\ u^{(0)} \end{bmatrix}$$

and if, furthermore, $u \in \ker(\Lambda_{s^{-2}\eta} \circ \pi_s^*)$ then

$$\mathcal{Q}_\lambda\, u = \begin{bmatrix} \Delta + \lambda^2 - c_2' & 2\,\mathrm{d} & 0 \\ -2\,\delta & \Delta + \lambda^2 - c_1' & 2\sqrt{2}\,\mathrm{d} \\ 0 & -2\sqrt{2}\,\delta & \Delta + \lambda^2 - c_0' \end{bmatrix} \begin{bmatrix} u^{(2)} \\ u^{(1)} \\ u^{(0)} \end{bmatrix}$$

with previously announced constants.

Illustration of proof for 2-cotensors

Let $f \in \dot{C}^\infty(X;\mathcal{E}^{(2)}) \cap \ker\Lambda \cap \ker\delta$ and define

$$\begin{bmatrix} u^{(2)} \\ u^{(1)} \\ u^{(0)} \end{bmatrix} = J^{-1}\rho^{\lambda+\frac{n}{2}-2} r_X\, \mathcal{P}^{-1}\, \rho^{-\lambda-\frac{n}{2}} J \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix}$$

Take $\mathrm{Re}\,\lambda \gg 1$ and $|\,\mathrm{Im}\,\lambda| \ll 1$. By Theorem 6

$$u^{(k)} \in \rho^{\lambda+\frac{n}{2}-2-2k} C^\infty_{\mathrm{even}}(\overline{X};\mathcal{E}^{(k)})$$

and by Proposition 6.31, $\mathcal{Q}_\lambda\, u = f$. Theorem 7 forces

$$\Lambda_{s^{-2}\eta}\left(u^{(2)} + \tfrac{ds}{s}\cdot u^{(1)} + \tfrac{1}{\sqrt{2}}(\tfrac{ds}{s})^2.u^{(0)}\right) = 0$$

hence $\Lambda\, u^{(2)} = -\sqrt{2}u^{(0)}$. And $\mathcal{Q}_\lambda\, u = f$ reads explicitly

$$\begin{bmatrix} \Delta + \lambda^2 - c_2 & 2\,\mathrm{d} & 0 \\ -2\,\delta & \Delta + \lambda^2 - c_1 & 2\sqrt{2}\,\mathrm{d} \\ 0 & -2\sqrt{2}\,\delta & \Delta + \lambda^2 - c_0' \end{bmatrix} \begin{bmatrix} u^{(2)} \\ u^{(1)} \\ u^{(0)} \end{bmatrix} = \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix}$$

Introducing the resolvents $\mathcal{R}^{(0)}_\lambda$ and $\mathcal{R}^{(1)}_\lambda$ provides

$$\begin{bmatrix} \Delta+\lambda^2-c_2+4\,\mathrm{d}\,\mathcal{R}^{(1)}_\lambda\,\delta & 0 & 0 \\ -2\,\delta & \Delta+\lambda^2-c_1+8\,\mathrm{d}\,\mathcal{R}^{(0)}_\lambda\,\delta & 0 \\ 0 & -2\sqrt{2}\,\delta & \Delta+\lambda^2-c_0' \end{bmatrix}\begin{bmatrix} u^{(2)} \\ u^{(1)} \\ u^{(0)} \end{bmatrix} = \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix}$$

and applying δ assuming that X is Einstein provides the homogeneous equation

$$\begin{bmatrix} \Delta+\lambda^2-c_2+4\,\delta\,\mathrm{d}\,\mathcal{R}^{(1)}_\lambda & 0 & 0 \\ -2\,\delta & \Delta+\lambda^2-c_1+8\,\delta\,\mathrm{d}\,\mathcal{R}^{(0)}_\lambda & 0 \\ 0 & -2\sqrt{2}\,\delta & \Delta+\lambda^2-c_0' \end{bmatrix}\begin{bmatrix} \delta\,u^{(2)} \\ \delta\,u^{(1)} \\ \delta\,u^{(0)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The lower triangular nature of this system implies $\delta\,u^{(k)}=0$ for all k. Hence the system $\mathcal{Q}_\lambda\,u=f$ collapses. So $u^{(0)}$ and $u^{(1)}$ vanish and by Remark 6.42,

$$u=u^{(2)}\in\rho^{\lambda+\frac{n}{2}-2}C^\infty_{\mathrm{even}}(\overline{X};\mathcal{E}^{(k)})$$

giving $(\Delta+\lambda^2-c_2)u=f$.

6.8 High Energy Estimates of Theorem 10

This chapter shows the meromorphic continuation of the resolvent of the Laplacian on symmetric tensors using microlocal techniques. This direction means one does not talk about introducing complex absorbers but rather studies the problem on a manifold with boundary. If one were to follow more closely the track established by Vasy, one obtains semiclassical estimates. We state these estimates.

On X, whose smooth structure at infinity is the even structure given by μ rather than ρ, we have the semiclassical spaces $H^s_{|\lambda|^{-1}}(X;\mathcal{E})$.

Theorem 10. *Suppose that X is an even asymptotically hyperbolic manifold which is non-trapping. Then the meromorphic continuation, written $\mathcal{Q}_\lambda^{-1}$ of the inverse of $\mathcal{Q}_\lambda$ initially acting on $L^2_s(X;\mathcal{E})$ has non-trapping estimates holding in every strip $|\operatorname{Re}\lambda|<C, |\operatorname{Im}\lambda|\gg 0$: for $s>\frac{1}{2}+C$*

$$\|\rho^{-\lambda-\frac{n}{2}+m}\,\mathcal{Q}_\lambda^{-1}\,f\|_{H^s_{|\lambda|^{-1}}(X;\mathcal{E})}\le C|\lambda|^{-1}\|\rho^{-\lambda-\frac{n}{2}+m-2}f\|_{H^{s-1}_{|\lambda|^{-1}}(X;\mathcal{E})}.$$

If X is furthermore Einstein, then restricting to symmetric 2-cotensors, the meromorphic continuation $\mathcal{R}_\lambda$ of the inverse of

$$\Delta-\frac{n(n-8)}{4}+\lambda^2$$

initially acting on $L^2(X;\mathcal{E}^{(2)})\cap\ker\Lambda\cap\ker\delta$ has non-trapping estimates holding in every strip $|\operatorname{Re}\lambda|<C, |\operatorname{Im}\lambda|\gg 0$: for $s>\frac{1}{2}+C$

$$\|\rho^{-\lambda-\frac{n}{2}+2}\,\mathcal{R}_\lambda\,f\|_{H^s_{|\lambda|^{-1}}(X;\mathcal{E}^{(2)})}\le C|\lambda|^{-1}\|\rho^{-\lambda-\frac{n}{2}}\,f\|_{H^{s-1}_{|\lambda|^{-1}}(X;\mathcal{E}^{(2)})}.$$

□

7. A Quantum-Classical Correspondence

This chapter is structured as follows. Section 7.1 reconsiders hyperbolic space. As this chapter considers strictly the constant curvature case, we introduce numerous objects not present in the setting of asymptotically hyperbolic manifolds, in particular its isometry group. All these objects are present in the original article showing a quantum-classical correspondence [DFG15] however notation has been significantly altered to be consistent both with this thesis and with a more Lie theoretic notation. Section 7.2 recalls a key result providing a definition of Ruelle resonances in open systems [DG16]. It also recalls the band structure of Ruelle resonances due to the Lie algebra commutation relations. Section 7.3 reproves a result on inversion of horosphere operators present in [DFG15, Lemma 4.2]. Proposition 7.10 restates this inversion emphasising the polynomial structure which allows the inversion result to be used in the presence of Jordan blocks. Section 7.4 defines quantum resonances as poles of the meromorphic inverse of the Laplacian obtained in the preceding chapter. Crucially, this section characterises in Lemma 7.12 the asymptotic structure of generalised quantum resonant states. It is an adaption of [GHW16, Proposition 4.1] however it also requires a subtle application of the structure of the operator $\mathcal{Q}_\lambda$ introduced in the previous chapter in order to deal with issues surrounding the divergence-free requirement of tensor valued resonant states. Section 7.5 introduces the appropriate boundary distributions and shows that the Poisson operator remains an isomorphism in the convex cocompact setting. Finally, Section 7.6 provides the proof of Theorem 11.

7.1 Hyperbolic Space

We recall the hyperbolic space as a submanifold of Minkowski space, introducing structures present in this constant curvature case. Enumerate the canonical basis of $\mathbb{R}^{1,n+1}$ by $e_0, \dots e_{n+1}$ and provide $\mathbb{R}^{1,n+1}$ with the indefinite inner product

$$\langle x, y\rangle := -x_0 y_0 + \sum_{i=1}^{n+1} x_i y_i.$$

Hyperbolic space, $\mathbb{H}^{n+1}$, a submanifold of $\mathbb{R}^{1,n+1}$, is

$$\mathbb{H}^{n+1} := \left\{ x \in \mathbb{R}^{1,n+1} \;\middle|\; \langle x, x\rangle = -1, x_0 > 0 \right\}$$

supplied with the Riemannian metric, g, induced from restriction of $\langle \cdot, \cdot \rangle$ and Levi-Civita connection ∇. The unit tangent bundle is

$$S\mathbb{H}^{n+1} := \left\{ (x, \xi) \;\middle|\; x \in \mathbb{H}^{n+1}, \xi \in \mathbb{R}^{1,n+1}, \langle \xi, \xi\rangle = 1, \langle x, \xi\rangle = 0 \right\}$$

Define the projection

$$\pi_S : S\mathbb{H}^{n+1} \to \mathbb{H}^{n+1} : (x, \xi) \mapsto x$$

and denote by

$$\varphi_t : \begin{cases} S\mathbb{H}^{n+1} & \to \quad S\mathbb{H}^{n+1} \\ (x, \xi) & \mapsto \quad (x \cosh t + \xi \sinh t, x \sinh t + \xi \cosh t) \end{cases}$$

the geodesic flow for $t \in \mathbb{R}$ with generator denoted A. That is,

$$A_{(x,\xi)} := (\xi, x).$$

The tangent space $\mathrm{T}S\mathbb{H}^{n+1}$ at (x, ξ) may be written

$$\mathrm{T}_{(x,\xi)}S\mathbb{H}^{n+1} := \left\{ (v_x, v_\xi) \in (\mathbb{R}^{1,n+1})^2 \;\middle|\; \langle x, v_x\rangle = \langle \xi, v_\xi\rangle = \langle x, v_\xi\rangle + \langle \xi, v_x\rangle = 0 \right\}.$$

It has a smooth decomposition, invariant under φ_{t*}

$$\mathrm{T}S\mathbb{H}^{n+1} = E^n \oplus E^s \oplus E^u$$

where

$$\begin{aligned} E^n{}_{(x,\xi)} &:= \{ (v_x, v_\xi) \mid (v_x, v_\xi) \in \mathrm{span}\{(\xi, x)\} \}, \\ E^s{}_{(x,\xi)} &:= \{ (v, -v) \mid \langle x, v\rangle = \langle \xi, v\rangle = 0 \}, \\ E^u{}_{(x,\xi)} &:= \{ (v, v) \mid \langle x, v\rangle = \langle \xi, v\rangle = 0 \} \end{aligned}$$

are respectively called the neutral, stable, unstable bundles (of φ_{t*}). (The latter two also being tangent to the positive and negative horospheres.) The dual space has a similar decomposition

$$\mathrm{T}^*S\mathbb{H}^{n+1} = E^{*n} \oplus E^{*s} \oplus E^{*u}$$

where E^{*n}, E^{*s}, E^{*u} are respectively the dual spaces to E^n, E^u, E^s. (They are the neutral, stable, unstable bundles of φ^*_{-t}.) Explicitly

$$\begin{aligned} E^{*n}{}_{(x,\xi)} &:= \{ (v_x, v_\xi) \mid (v_x, v_\xi) \in \mathrm{span}\{(\xi, x)\} \}, \\ E^{*s}{}_{(x,\xi)} &:= \{ (v, v) \mid \langle x, v\rangle = \langle \xi, v\rangle = 0 \}, \\ E^{*u}{}_{(x,\xi)} &:= \{ (v, -v) \mid \langle x, v\rangle = \langle \xi, v\rangle = 0 \} \end{aligned}$$

so we have canonical identifications

$$E^{n*} \simeq E^n \simeq \mathrm{span}\{A\}, \qquad E^{*s} \simeq E^u, \qquad E^{*u} \simeq E^s.$$

Consider the pullback bundle $\pi_S^*\mathrm{T}\mathbb{H}^{n+1} \to S\mathbb{H}^{n+1}$ equipped with the pullback metric, also denoted g. Define

$$\mathcal{E} := \left\{ (x, \xi, v) \in S\mathbb{H}^{n+1} \times \mathrm{T}_x\mathbb{H}^{n+1} \;\middle|\; \langle \xi, v\rangle = 0 \right\}$$

and

$$\mathcal{F} := \left\{ (x,\xi,v) \in S\mathbb{H}^{n+1} \times \mathrm{T}_x\mathbb{H}^{n+1} \,\middle|\, v \in \mathrm{span}\{\xi\} \right\}$$

so that

$$\pi_S^* \mathrm{T}\mathbb{H}^{n+1} = \mathcal{E} \oplus \mathcal{F}.$$

Appealing to Appendix A, we obtain the bundles $\mathrm{Sym}^m \mathcal{E}^*$ above $S\mathbb{H}^{n+1}$ and Lefschetz-type operators L, Λ.

There are canonical identifications from $\mathcal{E}$ to both E^s and E^u, which we denote by $\theta_\pm$:

$$\begin{matrix} \theta_+ : \mathcal{E} \to E^s : \\ \theta_- : \mathcal{E} \to E^u : \end{matrix} \qquad \theta_{\pm(x,\xi)}(v) := (v, \mp v).$$

Isometry group

The group $\mathrm{SO}(1, n+1)$ of linear transformations of $\mathbb{R}^{1,n+1}$ preserving $\langle \cdot,\cdot \rangle$ provides the group

$$G := \mathrm{SO}_0(1, n+1),$$

the connected component in $\mathrm{SO}(1, n+1)$ of the identity. Denote by $\gamma \cdot x$, multiplication of $x \in \mathbb{R}^{1,n+1}$ by $\gamma \in G$. Denote by E_{ij} is the elementary matrix such that $E_{ij} e_k = e_i \delta_{jk}$ and define the following matrices

$$R_{ij} := E_{ij} - E_{ji}, \qquad P_k := E_{0k} + E_{k0}$$

for $1 \le i,j,k \le n+1$. The Lie algebra, $\mathfrak{g}$, of G is then identified with

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

where

$$\mathfrak{k} := \mathrm{span}\{R_{ij}\}_{1 \le i,j \le n+1} \simeq \mathfrak{so}_{n+1}, \qquad \mathfrak{p} := \mathrm{span}\{P_k\}_{1 \le k \le n+1}.$$

An alternative description of $\mathfrak{g}$ may be obtained by defining

$$A := P_{n+1}, \qquad N_k^\pm := P_k \pm R_{n+1,k}$$

for $1 \le k \le n$. Then

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}_+ + \mathfrak{n}_-$$

where $\mathfrak{a} := \mathrm{span}\{A\}$ and

$$\mathfrak{m} := \mathrm{span}\{R_{ij}\}_{1 \le i,j \le n} \simeq \mathfrak{so}_n, \qquad \mathfrak{n}_\pm := \mathrm{span}\{N_k^\pm\}_{1 \le k \le n}.$$

The matrices introduced enjoy the following commutator relations, for $1 \le i,j \le n$

$$[A, N_i^\pm] = \pm N_i^\pm, \qquad [N_i^\pm, N_j^\pm] = 0, \qquad [N_i^+, N_j^-] = 2A\delta_{ij} + 2R_{ij},$$

while

$$[R_{ij}, A] = 0, \qquad [R_{ij}, N_k^{\pm}] = N_i^{\pm}\delta_{jk} - N_j^{\pm}\delta_{ik}.$$

Remark 7.1. If we define $\mathfrak{a}^{\perp} := \mathfrak{p}/\mathfrak{a}$ whence $\mathfrak{a}^{\perp} \simeq \{P_k\}_{1\le k\le n}$ then we may obtain identifications

$$\theta_{\pm} : \mathfrak{a}^{\perp} \to \mathfrak{n}_{\pm} : P_k \mapsto N_k^{\pm}.$$

Elements of the Lie algebra $\mathfrak{g}$ are identified with left invariant vector fields on G. The Lie algebras $\mathfrak{k}, \mathfrak{m}$ give Lie groups K, M considered subgroups of G. Now G acts transitively on both $\mathbb{H}^{n+1}$ and $S\mathbb{H}^{n+1}$ and the respective isotropy groups, for $e_0 \in \mathbb{H}^{n+1}$ and $(e_0, e_{n+1}) \in S\mathbb{H}^{n+1}$, are precisely K and M. Define projections

$$\pi_K : G \to \mathbb{H}^{n+1} : \gamma \mapsto \gamma \cdot e_0,$$
$$\pi_M : G \to S\mathbb{H}^{n+1} : \gamma \mapsto (\gamma \cdot e_0, \gamma \cdot e_{n+1}).$$

As A commutes with M, it descends to a vector field on $S\mathbb{H}^{n+1}$ via π_{M*}. It agrees with the generator of the geodesic flow justifying the notation. Similarly, the spans of $\{N_k^{+}\}_{1\le k\le n}$ and $\{N_k^{-}\}_{1\le k\le n}$ are each stable under commutation with M and via π_{M*} are respectively identified with the stable and unstable subbundles E^s, E^u

Equivariant sections

It is clear that distributions on $S\mathbb{H}^{n+1}$ may be considered as distributions on G which are annihilated by M. We denote such distributions

$$\mathcal{D}'(G)/\mathfrak{m} := \{ u \in \mathcal{D}'(G) \mid R_{ij}u = 0, 1 \le i, j \le n \}.$$

This is true for more general sections, in particular we have

Lemma 7.2. *Sections $\mathcal{D}'(S\mathbb{H}^{n+1}; \mathrm{Sym}^m \mathcal{E}^*)$ are equivalent to equivariant sections*

$$\mathcal{D}'(G; \mathrm{Sym}^m \mathbb{R}^n)/\mathfrak{m} := \left\{ \sum_{K\in\mathscr{A}^m} u_K e_K \;\middle|\; R_{ij}u_K = \sum_{\ell=1}^{k} \left(u_{\{k_\ell\to i\}K}\delta_{jk_\ell} - u_{\{k_\ell\to j\}K}\delta_{ik_\ell} \right), 1 \le i, j \le n \right\}.$$

Proof. It suffices to consider the case $m = 1$. Demanding that $u = \sum_{k=1}^{n} u_k e_k$ corresponds to a section of $\mathcal{E}^*$ requires precisely that

$$0 = R_{ij}u = \sum_{k=1}^{n} (R_{ij}u_k)e_k + u_k(R_{ij}e_k)$$
$$= \sum_{k=1}^{n} (R_{ij}u_k)e_k + u_k(e_i\delta_{jk} - e_j\delta_{ik})$$

for $1 \le i, j \le n$. Applying $e_k \lrcorner$ to this equation recovers $R_{ij}u_k = u_i\delta_{jk} - u_j\delta_{ik}$. □

A similar statement may be made for other (not necessarily symmetric) tensor bundles of $\mathcal{E}$.

Differential operators on $\mathcal{E}$

We introduce several operators on (sections of tensor bundles of) $\mathcal{E}$. As $\mathcal{E}$ may be viewed as a subbundle of $\mathbb{R}^{1,n+1}$ above $S\mathbb{H}^{n+1}$, let ∇^{flat} denote the induced connection (upon projection onto $\mathcal{E}$ of the flat connection on $\mathbb{R}^{1,n+1}$). Now

$$\nabla^{\text{flat}} : \mathcal{D}'(S\mathbb{H}^{n+1}; \mathcal{E}^*) \to \mathcal{D}'(S\mathbb{H}^{n+1}; \mathrm{T}^* S\mathbb{H}^{n+1} \otimes \mathcal{E}^*)$$

however if we restrict to differentiating in either only the stable or only the unstable bundles E^s, E^u, via composition with $\theta_\pm$, we obtain horosphere operators $\nabla_\pm := \nabla^{\text{flat}}_{\theta_\pm}$ and in general we obtain

$$\nabla_\pm : \mathcal{D}'(S\mathbb{H}^{n+1}; \otimes^m \mathcal{E}^*) \to \mathcal{D}'(S\mathbb{H}^{n+1}; \otimes^{m+1} \mathcal{E}^*).$$

Symmetrising this operator we get the (positive and negative) horosphere symmetric derivatives and their divergences

$$\begin{aligned} \mathrm{d}_\pm &: \mathcal{D}'(S\mathbb{H}^{n+1}; \mathrm{Sym}^m \mathcal{E}^*) \to \mathcal{D}'(S\mathbb{H}^{n+1}; \mathrm{Sym}^{m+1} \mathcal{E}^*), \\ \delta_\pm &: \mathcal{D}'(S\mathbb{H}^{n+1}; \mathrm{Sym}^{m+1} \mathcal{E}^*) \to \mathcal{D}'(S\mathbb{H}^{n+1}; \mathrm{Sym}^m \mathcal{E}^*), \end{aligned}$$

as well as the horophere Laplacians $\Delta_\pm := [\delta_\pm, \mathrm{d}_\pm]$.

Considering these operators acting on equivariant sections of the corresponding vector bundles we have

$$\nabla_\pm = \sum_{i=1}^n e_k \otimes \mathcal{L}_{N_k^\pm} : \mathcal{D}'(G; \otimes^m \mathbb{R}^n)/\mathfrak{m} \to \mathcal{D}'(G; \otimes^{m+1} \mathbb{R}^n)/\mathfrak{m}$$

where $\mathcal{L}$ is the Lie derivative. (The appearance of merely the Lie derivative is because $\nabla_\pm$ uses ∇^{flat} and $N_i^\pm e_j = -(e_0 + e_{n+1})\delta_{ij} \notin \mathbb{R}^n$ for $1 \le i, j \le n$.) Similarly

$$\mathrm{d}_\pm = \sum_{k=1}^n e_k \cdot \mathcal{L}_{N_k^\pm}, \qquad \delta_\pm = -\sum_{k=1}^n e_k \lrcorner \mathcal{L}_{N_k^\pm}, \qquad \Delta_\pm = -\sum_{k=1}^n \mathcal{L}_{N_k^\pm} \mathcal{L}_{N_k^\pm}$$

on $\mathcal{D}'(G; \mathrm{Sym}^m \mathbb{R}^n)/\mathfrak{m}$.

Continuing to consider equivariant sections we note that $\mathcal{L}_A$ acts as a first order differential operator $\mathcal{D}'(G; \mathrm{Sym}^m \mathbb{R}^n)/\mathfrak{m}$ due to the commutator relations (A commutes with M). As $Ae_i = 0$, there will be no ambiguity in denoting this operator simply A. From the perspective of sections directly on $\mathrm{Sym}^m \mathcal{E}$ we have

$$A := (\pi_S^* \nabla)_A : \mathcal{D}'(S\mathbb{H}^{n+1}; \mathrm{Sym}^m \mathcal{E}^*) \to \mathcal{D}'(S\mathbb{H}^{n+1}; \mathrm{Sym}^m \mathcal{E}^*)$$

since $\pi_{S*} A = \xi$ at $(x, \xi) \in S\mathbb{H}^{n+1}$.

There are numerous useful relations between these operators. On $\mathcal{D}'(S\mathbb{H}^{n+1})$ the operators $(\nabla_\pm)^m$ and $(\mathrm{d}_\pm)^m$ agree since $[N_i^\pm, N_j^\pm] = 0$. As in Appendix A, there are commutation relations with the Lefschetz-type operators

$$[\Lambda, \delta_\pm] = 0 = [\mathrm{L}, \mathrm{d}_\pm], \qquad [\Lambda, \mathrm{d}_\pm] = -2\,\delta_\pm, \qquad [\mathrm{L}, \delta_\pm] = 2\,\mathrm{d}_\pm\,.$$

Moreover, these operators have simple commutation relations with A

$$[A, \mathrm{d}_\pm] = \pm\, \mathrm{d}_\pm, \qquad [A, \delta_\pm] = \pm\, \delta_\pm, \qquad [A, \Delta_\pm] = \pm 2\Delta_\pm.$$

Several operators on hyperbolic space

The metric on $\mathrm{T}\mathbb{H}^{n+1}$ allows the standard constructions from Appendix A. We obtain the rough Laplacian

$$\nabla^*\nabla : C^\infty(\mathbb{H}^{n+1}; \mathrm{Sym}^m \mathrm{T}^*\mathbb{H}^{n+1}) \to C^\infty(\mathbb{H}^{n+1}; \mathrm{Sym}^m \mathrm{T}^*\mathbb{H}^{n+1})$$

which will be more convenient in this chapter than the Lichnerowicz Laplacian, however we record that on $\mathbb{H}^{n+1}$, the curvature operator takes the constant value $q(\mathrm{R}) = -m(n+m-1)$. The divergence is

$$\delta : C^\infty(\mathbb{H}^{n+1}; \mathrm{Sym}^m \mathrm{T}^*\mathbb{H}^{n+1}) \to C^\infty(\mathbb{H}^{n+1}; \mathrm{Sym}^{m-1} \mathrm{T}^*\mathbb{H}^{n+1})$$

and we continue to use the notation L, Λ for the Lefschetz-type operators.

Conformal boundary

Hyperbolic space is projectively compact, and we identify the boundary of its compactification with the forward light cone

$$\big\{ (t, ty) \,\big|\, t \in \mathbb{R}^+, y \in \mathbb{S}^n \big\} \subset \mathbb{R}^{1,n+1}.$$

Now $x \pm \xi$ belongs to this light cone for $(x, \xi) \in S\mathbb{H}^{n+1}$ and this defines maps

$$\Phi_\pm : S\mathbb{H}^{n+1} \to \mathbb{R}^+, \qquad B_\pm : S\mathbb{H}^{n+1} \to \mathbb{S}^n,$$

by declaring

$$x \pm \xi = \Phi_\pm(x, \xi)(1, B_\pm(x, \xi)).$$

The Poisson kernel is

$$P : \begin{cases} \mathbb{H}^{n+1} \times \mathbb{S}^n & \to \quad \mathbb{R}^+ \\ \qquad (x, y) & \mapsto \quad -\langle x, e_0 + y \rangle^{-1} \end{cases}$$

which permits the definition of

$$\xi_\pm : \begin{cases} \mathbb{H}^{n+1} \times \mathbb{S}^n & \to \quad S\mathbb{H}^{n+1} \\ \qquad (x, y) & \mapsto \quad (x, \mp x \pm P(x, y)(e_0 + y)) \end{cases}$$

This gives an inverse to $B_\pm(x, \cdot)$ in the sense that $B_\pm(x, \xi_\pm(x, \nu)) = \nu$ (implying that $B_\pm$ is a submersion). Moreover,

$$\Phi_\pm(x, \xi_\pm(x, y)) = P(x, y)$$

The isometry group G acts on conformal infinity. There are maps

$$T : G \times \mathbb{S}^n \to \mathbb{R}^+, \qquad U : G \times \mathbb{S}^n \to \mathbb{S}^n,$$

defined by

$$\gamma \cdot (1, y) = T_\gamma(y)(1, U_\gamma(y)).$$

Useful formulae are

$$A \circ \Phi_\pm = \pm\Phi_\pm, \qquad N_k^\pm(\Phi_\pm \circ \pi_M) = 0, \qquad B_\pm = \lim_{t\to\pm\infty} \pi_S \circ \varphi_t,$$

and

$$B_\pm(\gamma \cdot (x, \xi)) = U_\gamma(B_\pm(x, \xi)), \qquad \Phi_\pm(\gamma \cdot (x, \xi)) = T_\gamma(B_\pm(x, \xi))\Phi_\pm(x, \xi).$$

We introduce the map

$$\tau_\pm : \begin{cases} \mathcal{E}_{(x,\xi)} & \to \quad \mathrm{T}_{y:=B_\pm(x,\xi)}\mathbb{S}^n \\ \qquad v & \mapsto \quad v + \langle v, e_0\rangle e_0 - \langle v, y\rangle y \end{cases}$$

which isometrically identifies $\mathcal{E}_{(x,\xi)}$ with $\mathrm{T}_{B_\pm(x,\xi)}\mathbb{S}^n$. It has an inverse

$${\tau_\pm}^{-1} : \begin{cases} \mathrm{T}_{B^\pm(x,\xi)}\mathbb{S}^n & \to \quad \mathcal{E}_{(x,\xi)} \\ \qquad \zeta & \mapsto \quad \zeta + \langle \zeta, x\rangle(x \pm \xi) \end{cases}$$

and the adjoint of $\tau_\pm$ is denoted ${\tau_\pm}^*$. Restricting our attention to τ_- we note the following equivariance under G

$$\left(\tau_{-\gamma\cdot(x,\xi)}\right)^{-1}\left(U_{\gamma*|B_-(x,\xi)}(\zeta)\right) = \frac{1}{T_\gamma(B_-(x,\xi))}\,\gamma \cdot \left(\left(\tau_{-(x,\xi)}\right)^{-1}(\zeta)\right)$$

for $\zeta \in \mathrm{T}_{B^\pm(x,\xi)}\mathbb{S}^n$. The identification offered by τ_- permits a second important identification of distributions in the kernel of both A and ∇_- with boundary distributions. Define the operator

$$\mathcal{Q}_- : \begin{cases} \mathcal{D}'(\mathbb{S}^n; \otimes^m \mathrm{T}^*\mathbb{S}^n) & \to \quad \mathcal{D}'(S\mathbb{H}^{n+1}; \otimes^m \mathcal{E}^*) \\ \qquad \omega & \mapsto \quad (\otimes^m(\tau_-^*)).\omega \circ B_- \end{cases}$$

which restricts to a linear isomorphism

$$\mathcal{Q}_- : \mathcal{D}'(\mathbb{S}^n; \mathrm{Sym}_0^m\mathrm{T}^*\mathbb{S}^n) \to \mathcal{D}'(S\mathbb{H}^{n+1}; \mathrm{Sym}_0^m\mathcal{E}^*) \cap \ker A \cap \ker \nabla_-.$$

Moreover, suppose we define

$$u := (\Phi_-)^\lambda \mathcal{Q}_-\omega, \qquad \lambda \in \mathbb{C}, \omega \in \mathcal{D}'(\mathbb{S}^n; \mathrm{Sym}_0^m\mathrm{T}^*\mathbb{S}^n),$$

then u enjoys the following equivariance property for $\gamma \in G$

$$\left(\gamma^*(\Phi_-)^\lambda \mathcal{Q}_-\omega\right)_{(x,\xi)}(\eta_1, \ldots, \eta_m) = (\Phi_-)^\lambda_{(x,\xi)}\left((T_\gamma)^{\lambda+m}U_\gamma^*\omega\right)_{B_-(x,\xi)}(\tau_-\eta_1, \ldots, \tau_-\eta_m)$$

where $\eta_i \in \mathcal{E}_{(x,\xi)}$. So $\gamma^* u = u$ if and only if, for $y \in \mathbb{S}^n$,

$$U_\gamma^* \omega(y) = T_\gamma(y)^{-\lambda - m} \omega(y).$$

Upper half-space model

Hyperbolic space is diffeomorphic to the upper half-space model $\mathbb{U}^{n+1} := \mathbb{R}^+ \times \mathbb{R}^n$. We take its closure $\overline{\mathbb{U}^{n+1}}$ by considering $\mathbb{U}^{n+1} \subset \mathbb{R}^{n+1}$. Using coordinates $x = (\rho, y)$ for $\rho \in \mathbb{R}^+$, $y \in \mathbb{R}^n$ the metric takes the form

$$g = \frac{d\rho^2 + h}{\rho^2}$$

where h is now the standard metric on $\mathbb{R}^n$.

In this model of hyperbolic space, the map τ_-^{-1} has been explicitly calculated in [GMP10, Appendix A] under the guise of parallel transport in the 0-calculus of Melrose. For $y' \in \mathbb{R}^n$, $x = (\rho, y) \in \mathbb{U}^{n+1}$, we write $\xi_- := \xi_-(x, y')$ and $r := y - y'$. Then

$$\tau_-^{-1} : \begin{cases} \mathrm{T}_{y'}\mathbb{R}^n & \to \quad \mathcal{E}_{(x,\xi_-)} \\ \partial_{y_i} & \mapsto \quad \rho \left(\frac{-2\rho^2 r_j}{\rho^2 + r^2} \frac{d\rho}{\rho} + \sum_{j=1}^n \left(\delta_{ij} - \frac{2 r_i r_j}{\rho^2 + r^2} \right) \partial_{y_j} \right) \end{cases}$$

Therefore $\tau_-^* dy_i = \rho^{-1} dy_i$ if $r = 0$ and in general, for fixed y' and variable x,

$$\tau_-^* dy_i = \rho^{-1} \left(b\, \rho d\rho + \sum_{j=1}^n b_{ij} dy_j \right) \tag{7.1}$$

for $b, b_{ij} \in C^\infty_{\mathrm{even}}(\overline{\mathbb{U}^{n+1}})$.

The Poisson kernel reads (continuing to use the notation from the previous paragraph)

$$P(x, y') = \frac{\rho}{\rho^2 + r^2} (1 + |y|^2)$$

and so $\rho^{-1} P(x, y')$ is even in ρ and, for fixed y', is smooth on $\overline{\mathbb{U}^{n+1}}$ away from $x = (0, y')$.

Convex cocompact quotients

Consider a discrete subgroup Γ of $G = \mathrm{SO}_0(1, n+1)$ which does not contain elliptic elements. Denote by K_Γ the limit set of Γ. Via the compactification $\overline{\mathbb{H}^{n+1}} = \mathbb{H}^{n+1} \sqcup \mathbb{S}^n$, the limit set is the the set of accumulation points of an arbitrary Γ-orbit, and is a closed subset of $\mathbb{S}^n$. The hyperbolic convex hull of all geodesics in $\mathbb{H}^{n+1}$ whose two endpoints both belong to K_Γ is termed the convex hull. The quotient of the convex hull by Γ gives the convex core of $\Gamma \backslash \mathbb{H}^{n+1}$, that is, the smallest convex subset of $\Gamma \backslash \mathbb{H}^{n+1}$ containing all closed geodesics of $\Gamma \backslash \mathbb{H}^{n+1}$. The group Γ is called convex cocompact if its associated convex core is compact.

Let Γ be convex cocompact and define $X := \Gamma \backslash \mathbb{H}^{n+1}$ denoting the canonical projection by $\pi_\Gamma : \mathbb{H}^{n+1} \to X$. Then $SX = \Gamma \backslash S\mathbb{H}^{n+1}$ (with canonical projection also denoted by π_Γ). The constructions of the previous subsections descend to constructions on X and SX.

Furthermore, denote by $\Omega_\Gamma \subset \mathbb{S}^n$ the discontinuity set of Γ. Then $\Omega_\Gamma = \mathbb{S}^n \backslash K_\Gamma$ and $\overline{X} = \Gamma \backslash (\mathbb{H}^{n+1} \sqcup \Omega_\Gamma)$. Denote by δ_Γ the Hausdorff dimension of the limit set K_Γ.

We introduce the outgoing tail $K_+ \subset SX$ as

$$K_+ := \pi_\Gamma \left(B_-^{-1} \left(K_\Gamma \right) \right)$$

and remark that this may be interpreted as the set of points $(x, \xi) \in SX$ such that $\pi_S(\varphi_t(x, \xi))$ does not tend to $\partial \overline{X}$ as $t \to -\infty$.

Using the outgoing tail, we define the following restriction of the unstable dual bundle

$$E_+^* := E^{*u}|_{K_+}.$$

7.2 Ruelle Resonances

The operator A acts on $\mathrm{Sym}^m \mathcal{E}^*$ above SX. For $\lambda \in \mathbb{C}$ with $\mathrm{Re}\, \lambda > 0$, the operator $(A + \lambda)$ has an inverse acting on $L^2(SX; \mathrm{Sym}^m \mathcal{E}^*)$. By [DG16], this inverse admits a meromorphic extension to $\mathbb{C}$ as a family of bounded operators

$$\mathcal{R}_{A,m}(\lambda) : C_c^\infty(SX; \mathrm{Sym}^m \mathcal{E}^*) \to \mathcal{D}'(SX; \mathrm{Sym}^m \mathcal{E}^*).$$

Near a pole λ_0, called a Ruelle resonance (of tensor order m), the resolvent may be expressed as

$$\mathcal{R}_{A,m}(\lambda) = \mathcal{R}_{A,m}^{\mathrm{Hol}}(\lambda) + \sum_{j=1}^{J(\lambda_0)} \frac{(-1)^{j-1}(A + \lambda_0)^{j-1} \prod_{A,m}^{\lambda_0}}{(\lambda - \lambda_0)^{-j}}$$

where the image of the finite rank projector $\prod_{A,m}^{\lambda_0}$ is called the space of generalised Ruelle resonant states (of tensor order m). It is denoted

$$\begin{aligned} \mathrm{Res}_{A,m}(\lambda_0) &:= \mathrm{Im}\left(\textstyle\prod_{A,m}^{\lambda_0} \right) \\ &= \left\{ u \in \mathcal{D}'(SX; \mathrm{Sym}^m \mathcal{E}^*) \;\middle|\; \mathrm{supp}(u) \subset K^+,\, \mathrm{WF}(u) \subset E_+^*,\, (A + \lambda_0)^{J(\lambda_0)} u = 0 \right\} \end{aligned}$$

We filter this space by declaring

$$\mathrm{Res}_{A,m}^j(\lambda_0) := \left\{ u \in \mathrm{Res}_{A,m}(\lambda_0) \;\middle|\; (A + \lambda_0)^j u = 0 \right\}$$

saying that such states are of Jordan order (at most) j. Then

$$\mathrm{Res}_{A,m}(\lambda_0) = \cup_{j \geq 1} \mathrm{Res}_{A,m}^j(\lambda_0)$$

and the space of Ruelle resonant states is $\mathrm{Res}_{A,m}^1(\lambda_0)$.

Band structure

Consider now A acting on $\mathrm{Sym}^0 \mathcal{E}^*$. Let λ_0 be a Ruelle resonance (of tensor order 0) and consider (a non-zero) $u \in \mathrm{Res}_{A,0}(\lambda_0)$. As Ruelle resonances (of arbitrary tensor order) are contained in $\{ \lambda \in \mathbb{C} \mid \mathrm{Re}\, \lambda \leq 0 \}$, the commutator relation $[A, \mathrm{d}_-] = -\mathrm{d}_-$ implies that there exists $m \in \mathbb{N}_0$ such that $(\mathrm{d}_-)^m u \neq 0$ and $(\mathrm{d}_-)^{m+1} u = 0$. We say that u is in the m^{th} band. Precisely, we define

$$V_m^j(\lambda_0) := \left\{ u \in \mathrm{Res}_{A,0}^j(\lambda_0) \;\middle|\; u \in \ker(\mathrm{d}_-)^{m+1} \right\}$$

The m^{th} band may then be considered the quotient $V^j_m(\lambda_0)/V^j_{m-1}(\lambda_0)$ whence

$$\operatorname{Res}^j_{A,0}(\lambda_0) = \bigoplus_{m\in\mathbb{N}_0} \left(V^j_m(\lambda_0)/V^j_{m-1}(\lambda_0)\right). \tag{7.2}$$

The extent to which this band may be identified with Ruelle resonances of tensor order m is contained in

Proposition 7.3. *Consider $\lambda_0 \in \mathbb{C}$, a Ruelle resonance with $\operatorname{Re}\lambda_0 \le -1$. Consider also $m \in \mathbb{N}$ such that $\operatorname{Re}\lambda_0 + m \le 0$. Further, exclude the case m even with $\lambda_0 + m = 0$. Under these assumptions, we obtain the following short exact sequence*

$$0 \longrightarrow V^j_{m-1}(\lambda_0) \longrightarrow V^j_m(\lambda_0) \xrightarrow{(\mathrm{d}_-)^m} \operatorname{Res}^j_{A,m}(\lambda_0+m)\cap\ker\nabla_- \longrightarrow 0$$

Proof. Denote by $W^j_m(\lambda_0+m)$ the third space in the sequence $\operatorname{Res}^j_{A,m}(\lambda_0+m)\cap\ker\nabla_-$. The non-trivial step is showing surjectivity of $(\mathrm{d}_-)^m$. We decompose $W^j_m(\lambda_0+m)$ into eigenspaces of $\mathrm{L}\,\Lambda$. In particular we denote

$$W^j_{m,k}(\lambda_0+m) := W^j_m(\lambda+m)\cap\ker(\mathrm{L}\,\Lambda - 2k(n+2m-2k-2))$$

(An element of this space may be written $\mathrm{L}^k\, u^{(m-2k)}$ for $u^{(m-2k)} \in W^j_{m-2k}(\lambda_0+m)\cap\ker\Lambda$.) There exist differential operators (linear of order m)

$$K_k : W^j_{m,k}(\lambda+m) \to V^j_m(\lambda_0)$$

such that $(\mathrm{d}_-)^m \circ K_k = P_{m-2k,k}(A)$ where $P_{m-2k,k} = P_{r,k}$ is the polynomial from Proposition 7.10

$$P_{r,k}(A) = 2^{k+r} m!(r!)^2 \prod_{j=1}^{k}(A+r+j-1)(-2A+(n-2j)) \prod_{j=1}^{r}(A-n-j+2)$$

As $W^j_{m,k}(\lambda_0+m)$ is finite dimensional, it suffices to show injectivity of $(\mathrm{d}_-)^m\circ K_k$ which we do by induction on j. Consider $j=1$ in which case $(\mathrm{d}_-)^m\circ K_k = P_{m-2k,k}(-(\lambda_0+m))$ on $W^1_{m,k}(\lambda_0+m)$ which is non-zero by the conditions imposed on λ_0 and by Corollary 7.11. Consider now

$$u \in W^j_{m,k}(\lambda_0+m)\cap\ker((\mathrm{d}_-)^m\circ K_k).$$

By considering again a decomposition of the form $u = \sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\mathrm{L}^k\, u^{(m-2k)}$, then the fact that $(\mathrm{d}_-)^m\circ K_k$ is a polynomial in A, implies that it commutes with $(A+\lambda_0+m)$ hence

$$(A+\lambda_0+m)u^{(m-2k)} \in W^{j-1}_{m-2k}(\lambda_0+m)\cap\ker\Lambda\cap\ker((\mathrm{d}_-)^m\circ K_k)$$

which by the inductive hypothesis forces $u \in \ker(A+\lambda_0+m)$ and the case $j=1$ now implies $u=0$. $\square$

Proposition 7.4. *Consider $\lambda_0 \in -2\mathbb{N}\backslash(-\frac{n}{2}-\frac{1}{2}\mathbb{N}_0)$, a Ruelle resonance and set $m := -\lambda_0$. Then*

$$\operatorname{Res}^j_{A,m}(0)\cap\ker\nabla_- = 0$$

so in this case also, there is trivially a short exact sequence as in Proposition 7.3.

Proof. It suffices to prove the statement for $j = 1$. Suppose $u \in \mathrm{Res}^1_{A,m}(0) \cap \ker \nabla_-$ non-zero and decompose $u =: \mathrm{L}^k\, u^{(m-2k)}$ for $u^{(m-2k)} \in \mathrm{Res}^j_{A,m-2k}(0) \cap \ker \Lambda \cap \ker \nabla_-$. Consider first $u^{(0)}$. This is a Ruelle resonant state (of tensor order 0) on SX but by [DG16] the real part of a Ruelle resonance of tensor order 0 is not greater than $\delta_\Gamma - n < 0$. Considering the other components of u, define $\varphi^{(m-2k)} := \pi_{0*} u^{(m-2k)}$ for $m - 2k \neq 0$. By Proposition 7.15 this is an isomorphism

$$\pi_{0*} : \mathrm{Res}^1_{A,m-2k}(0) \cap \ker \Lambda \cap \ker \nabla_- \to \mathrm{Res}^1_{\Delta,m-2k}(n).$$

From [DS10, Lemma 8.2] and the discussion preceding [DFG15, Lemma 6.1] the L^2 spectrum of $\nabla^*\nabla$ acting on $\mathrm{Sym}^{m-2k}_0 \mathrm{T}^*X$ (for $m - 2k \neq 0$) is bounded below by $(n + m - 2k - 1)$. However $\varphi^{(m-2k)} \in \ker(\nabla^*\nabla - (m - 2k))$ and by Lemma 7.12, $\varphi^{(m-2k)} \in L^2(X; \mathrm{Sym}^{m-2k}_0 \mathrm{T}^*X)$ so this forces $\varphi^{(m-2k)} = 0$ as $m - 2k < n + m - 2k - 1$. □

To finish this section, we consider the decomposition of the set of vector-valued generalised resonant states considered in this subsection into eigenspaces of $\mathrm{L}\,\Lambda$. Then

$$\mathrm{Res}^j_{A,m}(\lambda_0 + m) \cap \ker \nabla_- = \bigoplus_{k=0}^{\lfloor \frac{m}{2} \rfloor} \mathrm{L}^k \left(\mathrm{Res}^j_{A,m-2k}(\lambda_0 + m) \cap \ker \Lambda \cap \ker \nabla_- \right) \tag{7.3}$$

as A commutes with the Lefschetz-type operators, and the condition $\ker \nabla_-$ is conserved (which may be concluded from considering the form of ∇_- acting on $\mathcal{D}'(G; \mathrm{Sym}^m \mathbb{R}^n)/\mathfrak{m}$).

7.3 Inverting Horosphere Operators

This section recalculates in notation consistent with this thesis the result of [DFG15, Subsection 4.3]. The key result is the polynomial A presented in Proposition 7.10 which has been used in the proof of Proposition 7.3.

In this chapter all appearances of $A, R_{ij}, N_k^\pm$ are to be interpreted as Lie derivatives. Define $R := \sum_{ij} e_i \cdot e_j \lrcorner R_{ij}$ so that on $\mathcal{D}'(S\mathbb{H}^{n+1}; \mathrm{Sym}^m \mathcal{E})$ we have $R = m(n + m - 2) - \mathrm{L}\,\Lambda$. As a warm up exercise, we have

Lemma 7.5. *On $\mathcal{D}'(S\mathbb{H}^{n+1}; \mathrm{Sym}^m_0 \mathcal{E}) \cap \ker \nabla_-$ we have*

$$\mathrm{d}_-\, \delta_+ = 2m(A - n - m + 2)$$

Proof. Performing the calculation on $\mathcal{D}'(G; \mathrm{Sym}^m_0 \mathbb{R}^n)$ subject to the necessary conditions,

$$\begin{aligned}
\mathrm{d}_-\, \delta_+ &= \sum_{i,j} (e_i \cdot N_i^-)(-e_j \lrcorner N_j^+) \\
&= \sum_{i,j} e_i \cdot e_j \lrcorner (-N_j^+ N_i^- + 2A\delta_{ij} - 2R_{ij}) \\
&= (2A \sum_i e_i \cdot e_i \lrcorner) - 2R \\
&= 2m(A - n - m + 2).
\end{aligned}$$

□

Developing this calculation into a more useful result, we have

Lemma 7.6. *On $\mathcal{D}'(S\mathbb{H}^{n+1}; \mathrm{Sym}_0^m \mathcal{E}) \cap \ker \nabla_-$ we have, for $1 \le k \le m$*

$$\mathrm{d}_-(\delta_+)^k = 2k(m-k+1)(\delta_+)^{k-1}(A-n-m+k+1)$$

Proof. We perform the calculation on $u = \sum_M u_M e_M \in \mathcal{D}'(G; \mathrm{Sym}_0^m \mathbb{R}^n)$ subject to the necessary conditions. First,

$$\begin{aligned}
\mathrm{d}_-(\delta_+)^k u &= \sum_{i\in\mathscr{A}, M\in\mathscr{A}^m} (e_i \cdot N_i^-) \prod_{\ell=1}^k \sum_{j_\ell\in\mathscr{A}} (-e_{j_\ell} \lrcorner N_{j_\ell}^+) u_M e_M \\
&= (-1)^k \sum_{i\in\mathscr{A}, J\in\mathscr{A}^k, M\in\mathscr{A}^m} (N_i^- N_J^+ u_M) e_i \cdot e_J \lrcorner e_M \\
&= (-1)^k \frac{m!}{(m-k)!} \sum_{i\in\mathscr{A}, I\in\mathscr{A}^{m-k}, J\in\mathscr{A}^k} (N_i^- N_J^+ u_{IJ}) e_{iI}
\end{aligned}$$

and similarly (to obtain the final equation displayed in this proof)

$$(\delta_+)^{k-1} u = (-1)^{k-1} \frac{m!}{(m-k+1)!} \sum_{i\in\mathscr{A}, I\in\mathscr{A}^{m-k}, C\in\mathscr{A}^{k-1}} (N_C^+ u_{iIC}) e_{iI}$$

Continuing the antepreceding display, we commute N_i^- past N_J^+ (note $N_i^- u_{IJ} = 0$ since $u \in \ker \nabla_-$)

$$\sum_{J\in\mathscr{A}^k} N_i^- N_J^+ u_{iJ} = \sum_{\substack{0\le\alpha<k \\ \alpha+1+\beta:=k}} \sum_{A\in\mathscr{A}^\alpha, j\in\mathscr{A}, B\in\mathscr{A}^\beta} N_A^+ [N_i^-, N_j^+] N_B^+ u_{IAjB}$$

and with $[N_i^-, N_j^+] = -2A\delta_{ij} + 2R_{ij}$ we develop in two steps the preceding display. For terms involving A, we use the relation $[A, N_b^+] = N_b^+$ to commute A to the right. The commutation relation implies $[A, N_B^+] = \beta N_B^+$ hence

$$A\delta_{ij} N_B^+ u_{IAjB} = N_B^+ (A+\beta) \delta_{ij} u_{IAjB}$$

For terms involving R_{ij} we again commute R_{ij} to the right

$$\begin{aligned}
\sum_{j\in\mathscr{A}, B\in\mathscr{A}^\beta} R_{ij} N_B^+ u_{IAjB} &= \sum_{j\in\mathscr{A}, B\in\mathscr{A}^\beta} \left(N_B^+ R_{ij} + \sum_{\ell=1}^\beta N_{b_1\dots b_{\ell-1}}^+ [R_{ij}, N_{b_\ell}^+] N_{b_{\ell+1}\dots b_\beta}^+ \right) u_{IAjB} \\
&= \sum_{j\in\mathscr{A}, B\in\mathscr{A}^\beta} \left(N_B^+ R_{ij} + \sum_{\ell=1}^\beta N_{b_1\dots b_{\ell-1}}^+ (\delta_{jb_\ell} N_i^+ - \delta_{ib_\ell} N_j^+) N_{b_{\ell+1}\dots b_\beta}^+ \right) u_{IAjB} \\
&= \sum_{B\in\mathscr{A}^\beta} N_B^+ (R_{ij} - \beta\delta_{ij}) u_{IAjB}
\end{aligned}$$

where the final line is obtained by remarking $\sum_{j,b_\ell} \delta_{jb_\ell} u_{IAjB} = 0$ as $u \in \ker \Lambda$, and by switching indices j, b_ℓ in the summation of the term involving $\delta_{ib_\ell} N_j^+$. We also note here that, as $u \in \ker \Lambda$ we may write

$$\sum_{j\in\mathscr{A}, D\in\mathscr{A}^{m-1}} R_{ij} u_{jD} = (n+m-2) \sum_{D\in\mathscr{A}^{m-1}} u_{iD}$$

which is used in obtaining the third line in the following display. The principal calculation thus continues (using also that $\sum_{0\leq\alpha<k} 2(k-1-\alpha) = k(k-1)$ in order to obtain the third line)

$$\begin{aligned}
\mathrm{d}_-(\delta_+)^k u &= (-1)^{k-1}\frac{m!}{(m-k)!}\sum_{\substack{0\leq\alpha<k\\ \alpha+1+\beta:=k}}\ \sum_{\substack{i,j\in\mathscr{A},I\in\mathscr{A}^{m-k},\\ A\in\mathscr{A}^{\alpha},B\in\mathscr{A}^{\beta}}} N^+_{AB}\left(2(A+2\beta)\delta_{ij}-2R_{ij}\right)u_{IAjB}e_{iI}\\
&= (-1)^{k-1}\frac{m!}{(m-k)!}\sum_{0\leq\alpha<k}\ \sum_{\substack{i,j\in\mathscr{A}\\ I\in\mathscr{A}^{m-k},C\in\mathscr{A}^{k-1}}} 2N^+_C\left((A+2(k-1-\alpha))\delta_{ij}-R_{ij}\right)u_{jIC}e_{iI}\\
&= (-1)^{k-1}\frac{m!}{(m-k)!}\sum_{\substack{i\in\mathscr{A},\\ I\in\mathscr{A}^{m-k},C\in\mathscr{A}^{k-1}}} 2kN^+_C\left(A+(k-1)-(n+m-2)\right)u_{iIC}e_{iI}\\
&= 2k(m-k)(\delta_+)^{k-1}(A-n-m+k+1)u. \qquad \square
\end{aligned}$$

The previous lemma provides upon induction

Lemma 7.7. *On $\mathcal{D}'(S\mathbb{H}^{n+1};\mathrm{Sym}^m_0\mathcal{E})\cap\ker\nabla_-$ we have*

$$(\mathrm{d}_-)^m(\delta_+)^m = 2^m(m!)^2\prod_{j=1}^{m}(A-n-j+2)$$

In order to treat symmetric tensors which are not purely trace-free we include

Lemma 7.8. *On $\mathcal{D}'(S\mathbb{H}^{n+1})\cap\ker(\nabla_-\circ(\mathrm{d}_-)^r)$ we have,*

$$(\mathrm{d}_-)^{r+2}\Delta_+ = (r+2)(r+1)\,\mathrm{L}\Big(2(A+r)(-2A+n-2)-\mathrm{L}\,\Lambda\Big)(\mathrm{d}_-)^r$$

Proof. We perform the calculation on $u\in\mathcal{D}'(G)$ subject to the necessary conditions (in particular $R_{ij}u=0$) and reproduce verbatim the beginning of [DFG15, Lemma 4.4]. First,

$$\begin{aligned}
(\mathrm{d}_-)^{r+2}\Delta_+ u &= -\sum_{i\in\mathscr{A},K\in\mathscr{A}^{r+2}}(N^-_K N^+_{ii}u)e_K\\
&= -\sum_{i\in\mathscr{A},K\in\mathscr{A}^{r+2}}([[N^-_K,N^+_i],N^+_i]u)e_K
\end{aligned}$$

since $[N^-_K,N^+_i]u=0$ as $u\in\ker(\nabla_-\circ(\mathrm{d}_-)^r)$. We now calculate $[N^-_K,N^+_i]$ precisely as in [DFG15, Lemma 4.4] however we take this opportunity to introduce a notation which will be advantageous for the presentation of this proof. We write

$$N^-_{k_1\ldots k_{\ell-1}}[N^-_{k_\ell},N^+_i]N^-_{k_{\ell+1}\ldots k_{r+2}} =: N^-_K\big|_{N^-_{k_\ell}\to[N^-_{k_\ell},N^+_i]}$$

with the natural extension to other situations. The calculation now reads

$$
\begin{aligned}
-[N_K^-, N_i^+] &= 2\sum_{\ell=1}^{r+2} N_K^-\big|_{N_{k_\ell}^- \to A\delta_{k_\ell i} - R_{k_\ell i}} \\
&= 2\sum_{\ell=1}^{r+2}\left(N_{\{k_\ell\to\}K}^-(A+\ell-r-2)\delta_{k_\ell i} - N_K^-\big|_{N_{k_\ell}^- \to R_{k_\ell i}}\right) \\
&= 2\sum_{\ell=1}^{r+2}\left(N_{\{k_\ell\to\}K}^-((A+\ell-r-2)\delta_{k_\ell i} - R_{k_\ell i}) - \sum_{s=\ell+1}^{r+2} N_{\{k_\ell\to\}K}^-\Big|_{N_{k_s}^- \to \delta_{ik_s}N_{k_\ell}^- - \delta_{k_\ell k_s}N_i^-}\right) \\
&= 2\sum_{\ell=1}^{r+2}\left(N_{\{k_\ell\to\}K}^-((A+\ell-r-2)\delta_{k_\ell i} - R_{k_\ell i}) - \sum_{s=\ell+1}^{r+2} \delta_{ik_s}N_{\{k_s\to\}K}^- - \delta_{k_\ell k_s}N_{\{k_\ell\to,k_s\to i\}K}^-\right) \\
&= 2\sum_{\ell=1}^{r+2}\left(N_{\{k_\ell\to\}K}^-((A-(r+1))\delta_{k_\ell i} - R_{k_\ell i}) + \sum_{s=\ell+1}^{r+2} \delta_{k_\ell k_s}N_{i\{k_\ell\to,k_s\to\}K}^-\right)
\end{aligned}
$$

and

$$
\sum_{i\in\mathscr{A}}[(A-(r+1))\delta_{k_\ell i} - R_{k_\ell i}, N_i^+] = -(n-2)N_{k_\ell}^+
$$

So, upon setting $v := (A-(r+1)-(n-2))u$, the principal calculation reads

$$
(\mathrm{d}_-)^{r+2}\Delta_+ u = 2\sum_{K\in\mathscr{A}^{r+2}}\sum_{\ell=1}^{r+2}\left([N_{\{k_\ell\to\}K}^-, N_{k_\ell}^+]v + \sum_{i\in\mathscr{A}}\sum_{s=\ell+1}^{r+2}\delta_{k_\ell k_s}[N_{i\{k_\ell\to,k_s\to\}K}^-, N_i^+]u\right)e_K
$$

and we calculate in two steps considering separately terms involving u, v. For v,

$$
2\sum_{K\in\mathscr{A}^{r+2}}\sum_{\ell=1}^{r+2}[N_{\{k_\ell\to\}K}^-, N_{k_\ell}^+]ve_K = 2\sum_{K\in\mathscr{A}^{r+2}}\sum_{\ell=1}^{r+2}\sum_{\substack{s=1\\ s\neq\ell}}^{r+2} N_{\{k_\ell\to\}K}^-\Big|_{N_{k_s}^- \to -2A\delta_{k_s k_\ell} + 2R_{k_s k_\ell}} ve_K
$$

considering terms involving A and $R_{k_s k_\ell}$ seperately. We shift A to the left (and use the observation that $r(r+1) = r\sum_{s=1,s\neq\ell}^{r+2} 1$)

$$
\begin{aligned}
2\sum_{K\in\mathscr{A}^{r+2}}\sum_{\substack{\ell,s=1\\ s\neq\ell}}^{r+2} N_{\{k_\ell\to\}K}^-\Big|_{N_{k_s}^- \to -2A\delta_{k_s k_\ell}} ve_K &= -2(2A+r)\sum_{K\in\mathscr{A}^{r+2}}\sum_{\substack{\ell,s=1\\ s\neq\ell}}^{r+2}\delta_{k_s k_\ell}N_{\{k_\ell\to,k_s\to\}K}^- ve_K \\
&= (r+2)(r+1)\,\mathrm{L}(-2)(2A+r)\sum_{R\in\mathscr{A}^r} N_R^- ve_R \\
&= (r+2)(r+1)\,\mathrm{L}(-2)(2A+r)(\mathrm{d}_-)^r v
\end{aligned}
$$

We shift $R_{k_s k_\ell}$ to the right

$$\sum_{K\in\mathscr{A}^{r+2}}\sum_{\substack{\ell,s=1\\ s\neq\ell}}^{r+2} N^-_{\{k_\ell\to\}K}\Big|_{N^-_{k_s}\to R_{k_s k_\ell}} ve_K = \sum_{K\in\mathscr{A}^{r+2}}\sum_{\substack{\ell,s=1\\ s\neq\ell}}^{r+2}\sum_{\substack{p=s+1\\ p\neq\ell}}^{r+2} N^-_{\{k_\ell\to,k_s\to\}K}\Big|_{N^-_{k_p}\to N^-_{k_s}\delta_{k_\ell k_p}-N^-_{k_\ell}\delta_{k_s k_p}} ve_K$$

$$= \sum_{K\in\mathscr{A}^{r+2}}\sum_{\substack{\ell,s=1\\ s\neq\ell}}^{r+2}\sum_{\substack{p=s+1\\ p\neq\ell}}^{r+2}\Big(N^-_{\{k_\ell\to,k_p\to\}K}\delta_{k_\ell k_p} - N^-_{\{k_s\to,k_p\to\}K}\delta_{k_s k_p}\Big) ve_K$$

$$= \sum_{\substack{\ell,s=1\\ s\neq\ell}}^{r+2}\sum_{\substack{p=s+1\\ p\neq\ell}}^{r+2}(1-1)\,\mathrm{L}(\mathrm{d}_-)^r v$$

$$= 0$$

Ultimately, shifting the appearances of A in the definition of v to the left,

$$2\sum_{K\in\mathscr{A}^{r+2}}\sum_{\ell=1}^{r+2}[N^-_{\{k_\ell\to\}K}, N^+_{k_\ell}]ve_K = (r+2)(r+1)\,\mathrm{L}(-2)(2A+r)(\mathrm{d}_-)^r v$$

$$= (r+2)(r+1)\,\mathrm{L}\Big((-2)(2A+r)(A-n+1)\Big)(\mathrm{d}_-)^r u$$

Considering the term not involving v in $(\mathrm{d}_-)^{r+2}\Delta_+ u$ we have

$$2\sum_{i\in\mathscr{A},K\in\mathscr{A}^{r+2}}\sum_{\substack{\ell,s=1\\ s>\ell}}^{r+2}\delta_{k_\ell k_s}[N^-_{i\{k_\ell\to,k_s\to\}K}, N^+_i]ue_K = (r+2)(r+1)\,\mathrm{L}\sum_{i\in\mathscr{A},R\in\mathscr{A}^r}[N^-_{iR}, N^+_i]ue_R$$

and we split the calculation of $\sum_R [N^-_{iR}, N^+_i]ue_R$ into three parts:

$$\sum_{i\in\mathscr{A},R\in\mathscr{A}^r}[N^-_{iR}, N^+_i]ue_R = -2nA(\mathrm{d}_-)^r u + \sum_{\ell=1}^{r} N^-_{iR}\big|_{N^-_{r_\ell}\to -2\delta_{r_\ell i}A+2R_{r_\ell i}}\, ue_R$$

Moving A to the left

$$\sum_{i\in\mathscr{A},R\in\mathscr{A}^r}\sum_{\ell=1}^{r} N^-_{iR}\big|_{N^-_{r_\ell}\to -2\delta_{r_\ell i}A}\, ue_R = -2\sum_{\ell=1}^{r}\sum_{R\in\mathscr{A}^r}(A+\ell)N^-_R e_R$$

$$= -r(2A+r+1)(\mathrm{d}_-)^r u$$

Moving $R_{r_\ell i}$ to the right

$$\begin{aligned}
\sum_{i\in\mathscr{A},R\in\mathscr{A}^r}\sum_{\ell=1}^{r} N^-_{iR}\Big|_{N^-_{r_\ell}\to 2R_{r_\ell i}} ue_R &= 2\sum_{i\in\mathscr{A},R\in\mathscr{A}^r}\sum_{\substack{\ell,s=1\\ s>\ell}}^{r} N^-_{i\{r_\ell\to\}R}\Big|_{N^-_{r_s}\to[R_{r_\ell i},N^-_{r_s}]} ue_R \\
&= 2\sum_{i\in\mathscr{A},R\in\mathscr{A}^r}\sum_{\substack{\ell,s=1\\ s>\ell}}^{r} N^-_{i\{r_\ell\to\}R}\Big|_{N^-_{r_s}\to N^-_{r_\ell}\delta_{ir_s}-N^-_i\delta_{r_\ell r_s}} ue_R \\
&= r(r-1)(\mathrm{d}_-)^r u - 2\sum_{i\in\mathscr{A},R\in\mathscr{A}^r}\sum_{\substack{\ell,s=1\\ s>\ell}}^{r} N^-_{ii\{r_\ell\to,r_s\to\}R}\delta_{r_\ell r_s} ue_R \\
&= r(r-1)(\mathrm{d}_-)^r u + \mathrm{L}\, r(r-1)\Delta_-(\mathrm{d}_-)^{r-2}u \\
&= r(r-1)(\mathrm{d}_-)^r u - \mathrm{L}\,\Lambda(\mathrm{d}_-)^r u
\end{aligned}$$

where in the last line we use

$$\begin{aligned}
\Lambda(\mathrm{d}_-)^r u &= \sum_{i\in\mathscr{A},R\in\mathbb{R}} e_i \lrcorner\, e_i \lrcorner\, N^-_R ue_R \\
&= \sum_{i\in\mathscr{A},R\in\mathscr{A}^r}\sum_{\ell=1}^{r} e_i \lrcorner\, N^-_{i\{r_\ell\to\}R} ue_{\{r_\ell\to\}R} \\
&= \sum_{i\in\mathscr{A},R\in\mathscr{A}^r}\sum_{\substack{\ell,s=1\\ s\neq\ell}}^{r} N^-_{ii\{r_\ell\to,r_s\to\}R} ue_{\{r_\ell\to,r_s\to\}R} \\
&= -r(r-1)\Delta_-(\mathrm{d}_-)^{r-2}u
\end{aligned}$$

Ultimately, the term not involving v in $(\mathrm{d}_-)^{r+2}\Delta_+ u$ reads

$$\begin{aligned}
&2\sum_{i\in\mathscr{A},K\in\mathscr{A}^{r+2}}\sum_{\substack{\ell,s=1\\ s>\ell}}^{r+2} \delta_{k_\ell k_s}[N^-_{i\{k_\ell\to,k_s\to\}K}, N^+_i]ue_K \\
&= (r+2)(r+1)\,\mathrm{L}\Big(-2nA - r(2A+r+1) + r(r-1) - \mathrm{L}\,\Lambda\Big)(\mathrm{d}_-)^r u \\
&= (r+2)(r+1)\,\mathrm{L}\Big(-2A(n+r) - r - \mathrm{L}\,\Lambda\Big)(\mathrm{d}_-)^r u
\end{aligned}$$

The principal calculation thus terminates

$$\begin{aligned}
(\mathrm{d}_-)^{r+2}\Delta_+ u &= (r+2)(r+1)\,\mathrm{L}\Big((-2)(2A+r)(A-n+1) - 2A(n+r) - r - \mathrm{L}\,\Lambda\Big)(\mathrm{d}_-)^r u \\
&= (r+2)(r+1)\,\mathrm{L}\Big(2(A+r)(-2A+n-2) - \mathrm{L}\,\Lambda\Big)(\mathrm{d}_-)^r u
\end{aligned}$$

□

The previous lemma provides upon induction using $[\Lambda,\mathrm{L}] = 2n + 4\deg$.

Lemma 7.9. *On $\mathcal{D}'(S\mathbb{H}^{n+1}) \cap \ker(\Lambda \circ (\mathrm{d}_-)^{r+2k}) \cap \ker(\nabla_- \circ (\mathrm{d}_-)^{r+2k})$ we have,*

$$(\mathrm{d}_-)^{r+2k}(\Delta_+)^k = 2^k(r+2k)!\,\mathrm{L}^k\prod_{j=1}^{k}(A+r+j-1)(-2A+(n-2j))(\mathrm{d}_-)^r$$

with the interpretation that the product takes the value 1 if $k=0$.

Applying first Lemma 7.7 and second Lemma 7.9 provides

Proposition 7.10. *Consider* $u \in \mathcal{D}'(S\mathbb{H}^{n+1}; \mathrm{Sym}^m \mathcal{E}) \cap \ker(\nabla_-)$ *decomposed such that* $u = \sum_k \mathrm{L}^k u^{(k)}$ *for* $u^{(k)} \in \mathcal{D}'(S\mathbb{H}^{n+1}; \mathrm{Sym}_0^r \mathcal{E}) \cap \ker(\nabla_-)$ *with* $r := m - 2k$*. On* $u^{(k)}$*,*

$$(\mathrm{d}_-)^{r+2k} (\Delta_+)^k (\delta_+)^r = \mathrm{L}^k P_{r,k}(A)$$

where $P_{r,k}(A)$ *is the following polynomial*

$$P_{r,k}(A) = 2^{k+r} m! (r!)^2 \prod_{j=1}^{k} (A + r + j - 1)(-2A + (n - 2j)) \prod_{j=1}^{r} (A - n - j + 2)$$

The following corollary is slightly weaker than what one could say by distinguishing cases dependent on the parity of n however it is sufficient for the correspondence.

Corollary 7.11. *Consider* $\lambda \in \mathbb{C}$ *with* $\lambda \notin -\frac{n}{2} - \frac{1}{2}\mathbb{N}_0$ *and* $\operatorname{Re} \lambda \leq -1$*. Consider also* $m \in \mathbb{N}$ *such that* $\operatorname{Re} \lambda + m \leq 0$*. Then for* $r, k \in \mathbb{N}_0$ *such that* $m = r + 2k$*, the value of the polynomial* $P_{r,k}(-(\lambda + m))$ *is non zero except in the single situation* $m \in 2\mathbb{N}$*,* $r = 0$*,* $k = \frac{m}{2}$*,* $\lambda + m = 0$*.*

Proof. We need to ensure that $\lambda + m$ does not belong to any of the following sets

$$\begin{aligned} S_1 :=& \{r, r+1, \dots, r+k-1\}, \\ S_2 :=& \{-\tfrac{n}{2} + 1, -\tfrac{n}{2} + 2, \dots, -\tfrac{n}{2} + k\}, \\ S_3 :=& \{-n - r + 2, -n - r + 3, \dots, -n + 1\}. \end{aligned}$$

So if $\operatorname{Im} \lambda \neq 0$ then such an exclusion is guaranteed hence the only possible problematic situations are when $\lambda \in \{-\frac{n}{2} + \frac{1}{2}, -\frac{n}{2} + 1, \dots, -1\}$. Considering S_1, as $\lambda + m \leq 0$, the only problematic situation is the one announced in the corollary when $\lambda + m = r = 0$ (forcing m to be even). The set S_2 poses no problem as $\lambda > -\frac{n}{2}$ and $m > k$. Similarly S_3 poses no problem as $\lambda > -n$. ☐

7.4 Quantum Resonances

The rough Laplacian $\nabla^*\nabla$ acts on $\mathrm{Sym}^m \mathrm{T}^* X$. For $s \in \mathbb{C}$ with $s \gg 1$, the operator $\nabla^*\nabla - s(n-s) - m$ has an inverse acting on $L^2(X; \mathrm{Sym}^m \mathrm{T}^* X)$. We introduce the short-hand

$$\mathcal{A}_s := (\nabla^*\nabla - s(n-s) - m).$$

Note that $\nabla^*\nabla$ commutes with both the trace Λ and divergence δ operators. By Theorem 9, the inverse of $\nabla^*\nabla - s(n-s) - m$, written $\mathcal{R}_{\Delta,m}(s)$, admits, upon restriction to the kernels of both Λ and δ, a meromorphic extension from $\operatorname{Re} s \gg 1$ to $\mathbb{C}$ as a family of bounded operators

$$\mathcal{R}_{\Delta,m}(s) : C_c^\infty(X; \mathrm{Sym}_0^m \mathrm{T}^* X) \cap \ker \delta \to \rho^{s-m} C_{\mathrm{even}}^\infty(\overline{X}; \mathrm{Sym}_0^m \mathrm{T}^* X) \cap \ker \delta \,.$$

(Here ρ is an even boundary defining function providing the conformal compactification $\overline{X}$.) Near a pole s_0, called a quantum resonance, the resolvent may be written

$$\mathcal{R}_{\Delta,m}(s) = \mathcal{R}_{\Delta,m}^{\mathrm{Hol}}(s) + \sum_{j=1}^{J(\lambda_0)} \frac{(\nabla^*\nabla - s_0(n - s_0) - m)^{j-1} \prod_{\Delta,m}^{s_0}}{(s(n-s) - s_0(n - s_0))^j}$$

where the image of the finite rank projector $\prod_{\Delta,m}^{\lambda_0}$ is called the space of generalised quantum resonant states (of tensor order m)

$$\mathrm{Res}_{\Delta,m}(s_0) := \mathrm{Im}\left(\prod_{\Delta,m}^{s_0}\right).$$

We filter this space by declaring

$$\mathrm{Res}^j_{\Delta,m}(s_0) := \left\{\varphi \in \mathrm{Res}_{\Delta,m}(s_0) \,\middle|\, (\nabla^*\nabla - s_0(n-s_0) - m)^j \varphi = 0\right\}$$

saying that such states are of Jordan order (at most) j. Then

$$\mathrm{Res}_{\Delta,m}(s_0) = \cup_{j\geq 1}\mathrm{Res}^j_{\Delta,m}(s_0)$$

and the space of quantum resonant states is $\mathrm{Res}^1_{\Delta,m}(s_0)$.

Vasy's operator

We require an asymptotic description of states in $\mathrm{Res}_{\Delta,m}(s_0)$. For this we use aspects of the construction of the meromorphic continuation of the resolvent. To this end, we recall several aspects of its proof given in the previous chapter.

Consider the Lorentzian cone $M := \mathbb{R}^+_s \times X$ with Lorentzian metric $\eta = -ds \otimes ds + s^2 g$. (In our setting where X is a quotient of hyperbolic space, this cone is Minkowski space locally.) Symmetric tensors decompose

$$\mathrm{Sym}^m \mathrm{T}^* M = \bigoplus_{k=0}^{m} a_k \left(\tfrac{ds}{s}\right)^{m-k} \cdot \mathrm{Sym}^k \mathrm{T}^* X, \qquad a_k := \tfrac{1}{\sqrt{(m-k)!}}$$

and the (Lichnerowicz) d'Alembertian $\square$ acts on symmetric m-tensors. A particular conjugation by s of $s^2\,\square$ behaves nicely relative to the preceding decomposition giving the operator

$$\mathbf{Q} := s^{\frac{n}{2}-m+2}\,\square\, s^{-\frac{n}{2}+m} = \nabla^*\nabla + (s\partial_s)^2 + \mathbf{D} + \mathbf{G}$$

for a first order differential operator $\mathbf{D}+\mathbf{G}$ on $\mathrm{Sym}^m\mathrm{T}^*M$. (Above $s\partial_s$ is considered a Lie derivative and, along with $\nabla^*\nabla$, acts diagonally on each factor $\left(\frac{ds}{s}\right)^{m-k} \cdot \mathrm{Sym}^k\mathrm{T}^*X$.) The b-calculus of Melrose permits this operator to be pushed to a family of operators, denoted $\mathcal{Q}_\lambda$, (holomorphic in the complex variable λ) acting on $\oplus_{k=0}^m \mathrm{Sym}^k\mathrm{T}^*X$ above X which takes the form

$$\mathcal{Q}_\lambda = \nabla^*\nabla + \lambda^2 + \mathcal{D} + \mathcal{G}$$

for a first order differential operator $\mathcal{D}+\mathcal{G}$. (A more precise description of $\mathcal{D}+\mathcal{G}$ will be given shortly. Also, we will ultimately set $s := \lambda + \frac{n}{2}$ to return to the conventions present in this chapter.)

Convex cocompact quotients of hyperbolic space are asymptotically even hyperbolic. So consider a boundary defining function, ρ, for the conformal compactification $\overline{X}$. Near $Y := \partial\overline{X}$, say on $U := (0,1)_\rho \times Y$, the metric may be written

$$g = \frac{d\rho^2 + h}{\rho^2}$$

where h is a family of Riemannian metrics on Y smoothly parametrised by $\rho \in [0,1)$ whose Taylor expansion at $\rho = 0$ contains only even powers of ρ. Again consider the Lorentzian cone $M = \mathbb{R}_s^+ \times X$ with metric η. The metric η degenerates at $\rho = 0$ however under the change of coordinates

$$t := s/\rho, \qquad \mu := \rho^2$$

the metric takes the following form on $\mathbb{R}_t^+ \times (0, \varepsilon^2)_\mu \times Y$

$$\eta = -\mu dt \otimes dt - \tfrac{1}{2} t(d\mu \otimes dt + dt \otimes d\mu) + t^2 h.$$

We extend the manifold X to a slightly larger manifold $X_e := ((-1, 0]_\mu \times Y) \sqcup X$ and use μ to provide a smooth structure explained precisely in Section 5.2. Moreover η is extended to $M_e := \mathbb{R}_t^+ \times X_e$ by extending h to a family of Riemannian metrics on Y smoothly parametrised by $\mu \in (-1, 1)$. We now follow the recipe given in the preceding paragraph. The Lichnerowicz d'Alembertian $\Box$ acts on symmetric m-tensors above M_e. Conjugating $t^2 \Box$ provides

$$\mathbf{P} := t^{\frac{n}{2} - m + 2} \Box t^{-\frac{n}{2} + m}$$

The b-calculus pushes this operator to a family of operators (holomorphic in the complex variable λ), termed "Vasy's operator" and denoted

$$\mathcal{P}_\lambda \in \mathrm{Diff}^2(X_e; \oplus_{k=0}^m \mathrm{Sym}^k \mathrm{T}^* X_e).$$

On U, the two families are related

$$\mathcal{P}_\lambda = \rho^{-\lambda - \frac{n}{2} + m - 2} J \, \mathcal{Q}_\lambda \, J^{-1} \rho^{\lambda + \frac{n}{2} - m}$$

for $J \in C^\infty(X; \mathrm{End}(\oplus_{k=0}^m \mathrm{Sym}^k \mathrm{T}^* X))$ whose entries are homogeneous polynomials in $\frac{d\rho}{\rho} \cdot$, upper triangular in the sense that $J(\mathrm{Sym}^{k_0} \mathrm{T}^* X) \subset \oplus_{k=k_0}^m \mathrm{Sym}^k \mathrm{T}^* X$, and whose diagonal entries are the identity.

There are meromorphic inverses for the operators $\mathcal{P}_\lambda$ and $\mathcal{Q}_\lambda$. We denote respectively these meromorphic inverses by

$$\mathcal{R}_{\mathcal{P},m}(\lambda) : C_c^\infty(X_e; \oplus_{k=0}^m \mathrm{Sym}^k \mathrm{T}^* X_e) \to C^\infty(X_e; \oplus_{k=0}^m \mathrm{Sym}^k \mathrm{T}^* X_e)$$

and

$$\mathcal{R}_{\mathcal{Q},m}(\lambda) : C_c^\infty(X; \oplus_{k=0}^m \mathrm{Sym}^k \mathrm{T}^* X) \to \rho^{\lambda + \frac{n}{2} - m} \oplus_{k=0}^m \rho^{-2k} C^\infty_{\mathrm{even}}(\overline{X}; \mathrm{Sym}^k \mathrm{T}^* X).$$

We do not need a complete description of the precise form of $\mathcal{Q}_\lambda$. With respect to the extent to which we require a precise description, we note its form upon restriction to $\mathrm{Sym}_0^m \mathrm{T}^* X$.

$$\mathcal{Q}_\lambda|_{\mathrm{Sym}_0^m \mathrm{T}^* X} = \begin{bmatrix} \nabla^*\nabla + \lambda^2 - \frac{n^2}{4} - m \\ -2\,\delta \end{bmatrix} : C^\infty(X; \mathrm{Sym}_0^m \mathrm{T}^* X) \to C^\infty(X; \oplus_{k=m-1}^m \mathrm{Sym}_0^k \mathrm{T}^* X)$$

which upon setting $s := \lambda + \frac{n}{2}$ provides

$$\mathcal{Q}_{s - \frac{n}{2}}|_{\mathrm{Sym}_0^m \mathrm{T}^* X} = \begin{bmatrix} \nabla^*\nabla - s(n-s) - m \\ -2\,\delta \end{bmatrix}.$$

In a similar spirit we record that

$$J|_{\oplus_{k=m-1}^{m}\mathrm{Sym}_0^k\mathrm{T}^*X} = \begin{bmatrix} 1 & \frac{d\rho}{\rho}\cdot \\ 0 & 1 \end{bmatrix}.$$

Lemma 7.12. *For $s_0 \in \mathbb{C}$ with $s_0 \neq \frac{n}{2}$, generalised quantum resonant states $\mathrm{Res}^j_{\Delta,m}(s_0)$ are precisely identified with*

$$\left\{ \varphi \in \bigoplus_{k=0}^{j-1} \rho^{s_0-m}(\log\rho)^k\, C^\infty_{\mathrm{even}}(\overline{X};\mathrm{Sym}_0^m\mathrm{T}^*X) \,\middle|\, \varphi \in \ker(\nabla^*\nabla - s_0(n-s_0) - m)^j \cap \ker\delta \right\}.$$

Proof. That a generalised resonant state has the prescribed form is reasonably direct. Indeed given $\varphi \in \mathrm{Im}\left(\prod^{s_0}_{\Delta,m}\right)$ there exists $\psi \in C^\infty_c(X;\mathrm{Sym}_0^m\mathrm{T}^*X)$ (which is divergence-free) such that $\varphi = \mathrm{Res}_{s_0}(\mathcal{R}_{\Delta,m}(s)\psi)$. By Theorem 9, we may write

$$\mathcal{R}_{\Delta,m}(s)\psi =: \rho^{s-m}\Psi_s \in \ker\delta$$

for Ψ a meromorphic family taking values in $C^\infty_{\mathrm{even}}(\overline{X};\mathrm{Sym}_0^m\mathrm{T}^*X)$. Supposing the specific Jordan order of φ to be $j \leq J(s_0)$, equivalently $\mathcal{A}^{j-1}_{s_0}\varphi \neq 0$ and $\varphi \in \ker\mathcal{A}^j_{s_0}$, implies Ψ has a pole of order j at s_0. Expanding ρ^{s-m} and Ψ_s in Taylor and Laurent series about s_0 respectively gives

$$\mathcal{R}_{\Delta,m}(s)\psi = \left(\rho^{s_0-m}\sum_{k=0}^{j-1}(\log\rho)^k\frac{(s-s_0)^k}{k!} + O((s-s_0)^j)\right)\left(\Psi^{\mathrm{Hol}}_s + \sum_{k=0}^{j}\frac{\Psi^{(k)}}{(s-s_0)^k}\right)$$

with Ψ^{Hol} (a holomorphic family) and $\Psi^{(k)}$ taking values in $C^\infty_{\mathrm{even}}(\overline{X};\mathrm{Sym}_0^m\mathrm{T}^*X)$. Extracting the residue gives the result that

$$\varphi \in \left(\oplus_{k=0}^{j-1}\rho^{s_0-m}(\log\rho)^k\, C^\infty_{\mathrm{even}}(\overline{X};\mathrm{Sym}_0^m\mathrm{T}^*X)\right)\cap\ker\delta\,.$$

For the converse statement we initially follow [GHW16, Proposition 4.1]. Suppose $\varphi \in \ker\mathcal{A}^j_{s_0}$ trace-free, divergence-free, and takes the required asymptotic form. We may suppose $\mathcal{A}^{j-1}_{s_0}\varphi \neq 0$. Set

$$\varphi^{(1)} := \mathcal{A}^{j-1}_{s_0}\varphi \in \rho^{s_0-m}C^\infty_{\mathrm{even}}(\overline{X};\mathrm{Sym}_0^m\mathrm{T}^*X)\cap\ker\delta\,.$$

For $k \in \{2,\dots,j\}$, there exist polynomials $p_{k,l}$ such that upon defining

$$\varphi^{(k)} := (n-2s_0)^{k-1}\mathcal{A}^{(j-k)}_{s_0}\varphi + \sum_{\ell=1}^{k-1}p_{k,l}(n-2s_0)\,\mathcal{A}^{(j-k+\ell)}_{s_0}\varphi \in \ker\Lambda\cap\ker\delta$$

we satisfy the condition, for $k \in \{1,\dots,j\}$,

$$\mathcal{A}_{s_0}\varphi^{(k)} - (n-2s_0)\varphi^{(k-1)} + \varphi^{(k-2)} = 0 \tag{7.4}$$

(with the understanding that $\varphi^{(0)} = \varphi^{(-1)} = 0$). Note that such a condition appears upon demanding

$$\mathcal{A}_s\varphi_s = O((s-s_0)^j), \qquad \varphi_s := \sum_{k=1}^{j}\varphi^{(k)}(s-s_0)^{k-1}$$

Define

$$\Phi_s := \sum_{k=1}^{j} \Phi^{(k)}(s-s_0)^{k-1}, \qquad \Phi^{(k)} := \rho^{-s_0+m} \sum_{\ell=0}^{k-1} \tfrac{(-\log\rho)^\ell}{\ell!} \varphi^{(k-\ell)}.$$

We claim that

$$\Phi^{(k)} \in C^\infty_{\text{even}}(\overline{X}; \mathrm{Sym}^m_0 \mathrm{T}^* X).$$

As $\Phi^{(k)}$ a priori belongs in the space $\oplus_{\ell=0}^{k-1} (\log\rho)^\ell \, C^\infty_{\text{even}}(\overline{X}; \mathrm{Sym}^m_0 \mathrm{T}^* X)$, it suffices to observe that

$$\mathcal{P}_{s_0-\frac{n}{2}} \, \Phi^{(k)} \in C^\infty_{\text{even}}(\overline{X}; \oplus_{k=m-1}^{m} \mathrm{Sym}^k \mathrm{T}^* X)$$

where

$$\rho^2 \, \mathcal{P}_{s_0-\frac{n}{2}} \, \Phi^{(k)} = \begin{bmatrix} 1 & \frac{d\rho}{\rho} \cdot \\ 0 & 1 \end{bmatrix} \rho^{-s_0+m} \begin{bmatrix} \mathcal{A}_{s_0} \\ -2\,\delta \end{bmatrix} \rho^{s_0-m} \Phi^{(k)}.$$

We perform the required calculation in the collar neighbourhood $U = (0,1)_\rho \times Y$ where the metric is of the form $g = \rho^{-2}(d\rho^2 + h)$ and with a frame $\{dy^i\}_{1\le i\le n}$ for T^*Y. Define $\rho^{-2}B \in C^\infty_{\text{even}}(X; \mathrm{End}(\mathrm{T}^*Y))$ by $Bdy^i := \sum_{jk} \frac{1}{2}(h^{-1})^{ij}(\rho\partial_\rho h_{jk})dy^k$ and extend it to $\rho^{-2}B \in C^\infty_{\text{even}}(X; \mathrm{End}(\mathrm{T}^*X))$ as a derivation with $Bd\rho := 0$. The Laplacian, on functions, takes the form

$$\Delta = -(\rho\partial_\rho)^2 + \rho^2 \Delta_h + (n - \mathrm{tr}_h B)\rho\partial_\rho. \tag{7.5}$$

We calculate $\rho^2 \, \mathcal{P}_{s_0-\frac{n}{2}} \, \Phi^{(k)}$. The first tedious step is

$$\begin{aligned}
&\rho^{-s_0+m} \, \mathcal{A}_{s_0} \, \rho^{s_0-m} \Phi^{(k)} \\
&= \rho^{-s_0+m} \left(\Delta - s_0(n-s_0) - m\right) \sum_{\ell=0}^{k-1} \frac{(-\log\rho)^\ell}{\ell!} \varphi^{(k-\ell)} \\
&= \rho^{-s_0+m} \sum_{\ell=0}^{k-1} \frac{1}{\ell!} \Big((-\log\rho)^\ell \, \mathcal{A}_{s_0} \, \varphi^{(k-\ell)} - 2\,\mathrm{tr}_g \Big(\nabla(-\log\rho)^\ell \otimes \nabla\varphi^{(k-\ell)} \Big) + (\Delta(-\log\rho)^\ell)\varphi^{(k-\ell)} \Big) \\
&= \rho^{-s_0+m} \sum_{\ell=0}^{k-1} \frac{1}{\ell!} \Big((-\log\rho)^\ell \, \mathcal{A}_{s_0} \, \varphi^{(k-\ell)} - 2\,\mathrm{tr}_g \Big(\nabla(-\log\rho)^\ell \otimes \nabla\varphi^{(k-\ell)} \Big) + \big(\Delta(-\log\rho)^\ell\big) \, \varphi^{(k-\ell)} \Big)
\end{aligned}$$

and we split this calculation up further into three parts. Treating the first part with (7.4),

$$\begin{aligned}
&\rho^{-s_0+m} \sum_{\ell=0}^{k-1} \frac{(-\log\rho)^\ell}{\ell!} \, \mathcal{A}_{s_0} \, \varphi^{(k-\ell)} \\
&= \rho^{-s_0+m} \sum_{\ell=0}^{k-1} \frac{(-\log\rho)^\ell}{\ell!} \Big((n-2s_0)\varphi^{(k-1-\ell)} - \varphi^{(k-2-\ell)} \Big) \\
&= (n-2s_0)\Phi^{(k-1)} - \Phi^{(k-2)}.
\end{aligned}$$

Treating the second part directly

$$\begin{aligned}
&\rho^{-s_0+m}\sum_{\ell=0}^{k-1}\frac{1}{\ell!}\left(-2\operatorname{tr}_g\left(\nabla(-\log\rho)^\ell\otimes\nabla\varphi^{(k-\ell)}\right)\right)\\
&=\rho^{-s_0+m}\sum_{\ell=0}^{k-1}\frac{(-\log\rho)^{\ell-1}}{(\ell-1)!}\left(2\nabla_{\rho\partial_\rho}\varphi^{(k-\ell)}\right)\\
&=\rho^{-s_0+m}\sum_{\ell=0}^{k-1}2\nabla_{\rho\partial_\rho}\left(\frac{(-\log\rho)^{\ell-1}}{(\ell-1)!}\varphi^{(k-\ell)}\right)-2\left(\nabla_{\rho\partial_\rho}\frac{(-\log\rho)^{\ell-1}}{(\ell-1)!}\right)\varphi^{(k-\ell)}\\
&=2\rho^{-s_0+m}\nabla_{\rho\partial_\rho}\left(\rho^{s_0-m}\Phi^{(k-1)}\right)+2\Phi^{(k-2)}\\
&=2\rho^{m}\nabla_{\rho\partial_\rho}\left(\rho^{-m}\Phi^{(k-1)}\right)+2s_0\Phi^{(k-1)}+2\Phi^{(k-2)}.
\end{aligned}$$

Treating the third part with (7.5)

$$\begin{aligned}
&\rho^{-s_0+m}\sum_{\ell=0}^{k-1}\frac{1}{\ell!}\left(\Delta(-\log\rho)^\ell\right)\varphi^{(k-\ell)}\\
&=\rho^{-s_0+m}\sum_{\ell=0}^{k-1}\frac{1}{\ell!}\left(-\ell(\ell-1)(-\log\rho)^{\ell-2}+(\operatorname{tr}_h B-n)\ell(-\log\rho)^{\ell-1}\right)\varphi^{(k-\ell)}\\
&=(\operatorname{tr}_h B-n)\Phi^{(k-1)}-\Phi^{(k-2)}.
\end{aligned}$$

Combining these calculations provides

$$\rho^{-s_0+m}\,\mathcal{A}_{s_0}\,\rho^{s_0-m}\Phi^{(k)}=(\operatorname{tr}_h B+2\rho^m\nabla_{\rho\partial_\rho}\rho^{-m})\Phi^{(k-1)}.$$

The second tedious step in calculating $\rho^2\,\mathcal{P}_{s_0-\frac{n}{2}}\,\Phi^{(k)}$ is (recall $\varphi^{(k-\ell)}\in\ker\delta$)

$$\begin{aligned}
&\rho^{-s_0+m}(-2\,\delta)\rho^{s_0+m}\Phi^{(k)}\\
&=\rho^{-s_0+m}\sum_{\ell=0}^{k-1}\frac{2}{\ell!}\operatorname{tr}_g\left(\nabla(-\log\rho)^\ell\otimes\varphi^{(k-\ell)}\right)\\
&=\rho^{-s_0+m}\sum_{\ell=0}^{k-1}\frac{(-\log\rho)^{\ell-1}}{(\ell-1)!}\left(-2\tfrac{d\rho}{\rho}\lrcorner\,\varphi^{(k-\ell)}\right)\\
&=-2\tfrac{d\rho}{\rho}\lrcorner\,\Phi^{(k-1)}
\end{aligned}$$

Combing the two previous calculations provides

$$\rho^2\,\mathcal{P}_{s_0-\frac{n}{2}}\,\Phi^{(k)}=\begin{bmatrix}1 & \frac{d\rho}{\rho}\,\cdot\\ 0 & 1\end{bmatrix}\begin{bmatrix}2\rho^m\nabla_{\rho\partial_\rho}\rho^{-m}+\operatorname{tr}_h B\\ -2\frac{d\rho}{\rho}\,\lrcorner\end{bmatrix}\Phi^{(k-1)}$$

which may be developed upon analysing the following term

$$\left(\rho^m\nabla_{\rho\partial_\rho}\rho^{-m}-\tfrac{d\rho}{\rho}\cdot\tfrac{d\rho}{\rho}\,\lrcorner\right)\Phi^{(k-1)}.$$

Recall the asymptotic structure of tensors which are evenly smooth at infinity described at the end of Subsection 6.1. Writing

$$\Phi^{(k-1)} = \sum_{\ell=0}^{m} \sum_{L \in \mathscr{A}^\ell} \Phi^{(k-1)}_{\ell,L} (\rho d\rho)^{m-\ell} dy^L, \qquad \Phi^{(k-1)}_{\ell,L} \in C^\infty(X),$$

and remarking $\nabla_{\rho\partial_\rho} \rho d\rho = 2\rho d\rho$ and $\nabla_{\rho\partial_\rho} dy^\ell = (1+B) dy^\ell$ gives

$$\begin{aligned}
&\left(\rho^m \nabla_{\rho\partial_\rho} \rho^{-m} - \tfrac{d\rho}{\rho} \cdot \tfrac{d\rho}{\rho} \lrcorner \right) \Phi^{(k-1)} \\
&= (-m + \rho\partial_\rho + 2(m-\ell) + (\ell + B) - (m-\ell)) \sum_{L \in \mathscr{A}^\ell} \Phi^{(k-1)}_{\ell,L} (\rho d\rho)^{m-\ell} dy^L \\
&= (\rho\partial_\rho + B)\Phi^{(k-1)}
\end{aligned}$$

where $\rho\partial_\rho$ is to be interpreted as a Lie derivative. This finally establishes that

$$\rho^2 \, \mathcal{P}_{s_0} \, \Phi^{(k)} = \begin{bmatrix} 2\rho\partial_\rho + 2B + \operatorname{tr}_h B \\ -2\frac{d\rho}{\rho} \cdot \end{bmatrix} \Phi^{(k-1)}$$

which by induction on k produces the desired claim that $\Phi^{(k)} \in C^\infty_{\text{even}}(\overline{X}; \operatorname{Sym}^m \mathrm{T}^* X)$.

We extend $\Phi^{(k)}$ smoothly onto compactly supported sections over X_e and apply $\mathcal{R}_{\mathcal{P},m}(s - \frac{n}{2})$ to $\mathcal{P}_{s-\frac{n}{2}} \, \Phi_s$. On X,

$$\begin{aligned}
\Phi_s &= \mathcal{R}_{\mathcal{P},m}(s - \tfrac{n}{2}) \, \mathcal{P}_{s-\frac{n}{2}} \, \Phi_s \\
&= \rho^{-s+m} J \mathcal{R}_{\mathcal{Q},m}(s - \tfrac{n}{2}) \begin{bmatrix} \mathcal{A}_s \\ -2\,\delta \end{bmatrix} \rho^{s-m} \Phi_s
\end{aligned}$$

whence upon unpacking the definition of Φ_s and the expansion of ρ^{s+m} in s about s_0 implies

$$\varphi_s + O((s-s_0)^j) = \mathcal{R}_{\mathcal{Q},m}(s - \tfrac{n}{2})(s-s_0)^j \psi_s$$

for ψ a holomorphic family taking values in $C^\infty_{\text{even}}(\overline{X}; \oplus_{k=m-1}^m \operatorname{Sym}^m \mathrm{T}^* X)$. Considering the term at order $(s-s_0)^{j-1}$ provides that $\varphi^{(j)}$ is in the image of $\prod_{\mathcal{Q},m}^{s_0 - \frac{n}{2}}$. As $\varphi^{(j)} \in C^\infty(X; \operatorname{Sym}^m_0 \mathrm{T}^* X) \cap \ker \delta$ and

$$\operatorname{Im}\left(\prod\nolimits_{\Delta,m}^{s_0}\right) = \operatorname{Im}\left(\prod\nolimits_{\mathcal{Q},m}^{s_0 - \frac{n}{2}}\right) \cap C^\infty(X; \operatorname{Sym}^m_0 \mathrm{T}^* X) \cap \ker \delta$$

we deduce that $\varphi^{(j)}$ is in the image of $\prod_{\Delta,m}^{s_0}$. Therefore $\mathcal{A}^k_{s_0} \, \varphi^{(j)}$ is also in said image for $k \leq j$ whence the definition of $\varphi^{(k)}$ provides the desired result that φ is in the image of $\prod_{\Delta,m}^{s_0}$. □

7.5 Boundary Distributions and the Poisson Operator

Define $\operatorname{Bd}_m(\lambda)$ to be the following set of boundary distributions

$$\operatorname{Bd}_m(\lambda) := \left\{ \omega \in \mathcal{D}'(\mathbb{S}^n; \operatorname{Sym}^m_0 \mathrm{T}^* \mathbb{S}^n) \;\middle|\; \operatorname{supp}(w) \subset K_\Gamma,\, U^*_\gamma \omega(y) = T_\gamma(y)^{-\lambda-m} \omega(y) \text{ for } \gamma \in \Gamma, y \in \mathbb{S}^n \right\}$$

Then for $\lambda_0 \in \mathbb{C}$ a resonance,

$$\pi_\Gamma^* \left(\mathrm{Res}^1_{A,m}(\lambda_0) \cap \ker \Lambda \cap \ker \nabla_-\right) = (\Phi_-)^{\lambda_0} \mathcal{Q}_- \left(\mathrm{Bd}_m(\lambda_0)\right).$$

The Poisson operator is defined via integration of the fibres of $\pi_S : S\mathbb{H}^{n+1} \to \mathbb{H}^{n+1}$. For $u \in \mathcal{D}'(S\mathbb{H}^{n+1}; \otimes^m \mathcal{E}^*)$ we define, for $x \in \mathbb{H}^{n+1}$,

$$(\pi_{0*}u)(x) := \int_{S_x\mathbb{H}^{n+1}} u(x,\xi) dS(\xi)$$

where integration of elements of $\otimes^m \mathcal{E}^*$ is performed by embedding them in $\otimes^m \mathrm{T}^*\mathbb{H}^{n+1}$. For $\lambda \in \mathbb{C}$, the Poisson operator may be now defined as

$$\mathcal{P}_\lambda : \begin{cases} \mathcal{D}'(\mathbb{S}^n; \mathrm{Sym}^m_0 \mathrm{T}^*\mathbb{S}^n) & \to & C^\infty(\mathbb{H}^{n+1}; \mathrm{Sym}^m_0 \mathrm{T}^*\mathbb{H}^{n+1}) \\ \omega & \mapsto & \pi_{0*}\left((\Phi_-)^\lambda \mathcal{Q}_- \omega\right) \end{cases}$$

There is a useful change of variables which allows the integral to be performed on the boundary $\mathbb{S}^n$. Specifically, upon introducing the Poisson kernel, we may write

$$\mathcal{P}_\lambda \, \omega(x) = \int_{\mathbb{S}^n} P(x,y)^{n+\lambda} \left(\otimes^m \tau_{-}{}^*_{(x,\xi_-)}\right) \omega(y) \, dS(y) \tag{7.6}$$

for $\xi_- = \xi_-(x,y)$.

Asymptotics of the Poisson operator

We start by recalling a weak expansion detailed in [DFG15, Lemma 6.8]. For this we appeal to the diffeomorphism ϕ used in Defintion 5.1. That is, take ρ an even boundary defining function and giving $\phi : [0,\varepsilon) \times \mathbb{S}^n \to \overline{\mathbb{H}^{n+1}}$ the diffeomorphism induced by the flow of the gradient $\mathrm{grad}_{\rho^2 g}(\rho)$. By implicitly using ϕ we identify a neighbourhood of the boundary of $\overline{\mathbb{H}^{n+1}}$ with $[0,\varepsilon)_\rho \times \mathbb{S}^n$. Given $\Psi \in C^\infty(\mathbb{S}^n; \mathrm{Sym}^m \mathrm{T}\mathbb{S}^n)$ we define for ρ small

$$\psi(\rho, y) := (\otimes^m \tau_{-(x,\xi_-)}) \Psi(y)$$

for $x = (\rho, y)$ and $\xi_- = \xi_-(x,y)$.

Lemma 7.13. *[DFG15] Let $\omega \in \mathcal{D}'(\mathbb{S}^n; \mathrm{Sym}^m \mathrm{T}^*\mathbb{S}^n)$ and $\lambda \in \mathbb{C} \backslash (-\frac{n}{2} - \frac{1}{2}\mathbb{N}_0)$. For each $y \in \mathbb{S}^n$, there exists a neighbourhood $U_y \subset \overline{\mathbb{H}^{n+1}}$ of y and an even boundary defining function ρ such that for any $\Psi \in C^\infty(\mathbb{S}^n; \mathrm{Sym}^m \mathrm{T}\mathbb{S}^n)$ with support contained in $U_y \cap \mathbb{S}^n$ and giving $\psi \in C^\infty((0,\varepsilon) \times \mathbb{S}^n; \mathrm{Sym}^m \mathrm{T}\mathbb{S}^n)$ as above, there exists $F_\pm \in C^\infty_{\mathrm{even}}([0,\varepsilon))$ such that*

$$\int_{\mathbb{S}^n} ((\mathcal{P}_\lambda \, \omega)(\rho, y), \psi(\rho, y)) \, dS(y) = \begin{cases} \rho^{-\lambda} F_-(\rho) + \rho^{n+\lambda} F_+(\rho), & \lambda \notin -\frac{n}{2} + \mathbb{N}; \\ \rho^{-\lambda} F_-(\rho) + \rho^{n+\lambda} \log(\rho) F_+(\rho), & \lambda \in -\frac{n}{2} + \mathbb{N}, \end{cases}$$

where dS is the measure obtained from the metric $\rho^2 g$ restricted to $\mathbb{S}^n$. Moreover, if ω and Ψ have distinct supports, then the expansion may be written

$$\begin{cases} \rho^{n+\lambda} F_+(\rho), & \lambda \notin -\frac{n}{2} + \mathbb{N}; \\ \rho^{n+\lambda} (\log(\rho) F_+(\rho) + F'_+(\rho)), & \lambda \in -\frac{n}{2} + \mathbb{N}, \end{cases}$$

for $F'_+ \in C^\infty_{\text{even}}([0,\varepsilon))$.

□

Remark 7.14. The evenness is a consequence of the even expansions of the Bessel functions appearing in the proof. The additional conclusion when ω and Ψ have distinct supports is due to Equation 6.31 in the proof as well as the final equation displayed in the proof. In particular, the differential operators (rather than pseudo-differential operators) which appear do not enlarge the supports of ω and Ψ. Finally, if ω and Ψ have supports with non-trivial intersection, then $F_-(0) \neq 0$.

Proposition 7.15. *For* $\lambda \in \mathbb{C}\backslash(-\frac{n}{2} - \frac{1}{2}\mathbb{N}_0)$*, the pushforward map* $\pi_{0*} : \mathcal{D}'(SX; \mathrm{Sym}^m \mathcal{E}^*) \to \mathcal{D}'(X; \mathrm{Sym}^m \mathrm{T}^* X)$ *restricts to a linear isomorphism of complex vector spaces*

$$\pi_{0*} : \mathrm{Res}^j_{A,m}(\lambda_0) \cap \ker \Lambda \cap \ker \nabla_- \to \mathrm{Res}^j_{\Delta,m}(\lambda_0 + n).$$

Proof. Consider $u^{(k)} \in \mathrm{Res}^k_{A,m}(\lambda_0) \cap \ker \Lambda \cap \ker \nabla_-$ for $1 \leq k \leq j$ such that $(A + \lambda_0)u^{(k)} = -u^{(k-1)}$ and $(A + \lambda_0)u^{(1)} = 0$. We may suppose that $u^{(k)} \neq 0$. We lift these generalised resonant states to $\tilde{u}^{(k)} := \pi_\Gamma^* u^{(k)}$ whose supports are contained in $\pi_\Gamma^{-1}(K_+)$. Define

$$\tilde{\varphi}^{(k)} := \pi_{0*}\tilde{u}^{(k)},$$
$$\varphi^{(k)} := \pi_{0*}u^{(k)}.$$

Now $\varphi^{(1)}$ is a quantum resonance. Indeed, the distribution $v^{(1)} := (\Phi_-)^{-\lambda_0}\tilde{u}^{(1)}$ is annihilated by A (as well as both Λ and ∇_-) so there exists $\omega^{(1)} \in \mathrm{Bd}_m(\lambda_0)$ such that $\tilde{u}^{(1)} = (\Phi_-)^{\lambda_0}\mathcal{Q}_- w^{(1)}$. The properties of the Poisson transformation imply that $\tilde{\varphi}^{(1)} = \mathcal{P}_{\lambda_0}\tilde{u}^{(1)}$ is trace-free, divergence-free and in the kernel of $(\Delta - s_0(n - s_0) - m)$ for $s_0 := \lambda_0 + n$. The same statement is true for $\varphi^{(1)}$. Considering the alternative definition for the Poisson operator (7.6), as well as the upper half-space model, we recall the structure of $\otimes^m \tau_-{}^*$ from (7.1) and that $\rho^{-1}P(x, y)$ is smooth except at $x = (0, y)$. Since $\omega^{(1)}$ has support contained in K_Γ disjoint from Ω_Γ (and $\overline{X} = \Gamma\backslash(X \sqcup \Omega_\Gamma)$) we conclude that $\varphi^{(1)} \in \rho^{s_0 - m}C^\infty_{\text{even}}(X; \mathrm{Sym}^m \mathrm{T}^* X)$. This is the characterisation of quantum resonances given in Lemma 7.12. Therefore, as claimed, $\varphi^{(1)}$ is a quantum resonance.

We now show that $\varphi^{(k)}$ is a generalised quantum resonant. Define

$$v^{(k)} := (\Phi_-)^{-\lambda_0}\sum_{\ell=1}^{k}\frac{(-\log \Phi_-)^{k-\ell}}{(k-\ell)!}\tilde{u}^{(\ell)}.$$

Then a direct calculation shows $A\tilde{v}^{(k)} = 0$ and, since $\mathrm{d}_-\,\Phi_- = 0$, it also follows that $\nabla_-\tilde{v}^{(k)} = 0$. So let $\omega^{(k)} \in \mathcal{D}'(\mathbb{S}^n; \mathrm{Sym}^m_0 \mathrm{T}^*\mathbb{S}^n)$ with $\mathcal{Q}_-\omega^{(k)} := v^{(k)}$ and note $\mathrm{supp}(w^{(k)}) \subset K_\Gamma$. Rewriting $\tilde{u}^{(k)}$ in terms of $\tilde{v}^{(k)}$,

$$\tilde{u}^{(k)} = (\Phi_-)^{\lambda_0}\sum_{\ell=1}^{k}\frac{(\log \Phi_-)^{k-\ell}}{(k-\ell)!}\tilde{v}^{(\ell)}$$

and observing that

$$\partial_\lambda^{(k-\ell)}\mathcal{P}_{\lambda_0}\,\omega^{(\ell)} = \pi_{0*}\left((\Phi_-)^{\lambda_0}(\log \Phi_-)^{k-\ell}\mathcal{Q}_- w^{(\ell)}\right)$$

we obtain

$$\tilde{\varphi}^{(k)} = \pi_{0*}\tilde{u}^{(k)} = \sum_{\ell=1}^{k} \frac{\partial_\lambda^{(k-\ell)} \mathcal{P}_{\lambda_0} w^{(\ell)}}{(k-\ell)!}.$$

Taylor expanding $(\Delta + \lambda(n+\lambda) - m)\,\mathcal{P}_\lambda(w^{(k-\ell)}) = 0$ about λ_0 implies

$$(\Delta + \lambda_0(n+\lambda_0) - m)\frac{\partial_\lambda^{(\ell)} \mathcal{P}_{\lambda_0} w^{(k-\ell)}}{\ell!} + (2\lambda_0 + n)\frac{\partial_\lambda^{(\ell-1)} \mathcal{P}_{\lambda_0} w^{(k-\ell)}}{(\ell-1)!} + \frac{\partial_\lambda^{(\ell-2)} \mathcal{P}_{\lambda_0} w^{(k-\ell)}}{(\ell-2)!} = 0.$$

By introducing (again) $s_0 := \lambda_0 + n$, we deduce that

$$(\Delta - s_0(n-s_0) - m)\tilde{\varphi}^{(k)} = -(2s_0 - n)\tilde{\varphi}^{(k-1)} - \tilde{\varphi}^{(k-2)}$$

with the interpretation that $\tilde{\varphi}^{(0)} = \tilde{\varphi}^{(-1)} = 0$. By injectivity of the Poisson operator, $\varphi^{(k)} \neq 0$. A similar expansion for $\delta\,\mathcal{P}_\lambda(w^{(k-\ell)}) = 0$ implies $\delta\,\tilde{\varphi}^{(k)} = 0$. Recalling the definition of the Poisson operator involving the Poisson kernel, we have $\partial_\lambda^k P(x,y)^{n+\lambda_0} = P(x,y)^{s_0}(\log P(x,y))^k$ and so, as with the case of $\varphi^{(1)}$, we conclude

$$\varphi^{(k)} \in \oplus_{\ell=0}^{k-1} \rho^{s_0-m}(\log\rho)^\ell\, C^\infty_{\mathrm{even}}(X; \mathrm{Sym}_0^m \mathrm{T}^*X).$$

and so it is a generalised quantum resonance $\varphi^{(k)} \in \mathrm{Res}^k_{\Delta,m}(\lambda_0 + n)$ by Lemma 7.12.

In order to show surjectivity of π_{0*}, consider $\varphi^{(j)} \in \mathrm{Res}^j_{\Delta,m}(s_0)$ for $s_0 := \lambda_0 + n$ and define $\varphi^{(k)}$ for $1 \leq k < j$ by $\varphi^{(k)} := \mathcal{A}_{s_0}^{j-k}\,\varphi^{(j)} \in \mathrm{Res}^k_{\Delta,m}(\lambda_0 + n)$ (recalling the definition $\mathcal{A}_s := (\Delta - s(n-s) - m)$). We may assume $\varphi^{(1)} \neq 0$. By modifying $\varphi^{(k)}$ via linear terms in $\varphi^{(\ell)}$ with $1 \leq \ell < k$, we may assume

$$(\Delta - s_0(n-s_0) - m)\varphi^{(k)} = -(2s_0 - n)\varphi^{(k-1)} - \varphi^{(k-2)}.$$

We lift these modified states from SX to $S\mathbb{H}^{n+1}$ defining $\tilde{\varphi}^{(k)} := \pi_\Gamma^*\varphi^{(k)}$ which also satisfy the preceding display.

We now prove by induction on $1 \leq k \leq j$ that there exist $\omega^{(k)} \in \mathcal{D}'(\mathbb{S}^n; \mathrm{Sym}_0^m \mathrm{T}^*\mathbb{S}^n)$ with $\mathrm{supp}(\omega^{(k)}) \subset K_\Gamma$ such that

$$\tilde{\varphi}^{(k)} = \sum_{\ell=1}^{k} \frac{\partial_\lambda^{(k-\ell)} \mathcal{P}_{\lambda_0} \omega^{(\ell)}}{(k-\ell)!} \qquad \text{and} \qquad U_\gamma^*\omega^{(k)} = (T_\gamma)^{-\lambda_0 - m} \sum_{\ell=1}^{k} \frac{(-\log T_\gamma)^{k-\ell}}{(k-\ell)!}\omega^{(\ell)}.$$

For $k = 1$, this states that for $\varphi^{(1)} \in \mathrm{Res}^1_{\Delta,m}(s_0)$, there exists $\omega^{(1)} \in \mathrm{Bd}_m(\lambda_0)$ with $\pi_\Gamma^*\varphi^{(1)} = \mathcal{P}_{\lambda_0}\,\omega$. To demonstrate this statement we remark that $\tilde{\varphi}^{(1)}$ is tempered on $\mathbb{H}^{n+1}$, (the proof follows ad verbum [GHW16, Lemma 4.2]), so the surjectivity of the Poisson transform [DFG15, Corollary 7.6] provides $\omega^{(1)} \in \mathcal{D}'(\mathbb{S}^n; \mathrm{Sym}_0^m \mathrm{T}^*\mathbb{S}^n)$ such that $\tilde{\varphi}^{(1)} = \mathcal{P}_{\lambda_0}\,\omega^{(1)}$. The equivariance property demanded of $\omega^{(1)}$ under Γ is satisfied as $\tilde{\varphi}^{(1)} = \pi_\Gamma^*\varphi^{(1)}$. It remains to confirm that $\mathrm{supp}(\omega^{(1)}) \subset K_\Gamma$. By Lemma 7.12, we have the asymptotics $\varphi^{(1)} \in \rho^{s_0-m} C^\infty_{\mathrm{even}}(X; \mathrm{Sym}^m \mathrm{T}^*X)$ and so, by Remark 7.14, it is only possible for the weak expansion of Lemma 7.13 to hold for arbitrary $\Psi \in C^\infty(\Omega_\Gamma; \mathrm{Sym}^m \mathrm{T}^*\mathbb{S}^n)$ if $\mathrm{supp}(\omega^{(1)}) \subset K_\Gamma$.

For the general situation $k > 1$ consider

$$\psi^{(k)} := \tilde{\varphi}^{(k)} - \sum_{\ell=1}^{k-1} \frac{\partial_\lambda^{(k-\ell)} \mathcal{P}_{\lambda_0} \omega^{(\ell)}}{(k-\ell)!}$$

which is in the kernel of $\mathcal{A}_{s_0}$ by a direct calculation. This gives, by the usual argument, a $\omega^{(k)} \in \mathcal{D}'(\mathbb{S}^n; \mathrm{Sym}_0^m \mathrm{T}^*\mathbb{S}^n)$ with $\mathrm{supp}(\omega^{(k)}) \subset K_\Gamma$ such that $\psi^{(k)} = \mathcal{P}_{\lambda_0}\,\omega^{(k)}$ and establishes the first desired equation. Now consider $(\gamma^* - 1)\psi^{(k)}$. As $(\gamma^* - 1)\tilde{\varphi}^{(k)} = 0$ and $\gamma^* \circ \mathcal{P}_\lambda = \mathcal{P}_\lambda \circ ((T_\gamma)^{\lambda+m} U_\gamma^*)$, the induction hypothesis gives

$$(\gamma^* - 1)\psi^{(k)} = -\,\mathcal{P}_{\lambda_0}\left((T_\gamma)^{\lambda_0+m} \sum_{\ell=1}^{k-1} \frac{(\log T_\gamma)^{k-\ell}}{(k-\ell)!} U_\gamma^* \omega^{k-\ell}\right)$$

alternatively as $\psi^{(k)} = \mathcal{P}_{\lambda_0}\,\omega^{(k)}$, the equivariance of $\mathcal{P}_\lambda$ implies

$$(\gamma^* - 1)\psi^{(k)} = \mathcal{P}_{\lambda_0}(((T_\gamma)^{\lambda_0+m} U_\gamma^* - 1)\omega^{(k)}).$$

From these two equations and the injectivity of the Poisson operator, we obtain the desired equivariance property for $U_\gamma^* \omega^{(k)}$.

We now may reproduce in reverse the beginning of the injectivity direction of this proof. Consider the following elements of $\mathcal{D}'(S\mathbb{H}^{n+1}; \mathrm{Sym}_0^m \mathcal{E}^*)$

$$v^{(k)} := \mathcal{Q}_- \omega^{(k)} \qquad \text{and} \qquad \tilde{u}^{(k)} := (\Phi_-)^{\lambda_0} \sum_{\ell=1}^{k} \frac{(\log \Phi_-)^{k-\ell}}{(k-\ell)!} \tilde{v}^{(\ell)}.$$

Then $\tilde{u}^{(k)}$ is annihilated by ∇_-. The equivariance property of $\omega^{(k)}$ implies that $(A + \lambda_0)\tilde{u}^{(k)} = -\tilde{u}^{(k-1)}$, that $(A + \lambda_0)\tilde{u}^{(1)} = 0$, and that $\gamma^* \tilde{u}^{(k)} = \tilde{u}^{(k)}$. So these distributions project down giving $u^{(k)} \in \mathcal{D}'(SX; \mathrm{Sym}_0^m \mathcal{E}^*)$. By the support properties of $\omega^{(k)}$, the support of $u^{(k)}$ is contained in K_+. Finally, elliptic regularity implies that the wave front sets of $u^{(k)}$ are contained in the annihilators of both E^n and E^u hence in $E^{*u}|_{K_+} = E_+^*$. This is the characterisation of Ruelle resonances so the equality $\pi_{0*} u^{(k)} = \varphi^{(k)}$ implies surjectivity of the Poisson operator. □

7.6 Proof of Theorem 11

Theorem 11. *Let $X = \Gamma\backslash\mathbb{H}^{n+1}$ be a smooth oriented convex cocompact hyperbolic manifold, and $\lambda_0 \in \mathbb{C}\backslash(-\frac{n}{2} - \frac{1}{2}\mathbb{N}_0)$. There exists a vector space linear isomorphism between Ruelle generalised resonant states*

$$\mathrm{Res}_{A,0}(\lambda_0)$$

and the following space of quantum generalised resonant states

$$\bigoplus_{m\in\mathbb{N}_0} \bigoplus_{k=0}^{\lfloor \frac{m}{2} \rfloor} \mathrm{Res}_{\Delta,m-2k}(\lambda_0 + m + n).$$

Proof. Generalised Ruelle resonant states are filtered by Jordan order

$$\begin{aligned}\mathrm{Res}_A(\lambda_0) &= \bigoplus_{j=1}^{J(\lambda_0)} \left(\mathrm{Res}_{A,0}^j(\lambda_0)/\mathrm{Res}_{A,0}^{j-1}(\lambda_0)\right) \\ &= \bigcup_{j=1}^{J(\lambda_0)} \mathrm{Res}_{A,0}^j(\lambda_0).\end{aligned}$$

Generalised Ruelle resonant states of Jordan order j, are filtered into bands via (7.2)

$$\mathrm{Res}^j_{A,0}(\lambda_0) = \bigoplus_{m\in\mathbb{N}_0} \left(V^j_{A,m}(\lambda_0)/V^j_{A,m-1}(\lambda_0)\right).$$

Each band m of Jordan order j is identified via Proposition 7.3 (and Proposition 7.4) with vector-valued generalised resonant states for the geodesic flow which are in the kernel of the unstable horosphere operator.

$$(\mathrm{d}_-)^m : V^j_{A,m}(\lambda_0)/V^j_{A,m-1}(\lambda_0) \to \mathrm{Res}^j_{A,m}(\lambda_0 + m) \cap \ker \nabla_-.$$

These generalised resonant states are decomposed via (7.3) according to their trace

$$\mathrm{Res}^j_{A,m}(\lambda_0 + m) \cap \ker \nabla_- = \bigoplus_{k=0}^{\lfloor \frac{m}{2} \rfloor} \mathrm{L}^k \left(\mathrm{Res}^j_{A,m-2k}(\lambda_0 + m) \cap \ker \Lambda \cap \ker \nabla_-\right).$$

Generalised resonant states of the geodesic flow which are in the kernels of the unstable horosphere operator and the trace operator are identified via Proposition 7.15 with generalised resonant states of the Laplacian acting on symmetric tensors

$$\pi_{0*} : \mathrm{Res}^j_{-X,m-2k}(\lambda_0 + m) \cap \ker \Lambda \cap \ker \nabla_- \to \mathrm{Res}^j_{\Delta,m-2k}(\lambda_0 + m + n). \qquad \square$$

Remark 7.16. As the proof of Theorem 11 shows, the isomorphism restricts to isomorphisms respecting the Jordan order of the generalised resonant states.

A. Symmetric Tensors

This appendix recalls some conventions for symmetric tensors on a vector space and some differential operators on Riemannian manifolds. It follows conventions established in [HMS16].

A.1 Linear Algebra

Let E be a vector space of dimension $n+1$ equipped with an inner product g. Use g to identify E with its dual space. Let $\{e_i\}_{0\leq i\leq n}$ be an orthonormal basis. We denote by $\mathrm{Sym}^m E$ the m-fold symmetric tensor product of E. Elements are symmetrised tensor products

$$u_1 \cdot \ldots \cdot u_m := \sum_{\sigma\in\Pi_m} u_{\sigma(1)} \otimes \ldots \otimes u_{\sigma(m)}, \qquad u_i \in E$$

where Π_m is the permutation group of $\{1,\ldots,m\}$. By linearity, this extends the operation $\cdot$ to a map from $\mathrm{Sym}^m E \times \mathrm{Sym}^{m'} E$ to $\mathrm{Sym}^{m+m'} E$. Define $\mathrm{Sym} E := \bigoplus_{m\in\mathbb{N}_0} \mathrm{Sym}^m E$ and introduce the map $\deg : \mathrm{Sym} E \to \mathrm{Sym} E$ by declaring $\deg u = mu$ for $u \in \mathrm{Sym}^m E$.

Some notation for finite sequences will be necessary for many calculations presented in this thesis. For a fixed positive integer m, denote by $\mathscr{A}^m$ the space of all sequences $K = k_1 \ldots k_m$ with $0 \leq k_r \leq n$. We write $\{k_r \to j\}K$ for the result of replacing the r^{th} element of K by j. If j is not present, this implies we remove the r^{th} element from K, while if k_r is not present, this implies we add j to K to obtain the sequence jK. This notation extends to replacing multiple indices at once. For example, $\{k_p \to, k_r \to\}K$ indicates we first remove the r^{th} element from K and then remove the p^{th} element from $\{k_r \to\}K$. We set

$$e_K := e_{k_1} \cdot \ldots \cdot e_{k_m} \in \mathrm{Sym}^m E, \qquad K = k_1 \ldots k_m \in \mathscr{A}^m.$$

The inner product takes the form $g = \frac{1}{2}\sum_{i=0}^n e_i \cdot e_i$, equivalently $g = \frac{1}{2}\sum_{i=0}^n e_{ii}$. For $u \in E$ we write u^m to denote the symmetric product of m copies of u. The inner product induces an inner product on $\mathrm{Sym}^m E$, also denoted by g, defined by

$$g(u_1 \cdot \ldots \cdot u_m, v_1 \cdot \ldots \cdot v_m) := \sum_{\sigma\in\Pi_m} g(u_1, v_{\sigma(1)}) \ldots g(u_m, v_{\sigma(m)}), \qquad u_i, v_i \in E.$$

For $u \in E$, the metric adjoint of the linear map $u\cdot : \mathrm{Sym}^m E \to \mathrm{Sym}^{m+1} E$ is the contraction $u \lrcorner : \mathrm{Sym}^{m+1} E \to \mathrm{Sym}^m E$ defined by

$$(u \lrcorner v)(w_1, \ldots, w_m) := v(u, w_1, \ldots, w_m), \qquad u, w_i \in E, v \in \mathrm{Sym}^m E.$$

Contraction and multiplication with the metric g define two additional linear maps:

$$\Lambda : \begin{cases} \mathrm{Sym}^m E & \to & \mathrm{Sym}^{m-2} E \\ u & \mapsto & \sum_{i=0}^n e_i \lrcorner e_i \lrcorner u \end{cases}$$

and

$$\mathrm{L} : \begin{cases} \mathrm{Sym}^m E & \to & \mathrm{Sym}^{m+2} E \\ u & \mapsto & \sum_{i=0}^n e_i \cdot e_i \cdot u \end{cases}$$

which are adjoint to each other. As the notation is motivated by standard notation from complex geometry, we will refer to these two operators as Lefschetz-type operators. Denote by

$$\mathrm{Sym}_0^m E := \ker\left(\Lambda : \mathrm{Sym}^m E \to \mathrm{Sym}^{m-2} E\right)$$

the space of trace-free symmetric tensors of degree m.

There are several algebraic commutator relations between the operators thus far introduced:

$$[\Lambda, \mathrm{L}] = 2(n+1) + 4\deg, \quad [\deg, \mathrm{L}] = 2\,\mathrm{L}, \quad [\deg, \Lambda] = -2\,\Lambda\,.$$

and for $u \in E$,

$$[u \lrcorner, \Lambda] = 0 = [u\,\cdot, \mathrm{L}], \quad [u \lrcorner, \mathrm{L}] = 2u\,\cdot, \quad [u\,\cdot, \Lambda] = -2u \lrcorner\,.$$

Also, $\deg = \sum_{i=0}^n e_i \cdot e_i \lrcorner$.

We have the decomposition

$$\mathrm{Sym}^m E = \bigoplus_{k=0}^{\lfloor \frac{m}{2} \rfloor} \mathrm{L}^k (\mathrm{Sym}_0^{m-2k} E)$$

and the operator $\mathrm{L}\,\Lambda$ preserves this decomposition. Indeed for $u \in \mathrm{Sym}_0^{m-2k} E$,

$$\begin{aligned} \mathrm{L}\,\Lambda\,\mathrm{L}^k u &= \sum_{\ell=1}^k \mathrm{L}^{\ell-1} [\mathrm{L}, \Lambda]\, \mathrm{L}^{k-\ell} u \\ &= \sum_{\ell=1}^k \mathrm{L}^{\ell} (2(n+1) + 4\deg)\, \mathrm{L}^{k-\ell} u \\ &= \left(\sum_{\ell=1}^k 2(n+1) + 4(m-2\ell) \right) \mathrm{L}^k u \\ &= 2k(n+2m-2k-1)\, \mathrm{L}^k u \end{aligned}$$

Therefore on $\mathrm{Sym}^m E$ we have the identification

$$\mathrm{L}^k\, \mathrm{Sym}_0^{m-2k} E = \ker\left(\mathrm{L}\,\Lambda - 2k(n+2m-2k-1) : \mathrm{Sym}^m E \to \mathrm{Sym}^m E\right).$$

An alternative dimension for the vector space

If the dimension of the vector space is n, then our convention is that an orthonormal basis is $\{e_i\}_{1\le i\le n}$ (it is enumerated from $i = 1$ rather than $i = 0$). Of course, this implies that when considering sequences, $\mathscr{A}^m$ denotes the space of all sequences $K = k_1 \dots k_m$ with $1 \le k_r \le n$.

Homogeneous polynomials

Symmetric tensors may be identified with homogeneous polynomials via the metric. Denote by $\mathrm{Pol}^m(E)$ the space of homogeneous polynomials of degree m on E and define $\mathrm{Pol}(E) := \bigoplus_{m\in\mathbb{N}_0} \mathrm{Pol}^m(E)$. To a given $u \in \mathrm{Sym}^m(E)$, we associate a homogeneous polynomial Pu such that

$$Pu(x) := g(u, x^m), \qquad x = \sum_{i=1}^{n} x_i e_i.$$

Let ∂_{x_i} denote the partial derivative in the direction e_i and by $\Delta_{\mathrm{Pol}(E)} := -\sum_{i=1}^n \partial_{x_i}^2$ the flat Laplacian. Then as $e_i \lrcorner = \partial_{x_i} \circ P$, we have

$$\Lambda = -\Delta_{\mathrm{Pol}(E)} \circ P.$$

In conclusion, symmetric tensors of degree m which are trace-free correspond to polynomials homogeneous of degree m which are harmonic.

A.2 Differential Geometry

Let (X, g) be a Riemannian manifold of dimension $n + 1$ and denote by ∇ the Levi-Civita connection. Identify the tangent bundle $\mathrm{T}X$ with the cotangent bundle T^*X and let $\{e_i\}_{0\le i\le n}$ be a local orthonormal frame.

Let the symmetrisation of the covariant derivative, called the symmetric differential, be denoted d:

$$\mathrm{d} : \begin{cases} C^\infty(X; \mathrm{Sym}^m \mathrm{T}X) & \to & C^\infty(X; \mathrm{Sym}^{m+1} \mathrm{T}X) \\ u & \mapsto & \sum_{i=0}^n e_i \cdot \nabla_{e_i} u \end{cases}$$

and, by δ, its formal adjoint, called the divergence:

$$\delta : \begin{cases} C^\infty(X; \mathrm{Sym}^{m+1} \mathrm{T}X) & \to & C^\infty(X; \mathrm{Sym}^{m} \mathrm{T}X) \\ u & \mapsto & -\sum_{i=0}^n e_i \lrcorner \nabla_{e_i} u \end{cases}$$

The two first-order operators behave nicely with L and Λ giving the following commutation relations [HMS16, Equation 8]:

$$[\Lambda, \delta] = 0 = [\mathrm{L}, \mathrm{d}], \quad [\Lambda, \mathrm{d}] = -2\,\delta, \quad [\mathrm{L}, \delta] = 2\,\mathrm{d}\,. \tag{A.1}$$

The rough Laplacian is denoted by $\nabla^*\nabla$:

$$\nabla^*\nabla : \begin{cases} C^\infty(X; \mathrm{Sym}^m \mathrm{T}X) & \to & C^\infty(X; \mathrm{Sym}^m \mathrm{T}X) \\ u & \mapsto & \nabla^*\nabla\, u \end{cases}$$

where ∇^* is the formal adjoint of $\nabla : C^\infty(X; \mathrm{Sym}^m \mathrm{T}X) \to C^\infty(X; \mathrm{T}X \otimes \mathrm{Sym}^m \mathrm{T}X)$. Equivalently

$$\nabla^* \nabla\, u = (-\operatorname{tr} \circ \nabla \circ \nabla)(u), \qquad u \in C^\infty(X; \mathrm{Sym}^m \mathrm{T}X)$$

where $\operatorname{tr} : C^\infty(X; \mathrm{T}X \otimes \mathrm{T}X) \to C^\infty(X)$ is a trace operator obtained from g by declaring $\operatorname{tr}(e_i \otimes e_j) = \delta_{ij}$ and is then extended to $\operatorname{tr} : C^\infty(X; \mathrm{T}X \otimes \mathrm{T}X \otimes \mathrm{Sym}^m \mathrm{T}X) \to C^\infty(X; \mathrm{Sym}^m \mathrm{T}X)$. For the Lichnerowicz Laplacian, we introduce the Riemann curvature tensor which will be denoted by R:

$$\mathrm{R}_{u,v}\, w = [\nabla_u, \nabla_v] w - \nabla_{[u,v]} w, \qquad u, v, w \in C^\infty(X; \mathrm{T}X)$$

and is extended to all tensor bundles as a derivation. On $\mathrm{Sym}^m \mathrm{T}X$ we introduce the following curvature endomorphism which is denoted by $q(\mathrm{R})$:

$$q(\mathrm{R}) u = \sum_{i,j=0}^{n} e_j \cdot e_i \lrcorner \mathrm{R}_{e_i, e_j}\, u, \qquad u \in \mathrm{Sym}^m \mathrm{T}X.$$

The Lichnerowicz Laplacian is denoted by Δ:

$$\Delta : \begin{cases} C^\infty(X; \mathrm{Sym}^m \mathrm{T}X) & \to & C^\infty(X; \mathrm{Sym}^m \mathrm{T}X) \\ u & \mapsto & (\nabla^* \nabla + q(\mathrm{R})) u \end{cases}$$

For local computations, we decompose symmetric m-tensors using the symmetrised basis elements:

$$u = \sum_{K \in \mathscr{A}^m} u_K e_K, \qquad u \in C^\infty(X; \mathrm{Sym}^m \mathrm{T}X), u_K \in C^\infty(X).$$

Bibliography

[AG11] Erwann Aubry and Colin Guillarmou. Conformal harmonic forms, Branson-Gover operators and Dirichlet problem at infinity. *J. Eur. Math. Soc. (JEMS)*, 13(4):911–957, 2011.

[AGH16] G. Arizmendi, A. L. Garcia-Pulido, and R. Herrera. A note on the geometry and topology of almost even-Clifford Hermitian manifolds. *ArXiv e-prints*, June 2016.

[AGI98] Bogdan Alexandrov, Gueo Grantcharov, and Stefan Ivanov. Curvature properties of twistor spaces of quaternionic Kähler manifolds. *J. Geom.*, 62(1-2):1–12, 1998.

[AH15] Gerardo Arizmendi and Rafael Herrera. Centralizers of spin subalgebras. *J. Geom. Phys.*, 97:77–92, 2015.

[AHS78] M. F. Atiyah, N. J. Hitchin, and I. M. Singer. Self-duality in four-dimensional Riemannian geometry. *Proc. Roy. Soc. London Ser. A*, 362(1711):425–461, 1978.

[AHS16] Gerardo Arizmendi, Rafael Herrera, and Noemi Santana. Almost even-Clifford hermitian manifolds with a large automorphism group. *Proc. Amer. Math. Soc.*, 144(9):4009–4020, 2016.

[BB82] Lionel Bérard-Bergery. Sur de nouvelles variétés riemanniennes d'Einstein. In *Institut Élie Cartan, 6*, volume 6 of *Inst. Élie Cartan*, pages 1–60. Univ. Nancy, Nancy, 1982.

[BEG94] T. N. Bailey, M. G. Eastwood, and A. R. Gover. Thomas's structure bundle for conformal, projective and related structures. *Rocky Mountain J. Math.*, 24(4):1191–1217, 1994.

[Bes08] Arthur L. Besse. *Einstein manifolds.* Classics in Mathematics. Springer-Verlag, Berlin, 2008. Reprint of the 1987 edition.

[BG05] Thomas Branson and A. Rod Gover. Conformally invariant operators, differential forms, cohomology and a generalisation of Q-curvature. *Comm. Partial Differential Equations*, 30(10-12):1611–1669, 2005.

[BHM$^+$15] Jean-Pierre Bourguignon, Oussama Hijazi, Jean-Louis Milhorat, Andrei Moroianu, and Sergiu Moroianu. *A spinorial approach to Riemannian and conformal geometry.* EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2015.

[Biq00] Olivier Biquard. Métriques d'Einstein asymptotiquement symétriques. *Astérisque*, (265):vi+109, 2000.

[BM11] Florin Belgun and Andrei Moroianu. Weyl-parallel forms, conformal products and Einstein-Weyl manifolds. *Asian J. Math.*, 15(4):499–520, 2011.

[Bur90] F. E. Burstall. Riemannian twistor spaces and holonomy groups. In *Twistors in mathematics and physics*, volume 156 of *London Math. Soc. Lecture Note Ser.*, pages 53–70. Cambridge Univ. Press, Cambridge, 1990.

[BVW15] Dean Baskin, András Vasy, and Jared Wunsch. Asymptotics of radiation fields in asymptotically Minkowski space. *Amer. J. Math.*, 137(5):1293–1364, 2015.

[Del99] Erwann Delay. Étude locale d'opérateurs de courbure sur l'espace hyperbolique. *J. Math. Pures Appl. (9)*, 78(4):389–430, 1999.

[Del02] Erwann Delay. Essential spectrum of the Lichnerowicz Laplacian on two tensors on asymptotically hyperbolic manifolds. *J. Geom. Phys.*, 43(1):33–44, 2002.

[Del07] Erwann Delay. TT-eigentensors for the Lichnerowicz Laplacian on some asymptotically hyperbolic manifolds with warped products metrics. *Manuscripta Math.*, 123(2):147–165, 2007.

[DFG15] Semyon Dyatlov, Frédéric Faure, and Colin Guillarmou. Power spectrum of the geodesic flow on hyperbolic manifolds. *Anal. PDE*, 8(4):923–1000, 2015.

[DG16] Semyon Dyatlov and Colin Guillarmou. Pollicott-Ruelle resonances for open systems. *Ann. Henri Poincaré*, 17(11):3089–3146, 2016.

[DGH08] Zindine Djadli, Colin Guillarmou, and Marc Herzlich. *Opérateurs géométriques, invariants conformes et variétés asymptotiquement hyperboliques*, volume 26 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 2008.

[DS10] N. S. Dairbekov and V. A. Sharafutdinov. Conformal Killing symmetric tensor fields on Riemannian manifolds. *Mat. Tr.*, 13(1):85–145, 2010.

[DZ16] Semyon Dyatlov and Maciej Zworski. Mathematical theory of scattering resonances. Version 0.1, 12 2016.

[ES83] James Eells and Simon Salamon. Constructions twistorielles des applications harmoniques. *C. R. Acad. Sci. Paris Sér. I Math.*, 296(15):685–687, 1983.

[FG85] Charles Fefferman and C. Robin Graham. Conformal invariants. *Astérisque*, (Numéro Hors Série):95–116, 1985. The mathematical heritage of Élie Cartan (Lyon, 1984).

[FG12] Charles Fefferman and C. Robin Graham. *The ambient metric*, volume 178 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2012.

[Fri01] Thomas Friedrich. Weak Spin(9)-structures on 16-dimensional Riemannian manifolds. *Asian J. Math.*, 5(1):129–160, 2001.

[FS11] Frédéric Faure and Johannes Sjöstrand. Upper bound on the density of Ruelle resonances for Anosov flows. *Comm. Math. Phys.*, 308(2):325–364, 2011.

[FT13] Frédéric Faure and Masato Tsujii. Band structure of the Ruelle spectrum of contact Anosov flows. *C. R. Math. Acad. Sci. Paris*, 351(9-10):385–391, 2013.

[GH16] A. L. Garcia-Pulido and R. Herrera. Rigidity and vanishing theorems for almost even-Clifford Hermitian manifolds. *ArXiv e-prints*, September 2016.

[GHW16] C. Guillarmou, J. Hilgert, and T. Weich. Classical and quantum resonances for hyperbolic surfaces. *ArXiv e-prints*, May 2016.

[GL91] C. Robin Graham and John M. Lee. Einstein metrics with prescribed conformal infinity on the ball. *Adv. Math.*, 87(2):186–225, 1991.

[GMP10] Colin Guillarmou, Sergiu Moroianu, and Jinsung Park. Eta invariant and Selberg zeta function of odd type over convex co-compact hyperbolic manifolds. *Adv. Math.*, 225(5):2464–2516, 2010.

[Gui05] Colin Guillarmou. Meromorphic properties of the resolvent on asymptotically hyperbolic manifolds. *Duke Math. J.*, 129(1):1–37, 2005.

[GZ95] Laurent Guillopé and Maciej Zworski. Polynomial bounds on the number of resonances for some complete spaces of constant negative curvature near infinity. *Asymptotic Anal.*, 11(1):1–22, 1995.

[HMS16] Konstantin Heil, Andrei Moroianu, and Uwe Semmelmann. Killing and conformal Killing tensors. *J. Geom. Phys.*, 106:383–400, 2016.

[Hör07] Lars Hörmander. *The analysis of linear partial differential operators. III.* Classics in Mathematics. Springer, Berlin, 2007. Pseudo-differential operators, Reprint of the 1994 edition.

[Lic61] André Lichnerowicz. Propagateurs et commutateurs en relativité générale. *Inst. Hautes Études Sci. Publ. Math.*, (10):56, 1961.

[LM89] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series.* Princeton University Press, Princeton, NJ, 1989.

[Maz88] Rafe Mazzeo. The Hodge cohomology of a conformally compact metric. *J. Differential Geom.*, 28(2):309–339, 1988.

[Mel93] Richard B. Melrose. *The Atiyah-Patodi-Singer index theorem*, volume 4 of *Research Notes in Mathematics.* A K Peters, Ltd., Wellesley, MA, 1993.

[Mel94] Richard B. Melrose. Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces. In *Spectral and scattering theory (Sanda, 1992)*, volume 161 of *Lecture Notes in Pure and Appl. Math.*, pages 85–130. Dekker, New York, 1994.

[MM87] Rafe R. Mazzeo and Richard B. Melrose. Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature. *J. Funct. Anal.*, 75(2):260–310, 1987.

[MP13] Andrei Moroianu and Mihaela Pilca. Higher rank homogeneous Clifford structures. *J. Lond. Math. Soc. (2)*, 87(2):384–400, 2013.

[MS11] Andrei Moroianu and Uwe Semmelmann. Clifford structure on Riemannian manifolds. *Adv. Math.*, 228(2):940–967, 2011.

[Nag02] Paul-Andi Nagy. On nearly-Kähler geometry. *Ann. Global Anal. Geom.*, 22(2):167–178, 2002.

[OR85] N. R. O'Brian and J. H. Rawnsley. Twistor spaces. *Ann. Global Anal. Geom.*, 3(1):29–58, 1985.

[Orn01] L. Ornea. Weyl structures on quaternionic manifolds. A state of the art. *ArXiv Mathematics e-prints*, May 2001.

[Pen77] Roger Penrose. The twistor programme. *Rep. Mathematical Phys.*, 12(1):65–76, 1977.

[PP01] S. J. Patterson and Peter A. Perry. The divisor of Selberg's zeta function for Kleinian groups. *Duke Math. J.*, 106(2):321–390, 2001. Appendix A by Charles Epstein.

[PP15] Maurizio Parton and Paolo Piccinni. The even Clifford structure of the fourth Severi variety. *Complex Manifolds*, 2:89–104, 2015.

[PPV15] M. Parton, P. Piccinni, and V. Vuletescu. Clifford systems in octonionic geometry. *ArXiv e-prints*, November 2015.

[Sal82] Simon Salamon. Quaternionic Kähler manifolds. *Invent. Math.*, 67(1):143–171, 1982.

[Sel56] A. Selberg. Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series. *J. Indian Math. Soc. (N.S.)*, 20:47–87, 1956.

[Vas13a] András Vasy. Microlocal analysis of asymptotically hyperbolic and Kerr-de Sitter spaces (with an appendix by Semyon Dyatlov). *Invent. Math.*, 194(2):381–513, 2013.

[Vas13b] András Vasy. Microlocal analysis of asymptotically hyperbolic spaces and high-energy resolvent estimates. In *Inverse problems and applications: inside out. II*, volume 60 of *Math. Sci. Res. Inst. Publ.*, pages 487–528. Cambridge Univ. Press, Cambridge, 2013.

[Vas17] András Vasy. Analytic continuation and high energy estimates for the resolvent of the Laplacian on forms on asymptotically hyperbolic spaces. *Adv. Math.*, 306:1019–1045, 2017.

[Wan09] F. Wang. Dirichlet-to-Neumann Map for Poincaré-Einstein Metrics in Even Dimensions. *ArXiv e-prints*, May 2009.

[Zwo16] Maciej Zworski. Resonances for asymptotically hyperbolic manifolds: Vasy's method revisited. *J. Spectr. Theory*, 6(4):1087–1114, 2016.

[Zwo17] Maciej Zworski. Mathematical study of scattering resonances. *Bulletin of Mathematical Sciences*, 7(1):1–85, 2017.