Twistor Spaces over Quaternionic-Kähler Manifolds

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Abstract

Let (M, g) be a quaternionic-Kähler manifold. This dissertation constructs the twistor space $\mathcal{Z}$ whose fibre is diffeomorphic to the 2-sphere. We prove, in the spirit of Bérard-Bergery, that the twistor space is integrable. If (M, g) is Ricci positive (it is necessarily Einstein) the twistor space is endowed with a metric such that $\pi : \mathcal{Z} \to M$ becomes a Riemannian submersion with totally geodesic fibres. It is shown that two possible scales for the fibre imply $\mathcal{Z}$ is Einstein. Moreover, for one of these scales, $\mathcal{Z}$ is also Kähler.

The twistor space construction is generalised in a new way to Riemannian manifolds carrying even Clifford structures. The twistor fibres are investigated geometrically and a natural curvature condition, in terms of the associated Clifford bundle morphism, is introduced. It is conjectured that, under this condition, the twistor space is also integrable. The rank 3 case coincides with the quaternionic-Kähler setting and is thus immediately true. The rank 4 case is also settled affirmatively.

Contents

1. Introduction

This dissertation investigates the construction and properties of the twistor space associated with a quaternionic-Kähler manifold, as well as a generalisation of this structure to Riemannian manifolds with even Clifford structures.

Interest in quaternionic-Kähler manifolds originates from several sources. Among such are the classification of holonomy groups and the consequences of self-duality in 4-dimensions. We briefly introduce and motivate these manifolds. In 1955 Berger [1] classified Riemannian manifolds in terms of their holonomy. For a non-symmetric, irreducible, simply-connected Riemannian manifold (M, g) of dimension n, precisely one of the following cases holds for the holonomy of g

1. $\mathrm{Hol}(g) = SO(n)$,
2. $n = 2m$ with $m \geq 2$, and $\mathrm{Hol}(g) = U(m)$ in $SO(2m)$,
3. $n = 2m$ with $m \geq 2$, and $\mathrm{Hol}(g) = SU(m)$ in $SO(4m)$,
4. $n = 4m$ with $m \geq 2$, and $\mathrm{Hol}(g) = Sp(m)$ in $SO(4m)$,
5. $n = 4m$ with $m \geq 2$, and $\mathrm{Hol}(g) = Sp(m) \cdot Sp(1)$ in $SO(4m)$,
6. $n = 7$ and $\mathrm{Hol}(g) = G_2$ in $SO(7)$,
7. $n = 8$ and $\mathrm{Hol}(g) = Spin(7)$ in $SO(8)$.

The assumptions are not restrictive as, for an arbitrary Riemannian manifold, one may work with $\mathrm{Hol}^0(g)$, and use the de Rham decomposition theorem as well as Cartan's classification of Riemannian symmetric spaces. Importantly, quaternionic-Kähler manifolds correspond to $\mathrm{Hol}(g) = Sp(m) \cdot Sp(1)$ in this classification and the initial local structure available to a differential geometer may be described as follows. Of course $\mathrm{Hol}(g) = U(m)$ corresponds to Kähler geometry and with it, a parallel complex structure J. The case $\mathrm{Hol}(g) = Sp(m)$ represents a specific case of Kähler geometry where the manifold possesses three parallel complex structures I, J, K behaving under composition as the standard quaternions i, j, k. This situation is too restrictive to include the (symmetric) quaternionic projective space $\mathbb{HP}^n$ which does not even admit an almost-complex structure. Indeed, quaternionic-Kähler geometry provides precisely the required generalisation. We take the following as our definition.

Definition 1.1. *A riemannian manifold (M, g) of dimension $4n \geq 8$ is quaternionic-Kähler if there exists a covering of M of open sets $\{U_i\}$ and, for each i, two almost complex structures I_i and J_i on U_i such that*

1. *g is Hermitian for I_i and J_i on U_i,*
2. *I_i and J_i anticommute ,*
3. *the local subbundle spanned by $\{I_i, J_i, K_i\}$ of End(TM) where $K_i = I_iJ_i$ is preserved by the connection induced from the Levi-Civita connection,*
4. *the subbundle of End(TM) spanned locally by $\{I_i, J_i, K_i\}$ of End(TM) is globally well-defined, independent of U_i.*

Note that $Sp(1) \cdot Sp(1) = SO(4)$ so the naive extension of the definition coincides with a generic oriented Riemannian manifold in dimension 4. There is a more appropriate extension involving the decomposition of the curvature tensor with which we will not concern ourselves. It suffices to say, the extension is equivalent to considering oriented manifolds which are self-dual and Einstein, a situation in which Penrose's original twistor space is a complex manifold.

A classic and important result is that quaternionic-Kähler manifolds are Einstein; a fact which is presented in Chapter 3. This decomposes their study into 3 distinct cases, depending on the sign (positive, negative or zero) of the scalar constant. We remark here that the Ricci flat case is precisely that of hyper-Kähler geometry. In all three cases, one may construct a complex manifold, the twistor space, which fibres over the original manifold. This was independently obtained by Salamon [13] and Bérard-Bergery [2] and is the content of Chapter 4. For the model space, projective quaternionic space, the twistor space gives a Hopf fibration $\pi : \mathbb{CP}^{2n+1} \to \mathbb{HP}^n$. In the case of positive scalar curvature, one can say more. There are two possible scales for the fibres for which the twistor space is Einstein and for one of these, it is also Kähler. This is the content of Chapter 5. More widely, the twistor space is known to be a projective Fano manifold equipped with a holomorphic contact structure and it is the result of LeBrun [9] that shows the two are equivalent. Finally we highlight the conjecture, proven in low dimensions, that all complete quaternionic-Kähler manifolds with positive scalar curvature are Wolf spaces. Early works on quaternionic-Kähler manifolds are [4, 5, 7] wherein one may find elementary properties. A recent account of the current field is provided by Salamon's essay [10].

The twistor space construction is a direct generalisation of Penrose's twistor construction for an orientated Riemannian manifold M of dimension 4. It is thus appropriate to recall here the construction, details of which may be found in Besse [2]. One observes that the Hodge $*$-operation acts on 2-forms as an involution and thus may be used to decompose 2-forms into self-dual and anti-self-dual forms. The twistor space $\pi : P \to M$ is then taken to be the unit sphere bundle of the 3-dimensional real vector bundle of anti-self-dual forms. The fibres are thus 2-spheres. Using the metric, one identifies 2-forms and skew-adjoint endomorphisms of the tangent bundle. Next, the Levi-Civita connection splits the tangent bundle of P such that $TP = \mathcal{H} \oplus \mathcal{V}$ where $\mathcal{V}$ is the tangent bundle along the fibres and $\mathcal{H}$ identifies, at each point, with the tangent space on the base manifold below via π_*. As each point of the twistor space is a complex structure on the tangent space below, one may partly define the complex structure on $\mathcal{H}$. The remainder of the definition, defining the complex structure on $\mathcal{V}$, is possible after identifying the fibres with $\mathbb{CP}^1$. Distinct from this construction is the notion that the manifold is half-conformally flat, a condition on the decomposition of the Weyl tensor also involving the Hodge $*$-operation. It is under precisely this hypothesis that one may show that the almost complex structure of the twistor space is integrable.

The final part of this dissertation considers even Clifford structures over Riemannian manifolds. This structure was introduced by Moroianu and Semmelmann [11]; the case of parallel even Clifford structures of rank 3 is precisely that of quaternionic-Kähler geometry. We are thus naturally led to consider possible generalisations of twistor spaces in this setting. The chapter initiates this study including an investigation of the geometrical structure of the twistor fibre, which is no longer simply identified with S^2. A curvature condition is introduced which naturally appears in [11] in order to further the study of the integrability of the twistor space. It is left open, for further work, as to whether the construction will ultimately give an integrable space in general.

1.1 Conventions

We recall below standard facts of differential and Riemannian geometry which will be required in following chapters. This is simply to establish our conventions and further details may be found in [2, 3].

Let (M, g) be a Riemannian manifold, $\pi : E \to M$ a vector bundle with connection ∇.

Lemma 1.2. *The Koszul formula characterises the unique torsion-free metric connection ∇ on TM. For vector fields X, Y, Z, one has*

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y)$$

Definition 1.3. *The curvature, R^E, on the vector bundle E is defined by*

$$R^E_{X,Y}s = [\nabla_X, \nabla_Y]s - \nabla_{[X,Y]}s$$

for vector fields X, Y on M, a section $s : M \to E$.

Remark 1.4. The connection ∇ induces connections on bundles associated with E. Such connections are also denoted by ∇. Consequently, these bundles also obtain a curvature tensor. In the particular case of the endomorphism bundle $\mathrm{End}(E)$, one has

$$R^{\mathrm{End}(E)}_{X,Y}A = [R^E_{X,Y}, A]$$

for vector fields X, Y on M, an endomorphism $A : M \to \mathrm{End}(E)$.

Definition 1.5. *The Ricci curvature r on the tangent bundle is defined by*

$$r(X, Y) = \sum_i g(R_{X,E_i}E_i, Y)$$

for vector fields X, Y on M, an orthonormal basis $\{E_i\}$.

Proposition 1.6. *Let S_ρ be the sphere of radius ρ in $\mathbb{R}^3$. With the standard metric g induced from $\mathbb{R}^3$, the Ricci curvature of S_ρ is $r = \rho^{-2}g$.*

Proposition 1.7. *The connection on E provides a splitting of the tangent bundle, $TE = \mathcal{H} \oplus \mathcal{V}$ where $\mathcal{V} = \ker \pi_*$. Let X, Y be vector fields on M with horizontal lifts denoted by X^*, Y^*. For a section $s : M \to E$ one has*

$$\mathcal{V}[X^*, Y^*]_s = -R^E_{X,Y}s$$

Proof. We recall from the study of connections on vector bundles that $[X^*, s] = \nabla_X s$ and $\nabla_X s = dsX - X^*$. The vector space structure of the fibres of E allow us to canonically extend the sections $\nabla_X s, \nabla_Y s$ to vertical vector fields on E whence $[\nabla_X s, \nabla_Y s] = 0$. As s provides an immersion of M into E we have $ds[X, Y] = [dsX, dsY]$. A direct calculation gives

$$\begin{aligned}
[X^*, Y^*] &= [X^* + \nabla_X s, Y^* + \nabla_Y s] - [X^*, \nabla_Y s] - [\nabla_X s, Y^*] - [\nabla_X s, \nabla_Y s] \\
&= [dsX, dsY] - [\nabla_X, \nabla_Y]s \\
&= \nabla_{[X,Y]}s + [X, Y]^* - [\nabla_X, \nabla_Y]s
\end{aligned}$$

As $\mathcal{V}[X^*, Y^*] = [X^*, Y^*] - [X, Y]^*$ the formula is established. □

Definition 1.8. *The Nijenhuis tensor of an almost complex structure J is*

$$4N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]$$

2. Riemannian Submersions

The results established in the following chapters rely on standard facts about Riemannian submersions. This notion, in particular the definitions of the T and A tensors, was introduced by O'Neill [12]. We announce precisely the required facts in order to facilitate our study of quaternionic-Kähler manifolds. In particular, many formulae appear in reduced form as we will soon assume that $T = 0$. Further details may be found in Besse [2].

Definition 2.1. *Let $\pi : (M, g) \to (B, \check{g})$ be a submersion of Riemannian manifolds. Let $\mathcal{V}$ be the vertical distribution given by $\ker \pi_*$ and $\mathcal{H}$ the orthogonal complement of $\mathcal{V}$ determined by g, that is, $TM = \mathcal{H} \oplus \mathcal{V}$. Then π is a Riemannian submersion if π_* induces an isometry from $\mathcal{H}_x$ to $T_b B$ for all $x \in M$ with $b = \pi(x)$.*

Definition 2.2. *The projection of the natural connection ∇ on M to each fibre gives a connection $\hat{\nabla}$ on each fibre. The submersion has totally geodesic fibres if $\hat{\nabla} = \nabla$.*

Definition 2.3. *A vector field on M is basic if it is horizontal and $\pi_* X_x = \pi_* X_y$ whenever $\pi(x) = \pi(y)$.*

Remark 2.4. Notationally, U, V will denote vertical fields on M, while X, Y will denote horizontal vector fields on M.

Definition 2.5. *The T tensor is defined by*

$$T_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F$$

for arbitrary vectors E, F.

Definition 2.6. *The A tensor is defined by*

$$A_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F$$

for arbitrary vectors E, F.

Lemma 2.7. *For vertical vectors U, V and horizontal vectors X, Y, one has $g(A_X Y, U) = -g(A_X U, Y)$.*

Proof.

$$0 = Xg(U, Y) = g(\nabla_X U, Y) + g(U, \nabla_X Y) = g(A_X U, Y) + g(A_X Y, U) \qquad \square$$

Proposition 2.8. *For horizontal vectors X, Y, one has $A_X Y = \frac{1}{2}\mathcal{V}[X, Y]$.*

Proof. This is Proposition 9.24 of Besse [2]. □

Proposition 2.9. *The submersion has totally geodesic fibres iff $T = 0$.*

Remark 2.10. From now on, only submersions with totally geodesic fibres will be considered.

Proposition 2.11. *The connection decomposes into horizontal and vertical components as*

$$\begin{aligned}
\nabla_U V &= && \hat{\nabla}_U V \\
\nabla_U X &= \mathcal{H}\nabla_U X && \\
\nabla_X U &= \quad A_X U && + \mathcal{V}\nabla_X U \\
\nabla_X Y &= \mathcal{H}\nabla_X Y + && A_X Y
\end{aligned}$$

for vertical vectors U, V, and horizontal vectors X, Y.

Definition 2.12. *Let $\{E_i\}$ be a local orthonormal frame for $\mathcal{H}$. We define*

$$
\begin{aligned}
(A_X, A_Y) &= \sum g(A_X E_i, A_Y E_i) \\
(AU, AV) &= \sum g(A_{E_i} U, A_{E_i} V) \\
\delta A &= -\sum (\nabla_{E_i} A)_{E_i}
\end{aligned}
$$

Definition 2.13. *The horizontal distribution $\mathcal{H}$ satisfies the Yang-Mills condition if $\delta A(X)$ is horizontal.*

Proposition 2.14. *The Ricci curvature of M is given by*

$$
\begin{aligned}
r(U, V) &= \hat{r}(U, V) + (AU, AV) \\
r(X, U) &= -(\delta A(X), U) \\
r(X, Y) &= \check{r}(X, Y) - 2(A_X, A_Y)
\end{aligned}
$$

where $\hat{r}$ is the Ricci curvature of the fibres, and $\check{r}$ is the pull-back of the Ricci curvature of the base manifold.

Proof. This is Proposition 9.36 of Besse [2] with the hypothesis that the fibres are totally geodesic. □

Corollary 2.15. *The total space M is Einstein with constant λ iff $\mathcal{H}$ is Yang-Mills and*

$$
\begin{aligned}
\hat{r}(U, V) + (AU, AV) &= \lambda g(U, V) \\
\check{r}(X, Y) - 2(A_X, A_Y) &= \lambda g(X, Y)
\end{aligned}
$$

Corollary 2.16. *Suppose that the fibres are Einstein with constant $\hat{\lambda}$ and that the base manifold is Einstein with constant $\check{\lambda}$. Then the total space M is Einstein with constant λ iff $\mathcal{H}$ is Yang-Mills and*

$$
\begin{aligned}
(AU, AV) &= (\lambda - \hat{\lambda}) g(U, V) \\
(A_X, A_Y) &= \tfrac{1}{2}(\check{\lambda} - \lambda) g(X, Y)
\end{aligned}
$$

3. Einstein Property

Let (M^{4n}, g) be a quaternionic-Kähler manifold with $n \geq 2$. We establish below directly that such manifolds are necessarily Einstein. Although a well established property of quaternionic-Kähler manifolds, this chapter is included as many intermediate calculations and results will prove useful in subsequent chapters. Recall Definition 1.1. We call the locally defined triplet of almost complex structures $\{I, J, K\}$ a local quaternionic frame (and do not notationally illustrate their chart dependence).

Lemma 3.1. *For a local quaternionic frame $\{I, J, K\}$,*

$$\begin{aligned} \nabla I &= - \alpha_3 \otimes J + \alpha_2 \otimes K \\ \nabla J &= +\alpha_3 \otimes I - \alpha_1 \otimes K \\ \nabla K &= -\alpha_2 \otimes I + \alpha_1 \otimes J \end{aligned}$$

for local 1-forms α_i.

Proof. For endomorphisms A, B, $D(AB) = (DA)B + A(DB)$ hence $0 = (DS)S + S(DS)$ for an almost complex structure S. Therefore if S is a local section of the twistor space, $\langle DS, S\rangle_E = 0$. □

Lemma 3.2. *For a local quaternionic frame $\{I, J, K\}$,*

$$\begin{aligned} [R_{X,Y}, I] &= - \omega_3(X,Y)J + \omega_2(X,Y)K \\ [R_{X,Y}, J] &= \omega_3(X,Y)I - \omega_1(X,Y)K \\ [R_{X,Y}, K] &= -\omega_2(X,Y)I + \omega_1(X,Y)J \end{aligned}$$

for local 2-forms ω_i and tangent vectors X, Y.

Proof. The calculation of $[R_{X,Y}, I]$ acting on an arbitrary tangent vector Z gives $[R_{X,Y}, I] = [\nabla_X, \nabla_Y]I - \nabla_{[X,Y]}I$ and it remains to follow through the calculation $[R_{X,Y}, I]Z$ using the structure of $\nabla I, \nabla J, \nabla K$. The resulting formulae for ω_i in terms of α_i are

$$\omega_1 = d\alpha_1 - \alpha_2 \wedge \alpha_3, \qquad \omega_2 = d\alpha_2 - \alpha_3 \wedge \alpha_1, \qquad \omega_3 = d\alpha_3 - \alpha_1 \wedge \alpha_2.$$ □

Proposition 3.3. *For a local quaternionic frame $\{I, J, K\}$ with 2-forms ω_i,*

$$\begin{aligned} \omega_1(X,Y) &= \frac{1}{n+2} r(IX, Y) \\ \omega_2(X,Y) &= \frac{1}{n+2} r(JX, Y) \\ \omega_3(X,Y) &= \frac{1}{n+2} r(KX, Y) \end{aligned}$$

for tangent vectors X, Y.

Proof. Evaluating $[R_{X,Y}, J]Z$ against KZ,

$$g(R_{X,Y}JZ, KZ) - g(JR_{X,Y}Z, KZ) = g(\omega_3(X,Y)IZ, KZ) - g(\omega_1(X,Y)KZ, KZ)$$

and using the equivariance of the metric with respect to J, K,

$$-\omega_1(X,Y)|Z|^2 = g(R_{X,Y}Z, IZ) + g(R_{X,Y}JZ, KZ) \tag{3.1}$$

We introduce an orthonormal frame $\{E_i\}$ whence so too is $\{JE_i\}$. As $g(R_{X,Y}\cdot,\cdot)$ is antisymmetric,

$$\sum g(R_{X,Y}E_i, IE_i) = \sum g(R_{X,Y}JE_i, KE_i)$$

whence

$$-2n\omega_1(X,Y) = \sum g(R_{X,Y}E_i, IE_i).$$

The pair symmetry for Riemannian curvature gives $g(R_{X,Y}E_i, IE_i) = g(R_{X,E_i}Y, IE_i) - g(R_{X,IE_i}Y, E_i)$ after applying the Bianchi symmetry. As before, the antisymmetry of $g(R_{X,Y}\cdot,\cdot)$ now implies

$$-n\omega_1(X,Y) = \sum g(R_{X,E_i}Y, IE_i) = -\sum g(IR_{X,E_i}Y, E_i)$$

Recalling the form of the commutator $[R_{X,Y}, I]$ we finish this calculation with

$$\begin{aligned} n\omega_1(X,Y) &= \sum g(-R_{X,E_i}IY + \omega_3(X,E_i)JY - \omega_2(X,E_i)KY, E_i) \\ &= -r(X,IY) + \omega_3(X,JY) - \omega_2(X,KY) \end{aligned}$$

Similarly calculations are done for ω_2, ω_3 culminating in

$$\begin{aligned} n\omega_1(X,IY) + \omega_2(X,JY) + \omega_3(X,KY) &= r(X,Y) \\ \omega_1(X,IY) + n\omega_2(X,JY) + \omega_3(X,KY) &= r(X,Y) \\ \omega_1(X,IY) + \omega_2(X,JY) + n\omega_3(X,KY) &= r(X,Y) \end{aligned}$$

Therefore

$$\omega_1(X,IY) = \omega_2(X,JY) = \omega_3(X,KY) = \frac{1}{n+2}r(X,Y)$$

and the result follows using the skew symmetry of ω_i and the symmetry of the Ricci tensor. □

Theorem 3.4. *A quaternionic-Kähler manifold is Einstein.*

Proof. By 3.1 and Proposition 3.3

$$\frac{1}{n+2}r(X,X)|Z|^2 = -g(R_{X,IX}Z, IZ) - g(R_{X,IX}JZ, KZ)$$

and we investigate the first term. By Lemma 3.2 and the symmetries of the curvature tensor

$$\begin{aligned} -g(R_{X,IX}Z, IZ) &= -g(R_{Z,IZ}X, IX) \\ &= g(JR_{Z,IZ}X, KX) \\ &= g(R_{Z,IZ}JX - \omega_3(Z,IZ)IX + \omega_1(Z,IZ)KX, KX) \\ &= g(R_{JX,KX}Z, IZ) + \omega_1(Z,IZ)|X|^2 \\ &= g(R_{JX,KX}Z, IZ) + \frac{1}{n+2}r(Z,Z)|X|^2 \end{aligned}$$

Replacing Z by JZ in the above calculation gives

$$-g(R_{X,IX}JZ, KZ) = g(R_{JX,KX}JZ, KZ) + \frac{1}{n+2}r(JZ,JZ)|X|^2$$

As before, by 3.1,

$$\frac{1}{n+2} r(JX, JX)|Z|^2 = -g(R_{JX,KX}Z, IZ) - g(R_{JX,KX}JZ, KZ)$$

hence

$$(r(X, X) + r(JX, JX))|Z|^2 = (r(Z, Z) + r(JZ, JZ))|X|^2$$

By Proposition 3.3 and the skew-symmetry of ω_2, we have $r(JX, Y) = -r(JY, X)$. On replacing Y by JY and using the symmetry of r, it follows that $r(JX, JY) = r(X, Y)$. Therefore we may advance the previous calculation and obtain

$$\frac{r(X, X)}{|X|^2} = \frac{r(Z, Z)}{|Z|^2}$$

for arbitrary X, Z hence there exists $\lambda \in \mathbb{R}$ such that $r(X, X) = \lambda|X|^2$. $\square$

4. Twistor Space

Let (M^{4n}, g) be a quaternionic-Kähler manifold with $n \geq 2$. Let E be the 3-dimensional subbundle of $\mathrm{End}(TM)$ spanned locally by a local quaternionic base $\{I, J, K\}$. The local basis $\{I, J, K\}$ defines a Euclidean structure on E by decreeing this basis to be orthogonal and each endomorphism to have norm ρ. (As any transition $\{I, J, K\} \to \{I', J', K'\}$ belongs to SO(3), such a structure is well-defined.) Denote this structure by $\langle \cdot, \cdot \rangle_E$.

Definition 4.1. *The twistor space of M is $\mathcal{Z}$, the sphere bundle of E of radius ρ with respect to $\langle \cdot, \cdot \rangle_E$.*

With $\pi : \mathcal{Z} \to M$ the natural projection, the fibres $\pi^{-1}(x)$ identify with the 2-sphere. For $a, b, c \in \mathbb{R}$ with $a^2 + b^2 + c^2 = 1$ and $\{I, J, K\}$ a local quaternionic frame, one verifies $aI + bJ + cK$ is also an almost-complex structure. Therefore elements $S \in \mathcal{Z}$ correspond to complex structures on $T_{\pi(S)}M$ and sections correspond to almost-complex structures on M.

4.1 Complex Structure

The natural connection on M induces a connection on the bundle of endomorphisms, also denoted ∇. By restriction ∇ is a connection on E by hypothesis. This gives a splitting of $TE = \mathcal{H}^E \oplus \mathcal{V}^E$ and of $T\mathcal{Z} = \mathcal{H} \oplus \mathcal{V}$. Note $\mathcal{H}^E = \mathcal{H}$.

Let $S \in \mathcal{Z}$ and $x = \pi(S)$. We have a canonical identification of $T_S E_x$ with E_x and an orthogonal decomposition $E_x = \mathrm{sp}\{S\} \oplus \mathrm{sp}\{S\}^\perp$ with respect to $\langle \cdot, \cdot \rangle_E$. As $\mathcal{V}_S = T_S \mathcal{Z}_x$, these give an identification

$$\mathcal{V}_S \text{ is identified with } \mathrm{sp}\{S\}^\perp.$$

The projection $\pi : \mathcal{Z} \to M$ gives an identification

$$\pi_* : \mathcal{H}_S \to T_x M$$

Definition 4.2. *Let $X + U$ be a tangent vector at S compatible with the decomposition $T_S\mathcal{Z} = \mathcal{H} \oplus \mathcal{V}$. The almost-complex structure $\mathcal{J}$ on the twistor space is defined by*

$$\mathcal{J}(X + U)_S = \pi_*^{-1} S \pi_* X + SU$$

Remark 4.3. One may also define an almost-complex structure $\mathcal{J}'$ by

$$\mathcal{J}'(X + U)_S = \pi_*^{-1} S \pi_* X - SU$$

however this structure is not integrable, as is clear from the proof of Proposition 4.8.

We now verify the Nijenhuis tensor N vanishes. This is done in stages by considering the action of N on horizontal and vertical vectors. For this, recall the notation of the chapter on submersions: U, V will denote vertical fields on $\mathcal{Z}$ while X, Y will denote horizontal vector fields on $\mathcal{Z}$. The notation will be modified again in the following chapter when we induce a metric on $\mathcal{Z}$, however in what proceeds, R will denote the Riemann curvature tensor of M while R^E will denote the curvature of E.

Proposition 4.4. *The Nijenhuis tensor vanishes when restricted to $\mathcal{V} \times \mathcal{V}$.*

Proof. As the bracket of vertical vectors is again a vertical fibre, this statement is a consequence of the identification of the fibres with $\mathbb{CP}^1$. □

Lemma 4.5. *For a basic vector field X and a vertical vector field U, one has $\mathcal{J}[U,X] = [\mathcal{J}U, X]$.*

Proof. As X is basic, $[U,X]$ is vertical. The result follows as the complex structure on the fibres is preserved by $\mathcal{H}$. □

Lemma 4.6. *For a horizontal vector field Y and a vertical vector field U, one has $\mathcal{J}\mathcal{V}[U,Y] = \mathcal{V}[\mathcal{J}U, Y]$*

Proof. We verify directly that both terms are tensorial in Y. So we may perturb Y to a basic vector field and apply Lemma 4.5. □

Lemma 4.7. *For a basic vector field X and a vertical vector field U, one has $\pi_*[U, \mathcal{J}X] = U\pi_*X$.*

Proof. We evaluate at $S \in \mathcal{Z}$ with $x = \pi(S)$. Noting that $Fl_0^U(S)\pi_*X$ lifts to a basic vector field,

$$\begin{aligned}
\pi_*[U, \mathcal{J}X]_S &= \pi_*\mathcal{L}_U(\mathcal{J}X) \\
&= \left.\frac{d}{dt}\right|_{t=0} \pi_* Fl_t^{U^*}(\mathcal{J}X)_{Fl_t^U(S)} \\
&= \left.\frac{d}{dt}\right|_{t=0} \pi_* Fl_t^{U^*}\pi_*^{-1}(Fl_t^U(S)\pi_*X) \\
&= \left.\frac{d}{dt}\right|_{t=0} Fl_t^U(S)\pi_*X \\
&= U\pi_*X
\end{aligned}$$
□

Proposition 4.8. *The Nijenhuis tensor vanishes when restricted to $\mathcal{V} \times \mathcal{H}$.*

Proof. Lemma 4.5 ensures $4N(U,X) = \mathcal{J}[U, \mathcal{J}X] - [\mathcal{J}U, \mathcal{J}X]$. By Lemma 4.6, this is horizontal. Under π_* we have, by Lemma 4.7 and the definition of $\mathcal{J}U$,

$$\begin{aligned}
\pi_*4N(U,X) &= \pi_*(\mathcal{J}[U,\mathcal{J}X] - [\mathcal{J}U, \mathcal{J}X]) \\
&= SU\pi_*X - SU\pi_*X
\end{aligned}$$
□

Lemma 4.9. *The horizontal component of the Nijenhuis tensor vanishes when restricted to $\mathcal{H} \times \mathcal{H}$.*

Proof. Let $x \in M$ and S a section of $\mathcal{Z}$ over $\Omega \subset M$ with $x \in \Omega$. For $y \in \Omega$ we write S_y for the complex structure on T_yM. Under the natural immersion, we view S as a local section of E. As $\mathcal{H}^E = \mathcal{H}$ we demand that $\nabla S = 0$ at x.

Let X, Y be basic vector fields on $\mathcal{Z}$ and let N^S be the Nijenhuis tensor of S. Restricting to S, we claim $N^S(\pi_*X, \pi_*Y) = \pi_*N(X,Y)$. To verify this, we will investigate one term of each Nijenhuis tensor, other calculations are similar. Recalling that S provides an immersion of Ω into $\mathcal{Z}$, the brackets in the following calculation are π-related. For $y \in \Omega$,

$$\begin{aligned}
\text{second term of } \pi_*4N(X,Y)_{S_y} &= \pi_*\mathcal{J}[\mathcal{J}X, Y] \\
&= S_y\pi_*[\mathcal{J}X, Y] \\
&= S_y\pi_*[\pi_*^{-1}S\pi_*X, Y] \\
&= S_y[S\pi_*X, \pi_*Y] \\
&= \text{second term of } 4N^S(\pi_*X, \pi_*Y)_y
\end{aligned}$$

It now suffices to show $N^S(\pi_*X, \pi_*Y)$ vanishes at x. For notational clarity, we will denote π_*X, π_*Y simply by X, Y respectively. The connection is torsion-free, that is, $[X,Y] = \nabla_XY - \nabla_YX$. A direct calculation using this fact gives the result.

$$\begin{aligned} 4N^S(X,Y) &= [X,Y] + S[SX,Y] + S[X,SY] - [SX,SY] \\ &= \nabla_X Y - \nabla_Y X + S(\nabla_{SX} Y - \nabla_Y SX) + S(\nabla_X SY - \nabla_{SY} X) - \nabla_{SX} SY + \nabla_{SY} SX \\ &= \nabla_X Y - \nabla_Y X + \nabla_{SX} SY + \nabla_Y X - \nabla_X Y - \nabla_{SY} SX - \nabla_{SX} SY + \nabla_{SY} SX = 0 \end{aligned}$$ □

Lemma 4.10. *For $x \in M$ let X, Y be tangent vectors at x and let $S \in \pi^{-1}(x)$. Then*

$$-[R_{X,Y}, S] - S[R_{SX,Y}, S] - S[R_{X,SY}, S] + [R_{SX,SY}, S] = 0.$$

Proof. It suffices to verify the claim in the case where $S = I$. Using Lemma 3.2 and Proposition 3.3 the result follows upon summing the following four calculations.

$$\begin{aligned} -(n+2)[R_{X,Y}, I] &= +r(KX,Y)J - r(JX,Y)K \\ -(n+2)I[R_{IX,Y}, I] &= -r(KX,Y)J + r(JX,Y)K \\ -(n+2)I[R_{X,IY}, I] &= r(JX,IY)J + r(KX,IY)K \\ (n+2)[R_{IX,IY}, I] &= -r(JX,IY)J - r(KX,IY)K \end{aligned}$$ □

Proposition 4.11. *The Nijenhuis tensor vanishes when restricted to $\mathcal{H} \times \mathcal{H}$.*

Proof. By Lemma 4.9 it suffices to consider the vertical component. By Proposition 1.7 and Remark 1.4

$$\begin{aligned} \mathcal{V}4N(X,Y)_S &= \mathcal{V}[X,Y] + \mathcal{J}\mathcal{V}[\mathcal{J}X,Y] + \mathcal{J}\mathcal{V}[X,\mathcal{J}Y] - \mathcal{V}[\mathcal{J}X,\mathcal{J}Y] \\ &= -R^E_{X,Y}S - \mathcal{J}R^E_{SX,Y}S - \mathcal{J}R^E_{X,SY}S + R^E_{SX,SY}S \\ &= -[R_{X,Y}, S] - S[R_{SX,Y}, S] - S[R_{X,SY}, S] + [R_{SX,SY}, S] \end{aligned}$$

which vanishes by Lemma 4.10. □

Theorem 4.12. *The twistor space is a complex manifold.*

Proof. The Nijenhuis tensor vanishes by Propositions 4.4, 4.8, 4.11. □

5. Kähler-Einstein Metric

Let $(M, \check{g})$ be quaternionic-Kähler with Einstein constant $\check{\lambda}$. As before, the tangent bundle of the twistor space decomposes $T\mathcal{Z} = \mathcal{H} \oplus \mathcal{V}$. We define a metric g on $\mathcal{Z}$. The horizontal distribution inherits the pull-back of the metric on M from the identification $\pi_* : \mathcal{H}_S \to T_x M$. Recall that the fibres $\mathcal{Z}_x$ are spheres in E_x of radius ρ with respect to $\langle \cdot, \cdot \rangle_E$. By restriction, this Euclidean structure induces a metric on the fibre $\mathcal{Z}_x$ and hence on $\mathcal{V}_S = T_S \mathcal{Z}_x$. Let $\check{r}$, $\hat{r}$ be the Ricci curvatures of M and the fibres respectively. We also denote by $\check{r}$ the pull-back of $\check{r}$ to $\mathcal{H}$. As before, U, V denote vertical vector fields while X, Y denote horizontal vector fields.

Remark 5.1. We do not explicitly detail $\hat{\lambda}$ (equivalently ρ) above as a function of $\check{\lambda}$. Indeed, two values exist for which $(\mathcal{Z}, g)$ is Einstein. These are

$$\hat{\lambda}_1 = \frac{1}{n+2}\check{\lambda} \quad \text{and} \quad \hat{\lambda}_2 = \frac{n+1}{n+2}\check{\lambda}$$

However for only the first of these values is $(\mathcal{Z}, g, \mathcal{J})$ Kähler. The obstruction for $\hat{\lambda}_2$ follows from the proof of Proposition 5.13.

Proposition 5.2. *The projection $\pi : (\mathcal{Z}, g) \to (M, \check{g})$ is a Riemannian submersion and g is Hermitian for the twistor space $(\mathcal{Z}, \mathcal{J})$. Moreover the fibres are totally geodesic and Einstein with constant $\hat{\lambda} = \rho^{-2}$.*

Proof. By construction the submersion is Riemannian and g is Hermitian for $\mathcal{J}$. Proposition 1.6 gives $\hat{\lambda} = \rho^{-2}$. It is clear that the T tensor vanishes for $T_X U$ and $T_X Y$. As $T_U V$ is horizontal, we calculate $g(T_U V, X)$ using the Koszul formula.

$$2g(T_U V, X) = 2g(\nabla_U V, X) = -Xg(U, V) - g([V, X], U) + g([X, U], V) = -(\mathcal{L}_X g)(U, V) = 0$$

A similar calculation for $T_U X$ gives the result. □

Definition 5.3. *For clarity in subsequent calculations we define the constant κ by*

$$\kappa = \frac{\check{\lambda}\rho^2}{2(n+2)} = \frac{1}{2(n+2)} \frac{\check{\lambda}}{\hat{\lambda}}$$

5.1 Einstein Metric

We now show that $(\mathcal{Z}, g)$ is Einstein by fulfilling the requirements of Corollary 2.16. Let $\{E_i\}$ and $\{F_i\}$ denote local orthonormal frames for $\mathcal{H}$ and $\mathcal{V}$ respectively. We assume that $\{E_i\}$ are basic and do not distinguish notationally between these vector fields and their projections onto M.

Lemma 5.4. *For U a vertical field, one has $(AU, AU) = 4n\kappa^2 \hat{\lambda} |U|^2$.*

Proof. Immediately we have

$$(AU, AU) = \sum_i g(A_{E_i} U, A_{E_i} U) = \sum_i |\mathcal{H}\nabla_{E_i} U|^2$$

We thus investigate $g(\mathcal{H}\nabla_{E_i}U, E_j)$ at $S \in \mathcal{Z}$. By the Koszul formula

$$g(\nabla_{E_i}U, E_j) = \frac{1}{2}\langle [R_{E_i,E_j}, S], U\rangle$$

We evaluate this at $S = I$ and $U = uJ$ for $u \in \mathbb{R}$.

$$g(\nabla_{E_i}U, E_j) = \frac{-1}{2(n+2)}\check{r}(KE_i, E_j)\langle J, uJ\rangle_E = -\kappa u\check{g}(KE_i, E_j)$$

Therefore, summing over i, j,

$$(AU, AU) = \sum_i |\mathcal{H}\nabla_{E_i}U|^2 = \sum_{i,j}|g(\nabla_{E_i}U, E_j)|^2 = 4n\kappa^2u^2$$

As $|U|^2 = u^2\rho^2$ the result follows. □

Lemma 5.5. *For X a horizontal vector, one has $(A_X, A_X) = 2\kappa^2\hat{\lambda}|X|^2$.*

Proof. Immediately we have

$$(A_X, A_X) = \sum_i g(A_XE_i, A_XE_i) = \sum_i |\mathcal{V}\nabla_XE_i|^2 = \sum_{i,j}|g(\nabla_XE_i, F_j)|^2$$

By the Koszul formula

$$2g(\nabla_XE_i, F_j) = -F_jg(X, E_i) + g([X, E_i], F_j)$$

As (A_X, A_Y) is tensorial, we may assume X is basic and may be written $X = xE_1$ for $x \in \mathbb{R}$. Whence

$$g(\nabla_XE_i, F_j) = \frac{1}{2}g([X, E_i], F_j) = -\frac{1}{2}g([R_{X,E_i}, S], F_j)$$

We evaluate this at $S = I$ using Lemma 3.2 and Proposition 3.3

$$g(\nabla_XE_i, F_j) = \frac{\check{\lambda}}{2(n+2)}g(\check{g}(KX, E_i)J - \check{g}(JX, E_i)K, F_i)$$

As $S = I$, $\rho F_j \in \{J, K\}$ and we observe

$$g(\nabla_XE_i, \rho^{-1}J) = \frac{\check{\lambda}\rho}{2(n+2)}\check{g}(KX, E_i) \quad \text{and} \quad g(\nabla_XE_i, \rho^{-1}K) = \frac{-\check{\lambda}\rho}{2(n+2)}\check{g}(JX, E_i)$$

Therefore the result follows from

$$(A_X, A_X) = \sum_{i,j}|g(\nabla_XE_i, F_j)|^2 = \frac{1}{2}\left(\frac{\check{\lambda}\rho}{n+2}\right)^2|X|^2$$ □

Proposition 5.6. *The horizontal distribution $\mathcal{H}$ is Yang-Mills.*

Proof. Let $S \in \mathcal{Z}_x$ for $x \in M$. Consider Riemann orthonormal coordinates about x which are lifted to a horizontal frame $\{E_i\}$. We investigate $\mathcal{V}(\delta A(X))$ for a horizontal vector X. First,

$$\delta A(X) = -\sum_i(\nabla_{E_i}A)_{E_i}X = \sum_i\left(-\nabla_{E_i}(A_{E_i}X) + A_{\nabla_{E_i}E_i}X + A_{E_i}(\nabla_{E_i}X)\right)$$

As this is tensorial, we assume without loss of generality that $X = E_1$. As the Christoffel symbols for the connection on M at x vanish, we have $\mathcal{H}\nabla_{E_i}E_i = 0$ at S. Therefore evaluating against a vertical vector U at S gives

$$g(\delta A(X), U) = -\sum_i g(\nabla_{E_i}(A_{E_i}X), U)$$

Using the Koszul formula, and evaluating at S leads to

$$g(\delta A(X), U) = -\frac{1}{2}\sum_i E_i g(A_{E_i}X, U) = \frac{1}{2}\sum_i g(R^E_{E_i,X}S, U)$$

We evaluate this at $S = I$ and $U = uJ$ for $u \in \mathbb{R}$. The use of Riemann normal coordinates along with Lemma 3.2 and Proposition 3.3 ensure the result.

$$\begin{aligned} g(\delta A(X), U) &= \frac{1}{2}\sum_i E_i \langle [R_{E_i,X}, S], uJ\rangle_E \\ &= -\frac{1}{2(n+2)}\sum_i E_i \left(\check{r}(K\pi_* E_i, \pi_* X)\langle J, uJ\rangle_E\right) \\ &= u\kappa \sum_i E_i \check{g}(K\pi_* E_i, \pi_* X) = 0 \end{aligned}$$

□

Recall Remark 5.1 in which we define two possible values for the Einstein constant $\hat{\lambda}$ of the fibres of the twistor space. The ratios are

$$\frac{\hat{\lambda}_1}{\check{\lambda}} = \frac{1}{n+2} \quad \text{and} \quad \frac{\hat{\lambda}_2}{\check{\lambda}} = \frac{n+1}{n+2}$$

Theorem 5.7. *The twistor space $(\mathcal{Z}, g)$ with $\hat{\lambda} = \hat{\lambda}_1$ or $\hat{\lambda} = \hat{\lambda}_2$ is Einstein.*

Proof. Consider Corollary 2.16. Following Proposition 5.6, it suffices to find $\lambda \in \mathbb{R}$ such that

$$(AU, AV) = (\lambda - \hat{\lambda})g(U, V) \quad \text{and} \quad (A_X, A_Y) = \tfrac{1}{2}(\check{\lambda} - \lambda)g(X, Y)$$

By the polarisation identity and Lemmata 5.4, 5.5, this amounts to finding λ such that

$$\lambda - \hat{\lambda} = \frac{n}{(n+2)^2}\frac{\check{\lambda}^2}{\hat{\lambda}} \quad \text{and} \quad \check{\lambda} - \lambda = \frac{1}{(n+2)^2}\frac{\check{\lambda}^2}{\hat{\lambda}}$$

A consistent solution is given by the zeros (for $\hat{\lambda}$) of the polynomial

$$\hat{\lambda}^2 - \check{\lambda}\hat{\lambda} + \frac{n+1}{(n+2)^2}\check{\lambda}^2$$

It is easy to carry out the calculations and arrive at explicit values for the Einstein constant λ of the twistor space in terms of $\check{\lambda}$. Precisely,

$$\lambda_1 = \frac{n+1}{n+2}\check{\lambda} \quad \text{and} \quad \lambda_2 = \frac{n^2+3n+1}{(n+2)(n+1)}\check{\lambda}.$$

However more appropriate is the ratio between the Einstein constant of the fibre and the Einstein constant for the base manifold; as given preceding the announcement of this theorem. □

5.2 Kähler Structure

We now show that, for $\check{\lambda} = \check{\lambda}_1$, the twistor space $(\mathcal{Z}, g, \mathcal{J})$ is Kähler. We investigate $\nabla\mathcal{J}$.

Proposition 5.8. *For vertical vectors U, V, one has $(\nabla_U \mathcal{J})V = 0$.*

Proof. As in Proposition 4.4, this is a consequence of the identification of the fibres with $\mathbb{CP}^1$ and a multiple of the Fubini-Study metric. □

Lemma 5.9. *For U a vertical vector, X a basic vector field, one has $\pi_*(\mathcal{J}\nabla_U X) = \kappa U \pi_* X$.*

Proof. Evaluating $\nabla_U X$ against a basic vector field Y and using the Koszul formula gives

$$2g(\nabla_U X, Y) = -g([X,Y],U) \quad \text{hence} \quad g(\nabla_U X, Y)_S = \frac{1}{2} g([R_{X,Y}, S], U)$$

for $S \in \mathcal{Z}$. We evaluate this at $S = I$ and $U = uJ$ for $u \in \mathbb{R}$. Lemma 3.2 and Proposition 3.3 give

$$\begin{aligned} g(\nabla_U X, Y)_I &= \frac{1}{2}\langle [R_{X,Y}, I], uJ\rangle_E \\ &= \frac{-\check{\lambda}}{2(n+2)} \check{g}(K\pi_* X, \pi_* Y)\langle J, uJ\rangle_E \\ &= -\kappa \check{g}(uK\pi_* X, \pi_* Y) \end{aligned}$$

Therefore $\pi_*(\nabla_U X)_I = -\kappa u K \pi_* X$ hence $\pi_*(\mathcal{J}\nabla_U X)_I = \kappa u J \pi_* X$. ☐

Lemma 5.10. *For U a vertical vector, X, Y basic vector fields, one has $g([\mathcal{J}X, Y], U) = 2\kappa \check{g}(U\pi_* X, \pi_* Y)$.*

Proof. As in previous calculations, we evaluate at $S \in Z$ to get

$$g([\mathcal{J}X, Y], U)_S = g(\mathcal{V}[\mathcal{J}X, Y], U)_S = -g(R^E_{SX,Y} S, U) = -g([R_{SX,Y}, S], U)$$

We evaluate this at $S = I$ and $U = uJ$ for $u \in \mathbb{R}$.

$$\begin{aligned} g([\mathcal{J}X, Y], U)_I &= \frac{1}{n+2} \check{r}(J\pi_* X, \pi_* Y)\langle J, uJ\rangle_E \\ &= \frac{\check{\lambda}\rho^2}{n+2} \check{g}(uJ\pi_* X, \pi_* Y) \\ &= 2\kappa \check{g}(U\pi_* X, \pi_* Y) \end{aligned}$$ ☐

Lemma 5.11. *For U a vertical vector, X, Y basic vector fields, one has $Ug(\mathcal{J}X, Y) = g([U, \mathcal{J}X], Y)$*

Proof. Evaluating at $S \in \mathcal{Z}$ we obtain directly that

$$\begin{aligned} Ug(\mathcal{J}X, Y) &= \left.\frac{d}{dt}\right|_{t=0} g(\mathcal{J}X, Y)_{Fl^U_t(S)} \\ &= \left.\frac{d}{dt}\right|_{t=0} \check{g}(Fl^U_t(S)\pi_* X, \pi_* Y) \\ &= \check{g}(U\pi_* X, \pi_* Y) \end{aligned}$$

and the result follows from Lemma 4.7. ☐

Lemma 5.12. *For U a vertical vector, X a basic vector field, one has $\pi_*(\nabla_U \mathcal{J}X) = (1-\kappa)U\pi_* X$.*

Proof. Evaluating $\nabla_U \mathcal{J}X$ against a basic vector field Y and using the Koszul formula gives

$$2g(\nabla_U \mathcal{J}X, Y) = Ug(\mathcal{J}X, Y) + g([U, \mathcal{J}X], Y) - g([\mathcal{J}X, Y], U)$$

By Lemmata 5.10, 5.11

$$2g(\nabla_U \mathcal{J}X, Y) = 2g([U, \mathcal{J}X], Y) - 2\kappa \check{g}(U\pi_* X, \pi_* Y)$$

and the result follows from Lemma 4.7. ☐

Proposition 5.13. *For U a vertical vector, X a horizontal vector, if $\hat{\lambda} = \hat{\lambda}_1$ then $(\nabla_U \mathcal{J})X = 0$.*

Proof. As the fibres are totally geodesic, $(\nabla_U \mathcal{J})X = \nabla_U(\mathcal{J}X) - \mathcal{J}\nabla_U X$ is horizontal. The equation is tensorial in X so we may assume that X is basic. Note the condition on $\hat{\lambda}$ is equivalent to $\kappa = \frac{1}{2}$. Therefore by Lemmata 5.9, 5.12 we get

$$\pi_*(\nabla_U \mathcal{J})X = (1-2\kappa)U\pi_* X = 0$$ ☐

Lemma 5.14. *For horizontal vectors X, Y, one has $A_X\mathcal{J}Y = \mathcal{J}A_XY$*

Proof. By Proposition 2.8, $A_XY = \frac{1}{2}\mathcal{V}[X,Y]$. So evaluating this at $S \in \mathcal{Z}$ we have $A_XY = -\frac{1}{2}[R_{X,Y}, S]$ by Proposition 1.7 and Remark 1.4. We now write X, Y for vectors on M. It thus suffices to show $[R_{X,SY}, S] = S[R_{X,Y}, S]$ which follows a similar argument to Lemma 4.10. Evaluating at $S = I$ gives

$$(n+2)[R_{X,IY}, I] = -\check{r}(KX, IY)J + \check{r}(JX, IY)K$$
$$(n+2)I[R_{X,Y}, I] = -\check{r}(JX, Y)J - \check{r}(KX, Y)K$$ □

Lemma 5.15. *For U a vertical vector, X a horizontal vector, one has $A_X\mathcal{J}U = \mathcal{J}A_XU$.*

Proof. By definition, A_XU is horizontal so evaluating against a horizontal vector Y and using Lemmata 2.7, 5.14 we obtain

$$g(A_X\mathcal{J}U, Y) = -g(\mathcal{J}U, A_XY) = g(U, \mathcal{J}A_XY) = g(U, A_X\mathcal{J}Y) = -g(A_XU, \mathcal{J}Y) = g(\mathcal{J}A_XU, Y)$$ □

Proposition 5.16. *For U a vertical vector, X a horizontal vector, one has $(\nabla_X\mathcal{J})U = 0$.*

Proof. By Proposition 2.11 and Lemma 5.15 we have

$$\begin{aligned}(\nabla_X\mathcal{J})U &= \nabla_X(\mathcal{J}U) - \mathcal{J}\nabla_XU \\ &= A_X\mathcal{J}U + \mathcal{V}\nabla_X(\mathcal{J}U) - \mathcal{J}A_XU - \mathcal{J}\mathcal{V}\nabla_XU \\ &= \mathcal{V}\nabla_X(\mathcal{J}U) - \mathcal{J}\mathcal{V}\nabla_XU\end{aligned}$$

We evaluate the first term against a vertical vector V and use the Koszul formula.

$$2g(\nabla_X(\mathcal{J}U), V) = Xg(\mathcal{J}U, V) + g([X, \mathcal{J}U], V) + g([X, V], \mathcal{J}U)$$

Similarly for the second term against V,

$$-2g(\mathcal{J}\nabla_XU, V) = 2g(\nabla_XU, \mathcal{J}V) = Xg(U, \mathcal{J}V) + g([X, U], \mathcal{J}V) + g([\mathcal{J}V, X], U)$$

As $(\nabla_X\mathcal{J})U$ is tensorial in X we assume X is basic. Lemma 4.5 and the compatibility of g with $\mathcal{J}$ now give the result. □

Proposition 5.17. *For horizontal vectors X, Y, one has $(\nabla_X\mathcal{J})Y = 0$.*

Proof. By Proposition 2.11 and Lemma 5.14 we have

$$\begin{aligned}(\nabla_X\mathcal{J})Y &= \nabla_X(\mathcal{J}Y) - \mathcal{J}\nabla_XY \\ &= \mathcal{H}\nabla_X(\mathcal{J}Y) + A_X\mathcal{J}Y - \mathcal{J}\mathcal{H}\nabla_XY - \mathcal{J}A_XY \\ &= \mathcal{H}\nabla_X(\mathcal{J}Y) - \mathcal{J}\mathcal{H}\nabla_XY\end{aligned}$$

Evaluating each term against a basic vector Z, and noting $-g(\mathcal{J}\nabla_XY, Z) = g(\nabla_XY, \mathcal{J}Z)$, gives

$$2g(\nabla_X\mathcal{J}Y, Z) = Xg(\mathcal{J}Y, Z) + \mathcal{J}Yg(Z, X) - Zg(X, \mathcal{J}Y) + g([X, \mathcal{J}Y], Z) - g([\mathcal{J}Y, Z], X) + g([Z, X], \mathcal{J}Y)$$
$$2g(\nabla_XY, \mathcal{J}Z) = Xg(Y, \mathcal{J}Z) + Yg(\mathcal{J}Z, X) - \mathcal{J}Zg(X, Y) + g([X, Y], \mathcal{J}Z) - g([Y, \mathcal{J}Z], X) + g([\mathcal{J}Z, X], Y)$$

Assume X, Y are basic as $(\nabla_X\mathcal{J})Y$ is tensorial. As in Lemma 4.9, let $x \in M$ and S a section of $\mathcal{Z}$ over $\Omega \subset M$ with $x \in \Omega$. For $y \in \Omega$ we write S_y for the complex structure on T_yM. Under the natural immersion, we view S as a local section of E. As $\mathcal{H}^E = \mathcal{H}$ we demand that $\nabla S = 0$ at x. Under this hypothesis, $(\nabla_X\mathcal{J})Y = (\nabla_{\pi_*X}S)\pi_*Y$. Indeed consider the preceding two equations. There are immediate relations such as $Xg(\mathcal{J}Y, Z) = \pi_*X\check{g}(S\pi_*X, \pi_*Y)$. Also, as $S : \Omega \to \mathcal{Z}$ provides an immersion of Ω, the terms involving brackets are dealt with, for example $\pi_*[X, \mathcal{J}Y] = [\pi_*X, S\pi_*Y]$. Finally, at $x \in M$, the condition $\nabla S = 0$

implies $\mathcal{J}Y g(Z,X) = S\pi_* Y \check{g}(\pi_* Z, \pi_* X)$ with a similar expression for $\mathcal{J}Zg(X,Y)$. Therefore, at $x \in M$

$$\begin{aligned} g((\nabla_X \mathcal{J})Y, Z) &= g(\nabla_X \mathcal{J}Y, Z) + g(\nabla_X Y, \mathcal{J}Z) \\ &= \check{g}(\nabla_{\pi_* X} S\pi_* Y, \pi_* Z) + \check{g}(\nabla_{\pi_* X}\pi_* Y, S\pi_* Z) \\ &= \check{g}((\nabla_{\pi_* X} S)\pi_* Y, \pi_* Z) = 0. \end{aligned}$$

□

Theorem 5.18. *The twistor space $(\mathcal{Z}, g, \mathcal{J})$ with $\hat{\lambda} = \hat{\lambda}_1$ is Kähler-Einstein.*

Proof. The complex structure is parallel by Propositions 5.8, 5.13, 5.16, 5.17. □

It is now not difficult to see that the non-integrable almost-complex structure defined in Remark 4.3 provides a nearly Kähler manifold.

Definition 5.19. *An almost Hermitian manifold (M, g, J) is nearly Kähler if ∇J is skew-symmetric.*

This definition amounts to showing $(\nabla_X J)X$ and $(\nabla_U \mathcal{J})U$ vanish for X a horizontal vector, U a vertical vector and also that $(\nabla_X J)U + (\nabla_U J)X$ vanishes. The first two statements are consequences of Propositions 5.8, 5.17, while the third is a consequence of Proposition 5.16.

6. Even Clifford Structures

This final chapter investigates a generalisation of the twistor space over quaternionic-Kähler manifolds and its integrability. We consider a structure introduced recently by Moroianu and Semmelmann [11]. More precisely even Clifford structures over Riemannian manifolds. In order to construct the associated twistor space we recall the appropriate setting.

6.1 Clifford Algebras

Standard facts on Clifford algebras may be found in [8]. However, in order to introduce the twistor space, it is appropriate to recall some of the associated structure. Throughout this section, let (V, q) be a finite dimensional vector space endowed with a positive definite quadratic form. The automorphism $\alpha : \mathrm{Cl}(V, q) \to \mathrm{Cl}(V, q)$ which extends the map $\alpha(v) = -v$ on V splits the Clifford algebra

$$\mathrm{Cl}(V, q) = \mathrm{Cl}^0(V, q) \oplus \mathrm{Cl}^1(V, q)$$

where $\mathrm{Cl}^i(V, q)$ is the eigenspace of α with eigenvalue $(-1)^i$. Then $\mathrm{Cl}^0(V, q)$ is the even Clifford algebra. There exists a natural isomorphism between the associated graded algebra of $\mathrm{Cl}(V, q)$ and the exterior algebra $\Lambda^* V$. In particular, we identify

$$\mathrm{Cl}^0(V, q) \simeq \bigoplus_{k \geq 0} \Lambda^{2k} V.$$

On the tensor algebra of V, there is an involution given on simple elements by the reversal of order, i.e., $v_1 \otimes \cdots \otimes v_k \mapsto v_k \otimes \cdots \otimes v_1$. This map descends to both the exterior algebra and the Clifford algebra. It is denoted $(\ )^t : \mathrm{Cl}(V, q) \to \mathrm{Cl}(V, q)$. We immediately get the following.

Lemma 6.1. *The transposition acts as the identity on $\Lambda^{4k} V$ and as by inversion on $\Lambda^{4k+2} V$.*

Proof. If $\{e_i\}$ is a basis for V then $e_{i_1} \wedge \cdots \wedge e_{i_{4k}}$ (where $i_j \neq i_k$ for $j \neq k$) may be grouped in $2k$ pairs $e_{i_{2j-1}} \wedge e_{i_{2j}}$ allowing the transposition to be returned to its original state up to a factor of $(-1)^{2k} = 1$. Similarly, $e_{i_1} \wedge \cdots \wedge e_{i_{4k+2}}$ (where $i_j \neq i_k$ for $j \neq k$) may be grouped in $2k + 1$ pairs allowing the transposition to be returned to its original state up to a factor of $(-1)^{2k+1} = -1$. $\square$

Under this transpose, we may split the even Clifford algebra into ± 1 eigenspaces which we will denote $\mathrm{Cl}^{0,\pm}(V, q)$. That is

$$\mathrm{Cl}^0(V, q) = \mathrm{Cl}^{0,+}(V, q) \oplus \mathrm{Cl}^{0,-}(V, q)$$

where

$$\mathrm{Cl}^{0,+}(V, q) \simeq \bigoplus_{k \geq 0} \Lambda^{4k} V \quad \text{and} \quad \mathrm{Cl}^{0,-}(V, q) \simeq \bigoplus_{k \geq 0} \Lambda^{4k+2} V.$$

6.2 Twistor Fibre

In order to understand the twistor space introduced in the following section, it is important to understand the fibres from an algebraic viewpoint. To this end, it is not important to continue the analysis with (V, q) and we will consider the standard Clifford algebras for $\mathbb{R}^r$ with the standard metric. We introduce the following definition.

Definition 6.2. *Given an even Clifford algebra Cl_r^0 the twistor fibre $\mathcal{Z}^r$ is defined by*

$$\mathcal{Z}^r = \{\sigma \in Cl_r^0 \,|\, \sigma^t \cdot \sigma = 1 \text{ and } \sigma^t + \sigma = 0\}$$

Immediately, from the above discussion we have the following characterisation.

Lemma 6.3. *The twistor fibre may be characterised by*

$$\mathcal{Z}^r = \{\sigma \in Cl_r^{0,-} \,|\, \sigma^2 = -1\}$$

Ultimately we want to consider an integrability question about a twistor space over a manifold. As in the quaternionic-Kähler setting, we would like the fibre to be a complex symmetric space. This requires an understanding of the representation theory of Clifford algebras. Due to the structure of Clifford algebras, the classification of even Clifford structures (introduced in the following section) and ease of reading, we will limit the analysis to ranks $3 \leq r \leq 10$. The following facts are discussed in [8]. We have the standard identifications for the even Clifford algebras Cl_r^0.

	3	4	5	6	7	8	9	10
Cl_r^0	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$

where $\mathbb{K}(n)$ denotes $n \times n$ matrices with values in the field $\mathbb{K}$.

Remark 6.4. There is a minor complication with cases $r = 4k$ where there are precisely 2 irreducible representations of Cl_r^0. This will lead to a splitting of the base manifold under a condition of parallelism to be discussed in the final section.

Identifying $\mathbb{H} \sim \mathbb{C}^2 \sim \mathbb{R}^4$ by

$$a + ib + jc + kd \sim (a + ib, c + id) \sim (a, b, c, d)$$

we obtain inclusions $\mathbb{H}(n) \subset \mathbb{C}(2n) \subset \mathbb{R}(4n)$. Moreover, the transpose on Cl_r introduced above corresponds, under these identifications, to the transpose in $\mathbb{R}(n)$ (and the conjugate transpose if considering $\mathbb{H}(n)$ or $\mathbb{C}(n)$ without the identification $\mathbb{H} \sim \mathbb{C}^2 \sim \mathbb{R}^4$.) Heuristically, recalling Definition 6.2, this means the twistor fibre becomes the intersection of a certain Lie group and its corresponding Lie algebra. We will consider the following three examples depending on whether the even Clifford algebra representation is real, complex, or quaternionic

$$\mathcal{Z}_{\mathbb{R}} = SO(d) \cap \mathfrak{o}(d), \qquad \mathcal{Z}_{\mathbb{C}} = U(d) \cap \mathfrak{u}(d), \qquad \mathcal{Z}_{\mathbb{H}} = Sp(d) \cap \mathfrak{sp}(d).$$

Here d corresponds to the real, complex, or quaternionic dimension of the associated even Clifford algebra. We will show that, in these three cases, we have symmetric spaces.

We consider first $\mathcal{Z}_{\mathbb{R}}$. Take the (left) adjoint action of $SO(d)$ on $\mathcal{Z}_{\mathbb{R}}$ (where d is even). That is, $U \cdot A = UAU^{-1}$ for $U \in SO(d)$ and $A \in \mathcal{Z}_{\mathbb{R}}$. It is clearly well-defined

$$(UAU^{-1})^t(UAU^{-1}) = UA^tAU^{-1} = 1 \quad \text{and} \quad (UAU^{-1})^t + (UAU^{-1}) = U(A^t + A)U^{-1} = 0.$$

The action is also transitive. Indeed for $A \in \mathcal{Z}_{\mathbb{R}}$ (which is a complex structure on $\mathbb{R}^d$ compatible with the standard metric) we may diagonalise A by some $U \in SO(d)$ such that

$$UAU^{-1} = J_d$$

where J_d is the standard complex structure on R^d with 2×2 entries on the diagonal of the form $J_2 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. From this, transitivity is now clear. Finally, as $J_d \in \mathcal{Z}_{\mathbb{R}}$ we compute the stabiliser

$$\mathrm{stab}_{J_d} = \{U \in SO(d) \,|\, [U, J_d] = 0\} = U(d/2).$$

We conclude $Z_{\mathbb{R}}$ may be characterised

$$\mathcal{Z}_{\mathbb{R}} = \frac{SO(d)}{U(d/2)}.$$

We consider second $\mathcal{Z}_{\mathbb{C}}$. Take the (left) adjoint action of $U(d)$ on $\mathcal{Z}_{\mathbb{C}}$. The action is again well-defined. Unlike before, the action is not transitive. Consider an element $A \in \mathcal{Z}_d$. As $A^2 = -1$, it may be diagonalised by $U \in U(d)$ such that the first k diagonal entries are i and the remaining $d-k$ diagonal entries are $-i$. It is now clear $\mathcal{Z}_{\mathbb{C}}$ decomposes into $d+1$ components (depending on the dimension of the i-eigenspace of a given matrix) and on each component $U(d)$ acts transitively. Moreover, if J_k denotes the diagonal matrix with the first k entries i and the second $d-k$ entries $-i$, it is clear that

$$\mathrm{stab}_{J_k} = \begin{pmatrix} U(k) & 0 \\ 0 & U(d-k) \end{pmatrix}$$

We conclude $\mathcal{Z}_{\mathbb{C}}$ has $d+1$ connected components and may be characterised

$$\mathcal{Z}_{\mathbb{C}} = \bigsqcup_{k=0}^{d} \frac{U(d)}{U(k) \times U(d-k)}.$$

We consider finally $\mathcal{Z}_{\mathbb{H}}$. Take the (left) adjoint action of $Sp(d)$ on $\mathcal{Z}_{\mathbb{H}}$. The action is again well-defined. The action is transitive however the demonstration is more involved. Precisely as in the quaternionic-Kähler setting we have the following. Identifying $\mathbb{R}^{4d} \sim \mathbb{H}^d$ (extending the identification above) and letting $\mathbb{H}$ act by right multiplication defines $\lambda : Sp(1) \hookrightarrow SO(4d)$ and $Sp(d)$ is the subgroup of $SO(4d)$ commuting with $\lambda(Sp(1))$. Let R^i, R^j, R^k be the images of i, j, k under λ. Take $A \in \mathcal{Z}_{\mathbb{H}}$ then immediately $A \in \mathrm{End}_{\mathbb{C},R^i}\mathbb{R}^{4d}$ where the notation indicates A is complex linear with respect to R^i. That is, $A \circ (a + bR^i) = (a + bR^i) \circ A$. As $A^2 = -1$ we have the decomposition of $\mathbb{R}^{4d}$

$$\mathbb{R}^{4d} = V_{R^i,A} \oplus V_{-R^i,A} \quad \text{where} \quad A|_{V_{\pm R^i,A}} = \pm R^i$$

We instantly have an isomorphism between $V_{\pm R^i,A}$ by considering $R^j : V_{R^i,A} \to V_{-R^i,A}$. Indeed we need only verify the map is well-defined. For $x \in V_{R^i,A}$,

$$AR^j x = R^j Ax = R^j R^i x = -R^i(R^j x)$$

so $R^j x \in V_{-R^i,A}$. In particular this ensures $\dim_{\mathbb{R}} V_{\pm R^i,A} = 2d$. If $B \in \mathcal{Z}_{\mathbb{H}}$ then we have a second decomposition

$$\mathbb{R}^{4d} = V_{R^i,B} \oplus V_{-R^i,B} \quad \text{where} \quad B|_{V_{\pm R^i,B}} = \pm R^i$$

Let $\{e_{l,A}\}$ (where $1 \leq l \leq 2d$) be an orthonormal basis for $V_{R^i,A}$. Using R^j, we extend this basis to an orthonormal basis $\{e_{l,A}, R^j e_{l,A}\}$ for $\mathbb{R}^{4d}$. First $V_{R^i,A}$ and $V_{R^i,B}$ are isomorphic. Second both $A|_{V_{R^i,A}} = R^i$ and $B|_{V_{R^i,B}} = R^i$. Therefore there exists an invertible $\tilde{U} \in \mathrm{Hom}_{\mathbb{C},R^i}(V_{R^i,A}, V_{R^i,B})$. Define an orthonormal basis for $V_{R^i,B}$ by $e_{l,B} = Ue_{l,A}$ for $1 \leq l \leq 2d$ and extend $\tilde{U}$ to $U \in SO(4d)$ by requiring $U(R^j e_{l,A}) = R^j e_{l,B}$. By construction $[U, R^i] = [U, R^j] = 0$ so $U \in Sp(d)$. Then $UAU^{-1} = B$. Indeed,

$$UAU^{-1} e_{l,B} = UAe_{l,A} = UR^i e_{l,A} = R^i U e_{l,A} = R^i e_{l,B} = Be_{l,B}$$

and

$$UAU^{-1} R^j e_{l,B} = UAR^j e_{l,A} = UR^j R^i e_{l,A} = -R^i R^j U e_{l,A} = -R^i R^j e_{l,B} = BR^j e_{l,B}.$$

To calculate the stabiliser of A, again consider the associated decomposition. If $UA = AU$ then U preserves the decomposition $\mathbb{R}^{4d} = V_{R^i,A} \oplus V_{-R^i,A}$. Indeed

$$A(Ue_{l,A}) = UAe_{l,A} = UR^i e_{l,A} = R^i(Ue_{l,A})$$

and

$$A(UR^j e_{l,A}) = UAR^j e_{l,A} = -UR^i R^j e_{l,A} = -R^i(UR^j e_{l,A}).$$

Therefore $U = \begin{pmatrix} \tilde{U}_1 & 0 \\ 0 & \tilde{U}_2 \end{pmatrix}$ where $\tilde{U}_i \in U_d$. Of course $\tilde{U}_2$ is determined by $\tilde{U}_1$ as $U \in Sp(d)$. That is $[U, R^j] = 0$ hence $U(R^j e_{l,A}) = R^j U e_{l,A}$. We conclude $\mathcal{Z}_{\mathbb{H}}$ may be characterised

$$\mathcal{Z}_{\mathbb{H}} = \frac{Sp(d)}{U(d)}.$$

In summary, we have the following classification of the twistor fibre when there exists precisely one (up to isomorphism) irreducible representation of Cl_r^0 for $3 \le r \le 10$.

	3	4	5	6	7	8	9	10
Cl_r^0	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$
$\mathcal{Z}^r$	$\frac{Sp(1)}{U(1)}$		$\frac{Sp(2)}{U(2)}$	$\bigsqcup_{k=0}^{5} \frac{U(4)}{U(k) \times U(4-k)}$	$\frac{SO(8)}{U(4)}$		$\frac{SO(16)}{U(8)}$	$\bigsqcup_{k=0}^{16} \frac{U(16)}{U(k) \times U(16-k)}$

Remark 6.5. It is a future project to investigate the complex structure on the twistor fibre. For the rest of this dissertation we will assume the following form for the complex structure. If $\sigma \in \mathrm{Cl}_r^0$ and $v \in T_\sigma \mathcal{Z}_0$ the complex structure is given by

$$\mathcal{J}_\sigma v = \sigma \cdot v$$

The verification of such a claim should come from the general theory of symmetric spaces as given in [6].

6.3 Even Clifford Structures

The naturality of the above objects enables us to carry them over to vector bundles. In particular, we may introduce the following.

Definition 6.6. *Let (M, g) be an n-dimensional simply connected orientated complete Riemannian manifold and let (E, h) be a rank $r \ge 3$ orientated Euclidean vector bundle over M. A rank r even Clifford structure, is the above data along with a Clifford morphism $\varphi : Cl^0(E, h) \to End(TM)$. That is, an algebra bundle morphism that sends $\Lambda^2 E$ into the bundle of skew-symmetric endomorphisms $End^-(TM)$.*

Definition 6.7. *An even Clifford structure is parallel if there is a metric connection ∇^E on (E, h) such that φ is connection preserving, i.e.*

$$\varphi(\nabla^E_X \sigma) = \nabla^g_X \varphi(\sigma)$$

for every tangent vector $X \in TM$ and section σ of $Cl^0(E, h)$ (here ∇^g denotes the Levi-Civita connection). The parallel even Clifford structure is flat if the connection ∇^E is flat.

Remark 6.8. This definition generalises quaternionic-Kähler manifolds, as explicitly mentioned in Example 2.7 of [11]. For a rank 3 parallel even Clifford structure, if $\{e_1, e_2, e_3\}$ is a local orthonormal basis of E, we define $I = \varphi(e_1 \cdot e_2)$, $J = \varphi(e_2 \cdot e_3)$, $K = \varphi(e_3 \cdot e_1)$ which gives a local quaternionic frame. Conversely if E' is the subbundle spanned locally by $\{I, J, K\}$, the Hodge isomorphism enables E to be constructed by identifying locally $E \simeq \Lambda^2 E \simeq E'$ with the second identification identical to the one above.

Extending these definitions, we introduce the twistor space.

Definition 6.9. *Consider a parallel even Clifford structure with notation as in Definition 6.6. Then the pre-twistor space $\mathcal{Z}^E$ is given by*

$$\mathcal{Z}^E = \{\sigma \in Cl^0(E, h) \mid \sigma^t \sigma = 1 \text{ and } \sigma^t + \sigma = 0\}$$

and the twistor space $\mathcal{Z}$ of M is the image of $\mathcal{Z}^E$ under φ, i.e.

$$\mathcal{Z} = \{J \in End(TM) \mid J = \varphi(\sigma) \text{ for some } \sigma \in Cl^0(E, h) \text{ such that } \sigma^t \sigma = 1 \text{ and } \sigma^t + \sigma = 0\}.$$

It is immediate that (local) sections of $\mathcal{Z}$ correspond to (local) almost complex structures (as the Clifford morphism sends -1 to minus the identity endomorphism). Clearly $\mathcal{Z}$ fibres naturally over M. For each $x \in M$, we have, from the preceding section, the algebraic twistor fibre $\mathcal{Z}^{E_x}$ associated with (E_x, h_x) whose image under φ is precisely $\mathcal{Z}_x$. Similarly, we have an alternative definition.

Lemma 6.10. *Consider a parallel even Clifford structure with notation as in Definition 6.6. Under the identification between $Cl^0(E,h)$ and $\bigoplus_{k\geq 0} \Lambda^{2k}E$. The twistor space may be characterised by*

$$\mathcal{Z} = \{J \in End(TM) \,|\, J = \varphi(\sigma) \text{ for some } \sigma \in \textstyle\bigoplus_{k\geq 0} \Lambda^{4k+2}E \subset Cl^0(E,h) \text{ such that } \sigma^2 = -1\}.$$

Remark 6.11. As in the preceding remark, this extends Definition 4.1. For rank 3, take $\{e_1, e_2, e_3\}$ an orthonormal basis for E_x where $x \in M$ and define I, J, K as in the preceding remark. As $\bigoplus_{k\geq 0} \Lambda^{4k+2}E = \Lambda^2 E$, and for $i \neq j$ we have $e_i \cdot e_j = -1$, the fibre $\mathcal{Z}_x$ consists of complex structures on T_xM of the form $aI + bJ + cK$ where $a, b, c \in \mathbb{R}$ with $a^2 + b^2 + c^2$.

As in the quaternionic-Kähler case, we introduce an almost complex structure on the twistor space of a parallel even Clifford structure. The metric connection on (E, h) gives a connection on $\Lambda^k E$ and the condition of parallelism ensures that the subbundle

$$F = \varphi\left(\mathrm{Cl}^{0,-}(E,h)\right) \subset \mathrm{End}(TM)$$

is preserved by the connection on the bundle of endomorphisms of the tangent bundle (induced by the Levi-Civita connection). We thus have a splitting

$$TF = \mathcal{H}^F \oplus \mathcal{V}^F$$

where $\mathcal{V}^F = \ker \pi_*$ where $\pi : F \to M$ is the natural projection. As before, this provides a splitting

$$T\mathcal{Z} = \mathcal{H} \oplus \mathcal{V}$$

where $\mathcal{H} = \mathcal{H}^F$ and $\mathcal{V} = \ker \pi_*$ for $\pi : \mathcal{Z} \to M$.

Let $S \in \mathcal{Z}$ and $x = \pi(S)$. The projection gives an identification $\pi_* : \mathcal{H}_S \to T_xM$ but unlike the quaternionic-Kähler setting, we use Remark 6.5 to define the complex structure on the fibre.

Definition 6.12. *Let $X + U$ be a tangent vector at S compatible with the decomposition $T_S\mathcal{Z} = \mathcal{H} \oplus \mathcal{V}$. The almost-complex structure $\mathcal{J}$ on the twistor space is defined by*

$$\mathcal{J}(X + U)_S = \pi_*^{-1} S \pi_* X + SU$$

Conjecture 6.13. *The Nijenhuis tensor vanishes when restricted to $\mathcal{V} \times \mathcal{V}$.*

Remark 6.14. As the bracket of vertical vectors is again a vertical fibre, this statement claims that the twistor fibre is naturally a complex manifold when endowed with the associated complex structure from Definition 6.12 and depends on Remark 6.5.

We have the following statements that hold immediately in the current setting with precisely the same proofs as given for the quaternionic-Kähler case.

Lemma 6.15. *For a basic vector field X and a vertical vector field U, one has $\mathcal{J}[U, X] = [\mathcal{J}U, X]$.*

Lemma 6.16. *For a horizontal vector field Y and a vertical vector field U, one has $\mathcal{J}\mathcal{V}[U, Y] = \mathcal{V}[\mathcal{J}U, Y]$*

Lemma 6.17. *For a basic vector field X and a vertical vector field U, one has $\pi_*[U, \mathcal{J}X] = U\pi_*X$.*

Proposition 6.18. *The Nijenhuis tensor vanishes when restricted to $\mathcal{V} \times \mathcal{H}$.*

Lemma 6.19. *The horizontal component of the Nijenhuis tensor vanishes when restricted to $\mathcal{H} \times \mathcal{H}$.*

We now consider the generalisation of the curvature condition in Lemma 4.10. Calculations for the claims made below are given in the proof of Proposition 2.10 of [11]. We announce this proposition below.

Proposition 6.20. *Consider a parallel non-flat even Clifford structure with notation as in Definition 6.6.*

1. *If $r \neq 4$ and $n \neq 8$ then*
   (a) *The curvature of E, viewed as a maps from $\Lambda^2 M$ to $End^- E$ is a non-zero constant times the metric adjoint of the Clifford map φ.*
   (b) *M is Einstein with non-vanishing scalar curvature and has irreducible holonomy.*
2. *If $r \neq 4$ and $n \neq 8$ then (a) implies (b).*

Lemma 6.21. *Let $S \in \mathcal{Z}$ with $x = \pi(S)$ and let $X, Y \in T_x M$. Suppose the curvature condition 1(a) of Proposition 6.20 holds. If $S = J_{ij} = \varphi(e_i \cdot e_j)$, i,j distinct, for some orthonormal basis $\{e_i\}$ of E_x then*

$$-[R_{X,Y}, S] - S[R_{SX,Y}, S] - S[R_{X,SY}, S] + [R_{SX,SY}, S] = 0$$

where R is the Riemann curvature tensor of the tangent bundle.

Proof. Let $\{e_i\}$ be a local orthonormal frame on E. This induces local endomorphisms on M defined by $J_{ij} = \varphi(e_i \cdot e_j)$. Denote by ω_{ij} the curvature forms of the connection on E with respect to $\{e_i\}$

$$R^E_{X,Y} e_i = \sum_{j=1}^{r} \omega_{ji}(X, Y) e_j$$

for tangent vectors X, Y on M. We note the following equivalence.

Lemma 6.22. *The curvature condition 1(a) of Proposition 6.20 is equivalent to the existence of a non-zero constant κ such that $\omega_{ij}(X, Y) = \kappa g(J_{ij}X, Y)$ for all $i \neq j$ and X, Y tangent vectors on M.*

The parallel condition on φ gives (for $i \neq j$)

$$[R_{X,Y}, J_{ij}] = \sum_{s=1}^{r} [\omega_{si}(X, Y) J_{sj} + \omega_{sj}(X, Y) J_{is}]$$

Using Lemma 6.22 we obtain

$$[R_{X,Y}, J_{ij}] = \kappa \sum_{s \neq i} g(J_{si}X, Y) J_{sj} + \kappa \sum_{s \neq j} g(J_{sj}X, Y) J_{is}$$

(Note the discrepancy between this equation and Equation (15) of [11].) We may efficiently write this as

$$[R_{X,Y}, J_{ij}] = \kappa \sum_{s \notin \{i,j\}} g(J_{si}X, Y) J_{sj} + g(J_{sj}X, Y) J_{is}.$$

We now investigate term by term the claimed formula of this lemma. One may verify

$$\begin{aligned}
-[R_{X,Y}, J_{ij}] &= \kappa \sum_{s \notin \{i,j\}} g(J_{si}X, Y) J_{sj} + g(J_{sj}X, Y) J_{is} \\
-[J_{ij} R_{J_{ij}X,Y}, J_{ij}] &= -\kappa \sum_{s \notin \{i,j\}} g(J_{sj}X, Y) J_{is} + g(J_{si}X, Y) J_{sj} \\
-[J_{ij} R_{X,J_{ij}Y}, J_{ij}] &= \kappa \sum_{s \notin \{i,j\}} g(J_{js}X, Y) J_{is} + g(J_{si}X, Y) J_{js} \\
[R_{J_{ij}X,J_{ij}Y}, J_{ij}] &= \kappa \sum_{s \notin \{i,j\}} g(J_{si}X, Y) J_{js} + g(J_{js}X, Y) J_{is}.
\end{aligned}$$

Indeed the calculations are relatively simple as i, j, k are distinct so distinct indices anti-commute (see Lemma 2.4 of [11]). Summing these four calculations now gives the result. $\square$

It is the basis of future work to investigate whether this preliminary result will be able to be generalised to the following more general and required statement.

Conjecture 6.23. *Let $S \in \mathcal{Z}$ with $x = \pi(S)$ and let $X, Y \in T_xM$. Suppose the curvature condition 1(a) of Proposition 6.20 holds. Then*

$$-[R_{X,Y}, S] - S[R_{SX,Y}, S] - S[R_{X,SY}, S] + [R_{SX,SY}, S] = 0.$$

Conjecture 6.24. *The Nijenhuis tensor vanishes when restricted to $\mathcal{H} \times \mathcal{H}$.*

Proof. The proof is precisely the proof of Proposition 4.11 replacing the references to Lemmata 4.9, 4.10 by Lemma 6.19 and Conjecture 6.23 $\square$

The future work will enable a statement on the integrability of the twistor space to be made. If Remark 6.5 is justified hence Conjecture 6.13 is true and if Conjecture 6.23 is also true, then indeed, the twistor space is a complex manifold.

6.4 Rank 4 Clifford Structures

The geometry of the algebraic twistor fibre of higher rank even Clifford structures is more complex than the rank 3 case (where we we saw the twistor bundle was a sphere subbundle of a vector subbundle of endomorphisms). However the rank 4 case may be fully described and also offers a model for the even Clifford structure when the even Clifford algebra possesses precisely two non-isomorphic representations. This situation follows relatively easily from results given in [11]. This is given below and is naturally split into two parts: algebraic and differential geometric. For compactness, we will introduce the structures immediately in the setting of a manifold.

Consider a parallel even Clifford structure with notation as in Definition 6.6 where E has rank 4. At a point $x \in M$ Take an orthonormal basis $\{e_i\}$ with positive orientation and denote the volume element $\omega = e_1 \cdot e_2 \cdot e_3 \cdot e_4$. We decompose $\Lambda^2 E_x$ into self-dual and anti-self dual forms and define the following orthonormal basis

$$\Lambda^2 E_x = \Lambda^2_+ E_x \oplus \Lambda^2_- E_x = \mathrm{sp}\{e_i^+\} \oplus \mathrm{sp}\{e_i^-\}$$

where

$$e_1^\pm = \tfrac{1}{2}\left(e_1 \wedge e_2 \pm e_3 \wedge e_4\right), \qquad e_2^\pm = \tfrac{1}{2}\left(e_1 \wedge e_3 \mp e_2 \wedge e_4\right), \quad e_3^\pm = \tfrac{1}{2}\left(e_1 \wedge e_4 \pm e_2 \wedge e_3\right).$$

Direct calculations establish the following.

Lemma 6.25. *The basis $\{e_i^\pm\}$ introduced above satisfies the following properties*

$$\begin{aligned}
\left(e_i^\pm\right)^2 &= \tfrac{1}{2}(-1 \pm \omega) \\
e_i^\pm \cdot e_j^\pm &= \pm\sigma e_k^\pm \qquad \textit{for } i,j,k \textit{ distinct and } \sigma \textit{ is the signature of the permutation } (1,2,3) \mapsto (i,j,k) \\
e_i^\pm \cdot e_j^\mp &= 0 \qquad \textit{for arbitrary } i,j \\
e_i^\pm \omega = \omega e_i^\pm &= \mp e_i^\pm \\
\omega^2 &= 1
\end{aligned}$$

The final property implies $\varphi(\omega)$ is an involution on T_xM. This splits T_xM into ± 1 eigenspaces $T_x^\pm M$. Importantly, this is compatible with the decomposition of $\Lambda^2 E_x$ in the sense that $\varphi(e_i^\pm)$ acts trivially on $T_x^\pm M$. Under the decomposition $T_xM = T_x^+ M \oplus T_x^- M$ we may thus write

$$\varphi(\omega) = (1, -1), \qquad \varphi(e_i^-) = (J_i^-, 0), \quad \varphi(e_i^+) = (0, J_i^+)$$

for appropriate complex structures $J_i^{\pm}$ acting on $T_x^{\mp}M$. The twistor fibre above x is thus, by Lemma 6.10, the image of elements

$$\sigma = \sum b_i^{\pm} e_i^{\pm}$$

such that $\sigma^2 = -1$. From Lemma 6.25 we have

$$\left(\sum b_i^{\pm} e_i^{\pm}\right)^2 = \left(\sum b_i^{+} e_i^{+}\right)^2 + \left(\sum b_i^{-} e_i^{-}\right)^2 = \sum \left(b_i^{\pm}\right)^2 \tfrac{1}{2}(-1 \pm \omega)$$

from which we deduce the two conditions $\sum (b_i)^+ = \sum (b_i)^- = 1$. The twistor fibre is thus a direct product of two spheres.

From a differential geometric viewpoint we can say more if we suppose that the Clifford structure is parallel. The first part of this statement is observed in [11]. Using the Hodge $*$-application, we see ω must be parallel with respect to the connection on E. Therefore $\varphi(\omega)$ is a parallel involution of TM which moreover commutes with the Clifford action (as may be observed from Lemma 6.25) and is a condition which generalises to other cases where there exist 2 non-isomorphic irreducible representations of the even Clifford algebra. By the de Rham decomposition theorem, M is a Riemannian product $M = M^+ \times M^-$. Furthermore, we reconsider the algebraic argument above replacing the orthonormal basis with a local orthonormal frame. It is clear that $J_i^{\pm}$ which come from $\varphi(e_i^{\pm})$ as introduced above, give local complex structures on $M^{\mp}$ which satisfy the quaternionic relations. The condition of parallelism ensures that they provide what were called local quaternionic frames in the previous chapters.

The conclusion is that for a rank 4 parallel even Clifford structure, the twistor space of M corresponds to the direct product of the twistor spaces of the two quaternionic-Kähler manifolds whose product gives M.

There is only one subtlety left in this case. It is possible that one of these quaternionic-Kähler manifolds is trivial. Indeed this is a consequence of the possibility that the splitting $T_xM = T_x^+M \oplus T_x^-M$ is trivial, hence the representation by φ on every tangent space T_xM is not faithful.

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