

# Construction of Conserved Quantities with the Conformal Tractor Calculus



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# Abstract

This dissertation applies the conformal tractor calculus in order to construct conserved quantities associated with scalar and electromagnetic fields defined on curved space-times. Several basic notions of differential geometry are introduced as well as a discussion of conformal geometry. Once the necessary background is established, many of the standard tools of the conformal tractor calculus are presented. We see that if the space-time is either conformally flat or conformally Einstein, the tractor calculus may be used to generate new symmetric divergence-free tensors from the standard energy-momentum tensors of a field described by a Lagrangian. The case of constructing such a tensor for an electromagnetic field from a massless scalar field is investigated.

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# Chapter 1

## Introduction

### 1.1 Motivation

A conformal manifold is a pair  $(M, c)$  where  $M$  is a smooth manifold and  $c$  is an equivalence class of pseudo-Riemannian metrics known only up to scale. That is, two metrics are conformally equivalent  $\hat{g} \sim g$  if and only if there exists a strictly positive function  $\Omega \in C^\infty(M)$  such that  $\hat{g} = \Omega^2 g$ . For two vectors  $X, Y \in TM$ , each with positive length, the angle between them (at each point)

$$\cos \theta = \frac{g(X, Y)}{\sqrt{g(X, X)g(Y, Y)}}$$

clearly remains invariant under the equivalence class.

Conformal invariance has long been important to physics [9]. A central aspect of research has focussed on constructing and classifying conformally invariant operators both in the conformally flat case as well as in general curved spaces [11, 14]. The tractor calculus, developed in this dissertation following [1, 6, 16], is considered the intuitive version of the theory of Cartan connections on principal bundles. The calculus allows the presentation of explicit formulae for differential operators as well as the ability to easily compute within a scale  $g \in c$  without breaking the conformal symmetry. In dimension 2, oriented conformal manifolds are precisely Riemann surfaces and thus (since Riemann surfaces are locally indistinguishable) possess no local invariants, however in higher dimensions such invariants do exist. These have been studied using the ambient method construction [13] as well as the tractor calculus [16].

As is clear from physics, conserved quantities are also of interest. The canonical example being conservation of energy, as well as linear and angular momentum which are consequences of Noether's theorem for conserved quantities derived from symmetries of a given system. Specifically, the vector fields obtained from contracting Killing vector fields with the energy-momentum tensor associated with a field attained from a Lagrangian. This dissertation was, in part, motivated by the desire to consider conformal analogues and extensions of this idea: whether one would be able to use the tractor calculus to develop new conserved quantities from the energy-momentum tensors. This was initiated by an observation (discussed in Section 5.3) made by my supervisor, Rod Gover, while working at the University of Brest

with Jean-Philippe Nicolas in 2012.

The reader is assumed to be familiar with the aspects of differential geometry found in [21, 23], in particular, the Levi-Civita connection and associated Riemann curvature tensor. One topic which is discussed in these texts as well as this dissertation is the Lie derivative; this has been included due to its geometrical importance to Killing vectors which, as mentioned above, lead to the conserved quantities associated with an energy-momentum tensor.

## 1.2 Outline of Dissertation

Chapter 2 introduces a selection of ideas from differential and pseudo-Riemannian geometry that will be required for work in this dissertation. The Lie derivative is presented from a geometric view by considering vector fields and their associated flows. Many fundamental tensors of Riemannian geometry are presented as well as several which play prominent roles in conformal geometry. Connection coupling is then discussed which is used abundantly throughout this dissertation, almost always without comment.

Chapter 3 develops ideas of conformal geometry. The conformal flat model is presented in order to motivate the extra 2 dimensions of the tractor bundle (a rank  $n + 2$  vector bundle central to the tractor calculus). This method has been chosen as it is slightly more concrete (although less general) than investigating the group structure of conformal (and more generally parabolic) geometries. Weight bundles are defined within the category of conformal manifolds (rather than the category of smooth manifolds). The necessary calculations are presented in order to investigate conformal transformations of the tensors introduced in chapter 2. This enables a discussion of the conformally invariant Laplacian however the methods used to attain this operator are too naive so we turn our attention to the tractor calculus.

Chapter 4, a significant part of this dissertation, develops much of the basic machinery of conformal tractor calculus. This includes the tractor bundle, its conformally invariant connection and compatible metric. The  $D$ -operator is derived via the double- $D$ -operator and parallel tractors are briefly mentioned.

Chapter 5 presents results using the tractor calculus. Attempting to construct conserved quantities related to the electromagnetic field equations, two ideas are presented. The second method applies in conformally Einstein settings rather than merely conformally flat settings so this idea is preferred and investigated. Finally this second method is considered in reverse and the necessary conditions are investigated for a construction of conserved quantities related to the scalar field equations.

## 1.3 Notation

This dissertation will use Penrose's abstract index notation. The full formalism is presented in [27] and a working definition is given in [32]. Two disadvantages of the conventional index notation and the index-free notation are avoided by using abstract indices. Index-free

notation is often useful when dealing with tensors of low rank however it is rarely possible to succinctly convey symmetries of high rank tensors. Moreover, calculations become particularly cumbersome if operations such as contraction are involved since, in such situations, multiple symbols are introduced to talk about objects which have been derived directly from a single tractor (consider the Riemann curvature, the Ricci curvature, and the scalar curvature). Of course, the other notations will occasionally find favour in this dissertation: the index free notation (Section 2.3) and the conventional index notation (Section 3.1).

Effectively, abstract index notation possesses all the advantages of conventional index notation without the significant drawback of requiring a choice of basis, thus one may be sure that any written formula is basis independent. In this context, for example, the Riemann curvature tensor is denoted  $R_{ab}{}^c{}_d$  and not simply  $R$ , that is, the indices are considered part of the tensor and do not merely offer the convenience of denoting the type of tensor one confronts. This allows the three curvatures mentioned above to be unambiguously denoted  $R_{ab}{}^c{}_d$ ,  $R_{ab}$ , and  $R$ . Riemannian metrics will be denoted  $g_{ab}$ , their inverses,  $g^{ab}$  (such that  $g_{ab}g^{bc} = \delta_a^c$ ), and, as is common practice, these will be used to raise and lower indices of tensors, often without comment.

Lower case Latin indices are used throughout for tensors; upper case Latin letters are used for tractor indices. The trivial bundle, tangent bundle, and cotangent bundles are denoted  $\mathcal{E}$ ,  $\mathcal{E}^a$ , and  $\mathcal{E}_a$  respectively. Higher tensor bundles are realised by appending the appropriate indices to  $\mathcal{E}$ . Similarly the tractor bundle and its dual are denoted  $\mathcal{E}^A$  and  $\mathcal{E}_A$  respectively. Further, the direct product notation is suppressed and  $\mathcal{E}_{aA}$  denotes  $\mathcal{E}_a \otimes \mathcal{E}_A$ , for example. Finally, we abuse notation by not distinguishing a vector bundle, say  $\mathcal{E}^a$ , from its space of smooth sections  $\Gamma(\mathcal{E}^a)$ . Consequently, we will write statements such as  $f \in \mathcal{E}$  implying  $f$  is a smooth real-valued function (on the manifold).

Square and round brackets are used to denote antisymmetrisation and symmetrisation. So for a covariant 2-tensor,  $T_{ab}$ , we have  $T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba})$  and  $T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba})$ , with the natural extension to tensors (and tractors) of higher rank.





# Chapter 2

## Differential Geometry and Pseudo-Riemannian Geometry

### 2.1 Lie Derivative

The Lie derivative is a natural and important operation on smooth manifolds. It is possible (see [27]) to generate the Lie derivative of a tensor with respect to a vector  $\xi$  uniquely as follows. The Lie derivative of a vector  $\chi$  is the Lie bracket  $\mathcal{L}_\xi\chi = [\xi, \chi]$ , and the Lie derivative of a function  $f$  is its directional derivative determined by  $\xi$ ,  $\mathcal{L}_\xi f = \xi(f)$ . The Lie derivative is then extended by requiring the Leibniz rule to apply. This method neglects the geometrical interpretation of the operation which is clear from the following (alternative but equivalent) construction.

#### Integral Curves and Flows

Integral curves of vector fields are smooth curves in the manifold whose tangent vector at each point agrees with the vector field. The collection of all integral curves associated with a vector field determine the flow of the vector field: a one parameter family of diffeomorphisms of open subsets of the manifold. The fundamental theorem on flows (see [24]) asserts that every smooth vector field determines a unique maximal integral curve starting at each point, and the collection of all such integral curves determines a unique maximal flow.

Following the notation of [22], if  $\xi$  is a vector field on  $M$  let  $Fl_t^\xi(x) = Fl^\xi(t, x) = c_x(t)$  where  $c_x : I_x \rightarrow M$  is the maximally defined integral curve of  $\xi$  starting at  $x \in M$  (i.e.  $c(0) = x$  and  $\dot{c} = \xi \circ c$ ). Then  $Fl^\xi$  is the flow of  $\xi$ . One says that the vector field is complete if  $I_x = \mathbb{R}$  for every  $x$  however in order to introduce the Lie derivative we need only a local result. The basic existence and uniqueness theorem of ordinary differential equation initial value problems ensures the domain  $I_x$  of each integral curve is an open set containing  $0 \in \mathbb{R}$  however the result is stronger. For each point  $x$ , there exists an open neighbourhood  $U$  containing  $x$  and an  $\epsilon > 0$  such that the local flow  $Fl^\xi$  along the integral curves of  $\xi$  is defined on  $U \times (-\epsilon, \epsilon)$  and for any  $t \in (-\epsilon, \epsilon)$ ,  $Fl_t^\xi$  is a diffeomorphism onto its image when restricted to  $U \times \{t\}$ . In particular we may use this diffeomorphism to pull back tensors of

arbitrary rank.

## Definition and Properties

The Lie derivative of a tensor  $T$  along a vector  $\xi$  at  $x \in M$  is defined by

$$\mathcal{L}_X T(x) = \left. \frac{d}{dt} \right|_{t=0} [(Fl_t^\xi)^* T](x).$$

Along the integral curve  $t \mapsto Fl^\xi(t, x)$  we pull back the values of the tensor to the fibre over  $x$  of the appropriate tensor bundle and then apply  $\left. \frac{d}{dt} \right|_{t=0}$ . The preceding result on local flows ensures that this is well-defined.

The Lie derivative satisfies a number of properties which are straightforward to verify. When applied to functions, it recovers the simple directional derivative and also satisfies the Leibniz property  $\mathcal{L}_\xi(S \otimes T) = \mathcal{L}_\xi S \otimes T + S \otimes \mathcal{L}_\xi T$  for tensors  $S, T$ . Acting on differential forms, it commutes with the exterior derivative  $d(\mathcal{L}_\xi \omega) = \mathcal{L}_\xi(d\omega)$  and satisfies Cartan's formula

$$\mathcal{L}_\xi = d \circ \iota_\xi + \iota_\xi \circ d$$

where  $\iota$  is the insertion operator. Finally (using abstract indices) we can express the Lie derivative of a tensor  $T^{a\dots b}_{c\dots d}$  along the vector field  $\xi^a$  by

$$\begin{aligned} \mathcal{L}_\xi T^{a\dots b}_{c\dots d} &= \xi^e \nabla_e T^{a\dots b}_{c\dots d} - T^{e\dots b}_{c\dots d} \nabla_e \xi^a - \dots - T^{a\dots e}_{c\dots d} \nabla_e \xi^b \\ &\quad + T^{a\dots b}_{e\dots d} \nabla_c \xi^e + \dots + T^{a\dots b}_{c\dots e} \nabla_d \xi^e \end{aligned}$$

where  $\nabla$  is any torsion-free connection on the tangent bundle.

When applied to a vector field, it coincides with the Lie bracket. It is this definition which reveals the geometrical interpretation of the Lie bracket of two vector fields: the Lie bracket is the directional derivative of the second vector field along the flow of the first. In fact for any tensor, the Lie derivative represents the infinitesimal dragging of the tensor along the integral curves of the vector. Since the Lie derivative and the Lie bracket coincide, the operator  $\mathcal{L}_\xi|_x$  is not tensorial in  $\xi$ , that is, it depends not only on the direction of  $\xi$  at  $x$  but also on the direction of  $\xi$  at neighbouring points. For this reason, the covariant derivative is preferred over the Lie derivative as a generalisation of the usual directional derivative on  $\mathbb{R}^n$ .

## Killing Vectors

A diffeomorphism  $\phi : M \rightarrow M$  is an isometry if it carries the metric into itself, that is, if pointwise  $g = \phi^* g$  (where  $\phi^*$  denotes the pull-back map induced by  $\phi$ ). If the flow of a vector  $k$  is a one-parameter group of isometries then  $k$  is called a Killing vector (or infinitesimal symmetry of the metric). In such a case, the local flow preserves the metric,  $(Fl_t^k)^* g = g$ , so the Lie derivative of the metric vanishes and it follows that  $2\nabla_{(a} k_{b)} = 0$ . This is Killing's equation and is a sufficient condition for  $k$  to be a Killing vector (see [21]).

Killing vectors on a manifold indicate the presence of symmetries within the system. They frequently lead to conservation properties and as an illustration, two examples are given.

**Proposition 2.1.** *Let  $k$  be a Killing vector field and  $c$  a geodesic in  $M$  with tangent vector field  $\dot{c}$ . Then  $g(k, \dot{c})$  is constant along  $c$ .*

*Proof.* Killing's equation implies  $g(\nabla_\xi k, \chi) + g(\xi, \nabla_\chi k) = 0$  for vector fields  $\xi, \chi$  so on setting  $\xi = \chi = \dot{c}$ ,  $g(\nabla_{\dot{c}} k, \dot{c}) = 0$ . Also,  $g(k, \nabla_{\dot{c}} \dot{c}) = 0$  so

$$\frac{d}{dt}g(k, \dot{c}) = g(\nabla_{\dot{c}} k, \dot{c}) + g(k, \nabla_{\dot{c}} \dot{c}) = 0 \quad \square$$

**Theorem 2.2.** *If  $T^{ab}$  is a symmetric tensor with  $\nabla_a T^{ab} = 0$  and  $k^a$  is a Killing vector field then  $\xi^a = T^{ab}k_b$  is divergence-free.*

*Proof.* First,  $\nabla_a \xi^a = (\nabla_a T^{ab})k_b + T^{ab}\nabla_a k_b = T^{ab}\nabla_a k_b$ . Killing's equation,  $\nabla_a k_b = -\nabla_b k_a$ , and the symmetry of  $T^{ab}$  give

$$T^{ab}\nabla_a k_b = -T^{ab}\nabla_b k_a = -T^{ba}\nabla_b k_a = -T^{ab}\nabla_a k_b.$$

Therefore  $\nabla_a \xi^a = -\nabla_a \xi^a$  hence  $\nabla_a \xi^a$  vanishes.  $\square$

Although the preceding result is easy to prove, its importance is paramount to this thesis and the construction of conserved quantities. Suppose a metric does admit a Killing vector field  $k^a$ . Then from a symmetric divergence-free tensor  $T^{ab}$  we construct the vector  $\xi^a = T^{ab}k_b$ . Then the quantity  $\nabla_a \xi^a$  vanishes. Moreover, if  $(M, g)$  is a compact orientable manifold (and so has volume measure  $dv$ ) with boundary  $\partial M$  (which will possess a surface measure  $d\sigma$  induced from  $dv$  and a choice of normal form  $n_a$ ), then Stokes' theorem ensures

$$\int_{\partial M} \xi^a d\sigma_a = \int_M \nabla_a \xi^a dv = 0$$

where  $d\sigma_a = n_a d\sigma$ .

This property is heavily exploited in physics. As discussed in [21], matter fields defined on space-time have associated energy-momentum tensors from which, in the presence of Killing vector fields, conservation laws result. As a simple example, the inhomogeneous Lorentz group is generated by the four translations and six rotations of (4-dimensional) Minkowski space. Contracting these Killing vector fields with the energy-momentum tensor of a particular matter field, the divergence-free vectors  $\xi^a$  are interpreted as the flow of energy (time translation vector), the flow of linear momentum (space translation vectors), and the flow of angular momentum (rotation vectors).

## 2.2 Riemann Tensor

### Definition and Symmetries

The Riemann tensor of a torsion-free connection is defined by the equation

$$R_{ab}{}^c{}_d \xi^d = 2\nabla_{[a}\nabla_{b]}\xi^c. \quad (2.2.1)$$

The tensor is skew in its first two indices so is conveniently thought of as a 2-form taking values in endomorphisms of the tangent bundle, a line of thinking which is consistent with curvature tensors defined on arbitrary vector bundles as will be shown below. In index free notation this definition is equivalent to

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z$$

for vector fields  $X, Y, Z$ .

Due to a number of symmetries possessed by the Riemann tensor, it has only  $\frac{1}{12}n^2(n^2 - 1)$  algebraically independent components rather than  $n^4$  like an arbitrary rank 4 tensor does on a manifold of dimension  $n$ . The first and most obvious is  $R_{(ab)}{}^c{}_d = 0$  which follows directly from the definition. In order to derive the Bianchi symmetry  $R_{[ab}{}^c{}_d] = 0$  we observe the following. First, if anti-symmetrisation is applied to a number of indices over which a subsequent anti-symmetrisation is also applied, then the first anti-symmetrisation may be ignored, for example  $S_{[[ab]c]} = S_{[abc]} = S_{[a[bc]]}$  for  $S_{abc} \in \mathcal{E}_{abc}$ . Second, the statement  $R_{[ab}{}^c{}_d] = 0$  may be proved locally by showing it vanishes when operating on exact 1-forms, that is by showing  $R_{[ab}{}^c{}_d]\nabla_c f = 0$  for arbitrary  $f \in \mathcal{E}$ . To this end, we require a formula for the tensor operating on 1-forms. Since we are considering a torsion-free connection,  $0 = 2\nabla_{[a}\nabla_{b]}(\omega_c \xi^c) = \omega_c 2\nabla_{[a}\nabla_{b]}\xi^c + \xi^c 2\nabla_{[a}\nabla_{b]}\omega_c$ , from which it is clear that

$$R_{ab}{}^c{}_d \omega_c = -2\nabla_{[a}\nabla_{b]}\omega_d. \quad (2.2.2)$$

for  $\omega_a \in \mathcal{E}_a$ . The Bianchi symmetry is now easy to prove. For  $f \in \mathcal{E}$ , the torsion-free property of the connection implies  $\nabla_{[a}\nabla_{b]}f = 0$  hence

$$R_{[ab}{}^c{}_d]\nabla_c f = -2\nabla_{[[a}\nabla_{b]}\nabla_{d]}f = -2\nabla_{[a}\nabla_{[b}\nabla_{d]}f = 0$$

verifying  $R_{[ab}{}^c{}_d] = 0$ . This result is also easily attained in index-free notation where one would view it as a consequence of the Jacobi identity.

The Bianchi identity  $\nabla_{[a}R_{bc]}{}^d{}_e = 0$  may also be attained in index-free notation (as a consequence of the Jacobi identity on vector fields and the differential operator  $\xi^a \nabla_a$ ) however the following derivation is given as it easily generalises to the curvature tensor on an arbitrary vector bundle. To this end we consider the action of  $2\nabla_{[a}\nabla_{b]}$  on  $\omega_c \xi^d$ . The Liebniz rule as

well as (2.2.1) and (2.2.2) give

$$\begin{aligned}
2\nabla_{[a}\nabla_{b]}(\omega_c\xi^d) &= \nabla_a\nabla_b(\omega_c\xi^d) - \nabla_b\nabla_a(\omega_c\xi^d) \\
&= \nabla_a(\xi^d\nabla_b\omega_c + \omega_c\nabla_b\xi^d) - \nabla_b(\xi^d\nabla_a\omega_c + \omega_c\nabla_a\xi^d) \\
&= \xi^d\nabla_a\nabla_b\omega_c + \omega_c\nabla_a\nabla_b\xi^d - \xi^d\nabla_b\nabla_a\omega_c - \omega_c\nabla_b\nabla_a\xi^d \\
&= \omega_c2\nabla_{[a}\nabla_{b]}\xi^d + \xi^d2\nabla_{[a}\nabla_{b]}\omega_c \\
&= \omega_c\xi^e R_{ab}{}^d{}_e - R_{ab}{}^e{}_c\omega_e\xi^d.
\end{aligned}$$

(It is clear how this extends to arbitrary tensors, in fact, we will need to consider  $2\nabla_{[a}\nabla_{b]}g_{cd}$  for the final two symmetries.) We use the preceding equation and the Bianchi symmetry to see

$$\begin{aligned}
\nabla_{[a}(R_{bc]}{}^d{}_e\xi^e) &= 2\nabla_{[a}\nabla_{[b}\nabla_{c]}\xi^d \\
&= 2\nabla_{[[a}\nabla_{b]}\nabla_{c]}\xi^d \\
&= (\nabla_{[c}\xi^e)R_{ab]}{}^d{}_e - R_{[ab}{}^e{}_c]\nabla_e\xi^d \\
&= (\nabla_{[c}\xi^e)R_{ab]}{}^d{}_e.
\end{aligned}$$

From this, the Leibniz rule gives the Bianchi identity

$$(\nabla_{[a}R_{bc]}{}^d{}_e)\xi^e = \nabla_{[a}(R_{bc]}{}^d{}_e\xi^e) - (\nabla_{[a}\xi^e)R_{bc]}{}^d{}_e = (\nabla_{[c}\xi^e)R_{ab]}{}^d{}_e - (\nabla_{[a}\xi^e)R_{bc]}{}^d{}_e = 0.$$

If a metric is present, in which case we will assume that  $\nabla$  is the Levi-Civita connection, applying the commutator of two derivatives gives

$$0 = 2\nabla_{[a}\nabla_{b]}g_{cd} = -R_{ab}{}^e{}_c g_{ed} - R_{ab}{}^e{}_d g_{ce} = -2R_{ab(cd)}.$$

The final interchange symmetry  $R_{abcd} = R_{cdab}$  is quickly attained by realising the Bianchi symmetry and  $R_{ab(cd)} = 0$  imply  $R_{[abc]d} = 0$  so

$$\begin{aligned}
2R_{abcd} &= R_{abcd} + R_{badc} \\
&= -R_{bcad} - R_{cabd} - R_{adbc} - R_{dbac} \\
&= -R_{dacb} - R_{acdb} - R_{cbda} - R_{bdca} \\
&= R_{cdba} + R_{dcba} \\
&= 2R_{cdba}.
\end{aligned}$$

## Riemann Tensor Decomposition

In dimensions  $n \geq 3$ , there is a well known invariant decomposition of the Riemann tensor (in its covariant form  $R_{abcd}$ ) using the Kulkarni-Nomizu product of two tensors in  $\mathcal{E}_{(ab)}$  defined by

$$(S \otimes T)_{abcd} = 2S_{a[c}T_{d]b} - 2S_{b[c}T_{d]a}.$$

In order to detail this decomposition, we note two tensors that are immediately attained from the Riemann tensor. The Ricci tensor is the symmetric covariant 2-tensor obtained from contracting on the first and third indices,  $R_{ab} = R_{ca}{}^c{}_b$ , and the Ricci scalar is obtained by contracting the Ricci tensor in the presence of a metric  $R = g^{ab}R_{ab}$ .

The space of covariant 4-tensors with the four symmetries of the Riemann tensor (excluding the Bianchi identity) may be decomposed into two complementary subspaces; the first consisting of covariant totally trace-free 4-tensors with the given symmetries and the second consisting of the image of symmetric covariant 2-tensors under the map  $S_{ab} \mapsto (S \otimes g)_{abcd}$  (see [2]). Importantly this gives the decomposition of the Riemann tensor  $R_{abcd} = C_{abcd} + (P \otimes g)_{abcd}$  as

$$R_{abcd} = C_{abcd} + 2P_{a[c}g_{d]b} - 2P_{b[c}g_{d]a} \quad (2.2.3)$$

where  $C_{abcd}$  is the Weyl tensor and

$$P_{ab} = \frac{1}{n-2} \left( R_{ab} - \frac{R}{2(n-1)}g_{ab} \right)$$

is the Schouten tensor (a trace adjusted multiple of the Ricci tensor). It is also necessary in conformal geometry to define the contraction of the Schouten tensor which will be called the Schouten scalar and denoted  $P$  consistent with the Ricci tensor and associated scalar. Notice that the Ricci tensor may be recovered from the Schouten tensor (in dimensions  $n \geq 3$ ). Contracting the formula for the Schouten tensor above gives

$$R = 2(n-1)P, \quad (2.2.4)$$

$$R_{ab} = (n-2)P_{ab} + Pg_{ab}. \quad (2.2.5)$$

Finally, the Bianchi identity gives two useful identities for the Schouten tensor [29]

$$\begin{aligned} \nabla_c C_{ab}{}^c{}_d &= 2(n-3)\nabla_{[a}P_{b]d}, \\ \nabla^b P_{ab} &= \nabla_a P. \end{aligned} \quad (2.2.6)$$

As we shall see the Weyl tensor is conformally invariant and thus plays a natural role in conformal geometry. A well known result in conformal geometry is that a manifold of dimension  $n \geq 4$  is locally conformally flat if and only if the Weyl tensor vanishes (local conformal flatness means there exists a local coordinate system in which the metric is proportional to a constant tensor). In dimensions  $n \leq 3$ , the Weyl curvature vanishes and conformal flatness is measured by the vanishing of  $\nabla_{[a}P_{b]c}$  (which is, up to index placement and scale, the Cotton-York tensor.) Finally, in two dimensions  $R_{abcd} = K(g_{ac}g_{bd} - g_{bc}g_{ad})$  where  $K$  is the Gauss curvature.

## 2.3 Vector Bundles and Connection Coupling

We introduce the notion of connections on vector bundles and then discuss coupled connections which are central to the application of the tractor calculus. As is common in the literature, we will write  $D : E \rightarrow F$  for a differential operator  $D$  acting between vector bundles  $E, F$ . In particular  $D$  takes smooth sections of  $E$  to smooth sections of  $F$  and is not, in general, a vector bundle homomorphism.

Let  $E$  be a real vector bundle over a manifold  $M$ . For consistency of notation, we will introduce connections using abstract indices, therefore let  $\mathcal{E}^\Phi$  denote the bundle  $E$ . A connection on  $E$  is an  $\mathbb{R}$ -linear differential operator

$$\nabla_a^E : \mathcal{E}^\Phi \rightarrow \mathcal{E}_a^\Phi$$

which satisfies the Leibniz rule

$$\nabla_a^E(fU^\Phi) = (\nabla_a^M f)U^\Phi + f\nabla_a^E U^\Phi$$

for  $U^\Phi \in \mathcal{E}^\Phi$ ,  $f \in \mathcal{E}$ . Here  $\nabla_a^M$  is any torsion-free affine connection (hence  $\nabla_a^M f$  is just the 1-form determined by the differential of  $f$ ). In the context of this dissertation,  $\nabla_a^M$  will be the Levi-Civita connection of a representative metric in the conformal class. This notation of appending the bundle as a superscript to  $\nabla$  is extremely cumbersome. In future sections it will not be used however this section aims to introduce coupled connections, and in this respect, the extra clarity associated with this notation will be useful.

The connection on  $E$  induces connections on all tensor powers of  $E$  and its dual by requiring the Leibniz rule to hold and extending linearly. For example, the dual connection  $\nabla_a^E : \mathcal{E}_\Phi \rightarrow \mathcal{E}_{a\Phi}$  is defined by requiring

$$U^\Phi \nabla_a^E V_\Phi = \nabla_a^M (V_\Phi U^\Phi) - V_\Phi \nabla_a^E U^\Phi$$

to hold for all  $U^\Phi \in \mathcal{E}^\Phi$ . The concept of linearity is used to extend the connection to  $\mathcal{E}^{\Phi\Psi}$  for example. More explicitly, all sections of  $\mathcal{E}^{\Phi\Psi}$  are generated by sections of the form  $U^\Phi V^\Psi$  so in order to define  $\nabla_a^E : \mathcal{E}^{\Phi\Psi} \rightarrow \mathcal{E}_a^{\Phi\Psi}$ , it suffices to require

$$\nabla_a^E(U^\Phi V^\Psi) = (\nabla_a^E U^\Phi)V^\Psi + U^\Phi(\nabla_a^E V^\Psi).$$

from which we may linearly extend the connection to act on all sections of  $\mathcal{E}^{\Phi\Psi}$ .

Suppose the bundle  $E$  is equipped with a bundle metric, denoted  $g_{\Phi\Psi}$ , which is preserved by the connection. (This will occur in the tractor setting.) Then raising and lowering bundle indices commutes with the connection. To understand this, preservation of the metric,  $\nabla_a^E g_{\Phi\Psi} = 0$  is equivalent to

$$\xi^a \nabla_a^M (g_{\Phi\Psi} U^\Phi V^\Psi) = g_{\Phi\Psi} (\xi^a \nabla_a^M U^\Phi) V^\Psi + g_{\Phi\Psi} U^\Phi (\xi^a \nabla_a^M V^\Psi)$$

for  $\xi^a \in \mathcal{E}^a$  and bundle sections  $U^\Phi, V^\Psi$ . The consequence of this is

$$\nabla_a^E U_\Psi = \nabla_a^E (g_{\Phi\Psi} U^\Phi) = g_{\Phi\Psi} \nabla_a^E U^\Phi$$

for  $U^\Phi \in \mathcal{E}^\Phi$  which simplifies calculations considerably.

Suppose  $E, F$  are two (real) vector bundles with respective connections  $\nabla_a^E, \nabla_a^F$ . As before, associate upper case Greek indices with  $E$ , and now associate upper case Latin indices with  $F$ . So  $E = \mathcal{E}^A$  for example. Then the coupled connection is the differential operator  $\nabla_a^{\mathcal{C}} : \mathcal{E}^{\Phi A} \rightarrow \mathcal{E}_a^{\Phi A}$  determined by

$$\nabla_a^{\mathcal{C}}(U^\Phi V^A) = (\nabla_a^E U^\Phi) V^A + U^\Phi (\nabla_a^F V^A) \quad (2.3.1)$$

where  $U^\Phi \in \mathcal{E}^\Phi, V^A \in \mathcal{E}^A$  (and then extending this result linearly to all sections). Extending the idea above in the obvious manner, this induces connections on tensor products of bundles associated to  $E$  and  $F$ . From now on this excessive notation will be abandoned. We will avoid the superscript on the connections and write all connections simply as  $\nabla_a$ . Context will determine whether connections have been coupled as well as determining which connections were originally present.

## Curvature

Interestingly, this idea of coupling connections explains the invariance of the tractor curvature to be discussed in section 4.4. On first sight, the invariance is slightly surprising however the invariance is ensured by the following argument. The exterior differential operator on  $k$ -forms  $d : \Lambda^k \rightarrow \Lambda^{k+1}$  is, of course, a natural operation on the smooth manifold  $M$ , independent of any metric or conformal structure. In the presence of a connection  $\nabla : E \rightarrow \Lambda^1 \otimes E$ , we attain induced operators

$$d^\nabla : \Lambda^k \otimes E \rightarrow \Lambda^{k+1} \otimes E$$

determined by

$$d^\nabla(\omega \otimes U) = d\omega \otimes U + (-1)^k \omega \wedge \nabla U$$

for  $\omega \in \Gamma(\Lambda^k)$  and  $U \in \Gamma(E)$ . Since  $d(f\omega) = df \wedge \omega + f d\omega$  for  $f \in C^\infty(M)$  we have

$$\begin{aligned} d^\nabla(f\omega \otimes U) &= df \wedge \omega \otimes U + f d\omega \otimes U + (-1)^k f \omega \wedge \nabla U \\ &= f d^\nabla(\omega \otimes U) + df \wedge \omega \otimes U \end{aligned}$$

so (by extending linearly)  $d^\nabla$  satisfies a Leibniz rule for all sections of  $\Lambda^k \otimes E$ .

Considering the composition

$$E \xrightarrow{\nabla} \Lambda^1 \otimes E \xrightarrow{d^\nabla} \Lambda^2 \otimes E$$



we have

$$\begin{aligned}
(d^\nabla \circ \nabla)(fU) &= d^\nabla(df \otimes U + f\nabla U) \\
&= d^2 f \otimes U - df \wedge \nabla U + f(d^\nabla \circ \nabla)(U) + df \wedge U \\
&= f(d^\nabla \circ \nabla)(U)
\end{aligned}$$

hence the composition function is linear over functions. This determines the curvature of  $\nabla : E \rightarrow \Lambda^1 \otimes E$  which is conventionally written  $\kappa \in \Gamma(\Lambda^2 \otimes \text{End}(E))$ .

Translating this idea into abstract indices, consider coupling the connection on  $E$  to a torsion-free connection on the tangent bundle (equivalently, the cotangent bundle). This allows second derivatives of a section of  $E$  to be taken and the curvature  $\kappa_{ab}{}^\Phi{}_\Psi$  is determined by

$$\kappa_{ab}{}^\Phi{}_\Psi U^\Psi = (\nabla_a \nabla_b - \nabla_b \nabla_a) U^\Phi.$$

In Section 4.4, the second derivative of a tractor will be taken by picking a metric in the conformal class and coupling the tractor connection to the associated Levi-Civita connection. The result that the curvature obtained remains conformally invariant is then a consequence of the observation that by skewing we deal with the exterior derivative  $d$ , an operator independent of the chosen metric.



# Chapter 3

## Conformal Geometry

This chapter gives a brief introduction to conformal geometry and the necessary objects to proceed with the tractor calculus. First, equivalent definitions of a conformal manifold are presented leading to a presentation of the flat model: the  $n$ -dimensional sphere with conformal structure descending from an ambient  $(n + 2)$ -dimensional Minkowski space. The weight bundles, which are necessary for the tractor calculus, are then constructed as associated bundles to the ray subbundle  $\mathcal{Q}$  (defined below). The conformal transformations of many of the tensors from the preceding chapter as well as the Levi-Civita connection are then calculated. It is clear from the transformation laws that the weight bundles enable a more concise treatment of these structures in the conformal setting. Finally we digress to a naive derivation of the Yamabe operator, the conformal Laplacian. The desire to find other (conformally) invariant operators motivates the tractor calculus of the following chapter.

### 3.1 Flat Model

Consider a conformal manifold  $(M, c)$  where  $c$  is an equivalence class of Riemannian metrics where the equivalence relation:  $\hat{g} \sim g$  means  $\hat{g} = \Omega^2 g$  for some smooth positive function  $\Omega$ . As this section will consider the flat model of conformal geometry, we will assume the metrics in the conformal class have Riemannian signature. Let  $\mathcal{Q}$  denote the bundle whose smooth sections are metrics from the conformal class  $c$ . Notice that above each point  $x \in M$ , the conformally related metrics differ by a positive real number so  $\mathcal{Q}$  is a ray subbundle of  $\mathcal{E}_{(ab)}$ .

The flat model of conformal geometry is best described on the sphere  $\mathbb{S}^n$ . Its conformal structure is then the class of metrics conformally related to the standard metric on  $\mathbb{S}^n$  (i.e. the metric induced on the sphere by considering the sphere embedded in  $\mathbb{R}^{n+1}$  with canonical metric). It is clear why  $\mathbb{S}^n$  rather than  $\mathbb{R}^n$  is the appropriate manifold as follows: recall that conformal transformations on  $\mathbb{R}^n$  are Euclidean motions, dilations, and inversions in spheres. Specifically, the maps for inversions are not globally defined on  $\mathbb{R}^n$  as the centers of the spheres go to  $\infty$ . This is avoided by considering the one-point compactification of  $\mathbb{R}^n$ , that is  $\mathbb{S}^n$ . Moreover, the standard metric on  $\mathbb{S}^n$  is easily seen to be conformally related to  $\mathbb{R}^n$  with its canonical metric via stereographic projection [10]. Although this

is a satisfactory definition of the flat model, it gives no motivation for why a conformally invariant connection may be found on a vector bundle of rank  $n + 2$ . To this end, a much more appropriate realisation of the flat model exists: the conformal structure is induced on the sphere by considering  $(n + 2)$ -dimensional Minkowski space, that is,  $\mathbb{R}^{n+2}$  equipped with a flat Lorentzian metric. (The calculations for the results mentioned below may be found, from a slightly alternate viewpoint, in [28].)

To describe the flat model, we introduce the following notation. Consider coordinates  $(x_0, x_1, \dots, x_{n+1})$  in  $\mathbb{R}^{n+2}$  such that the Lorentzian metric is

$$\eta = -d(x^0)^2 + \sum_{\alpha=1}^{n+1} d(x^\alpha)^2.$$

Let  $\mathcal{N}$  denote the light cone excluding the origin  $\mathcal{N} = \{x \in \mathbb{R}^{n+2} \setminus \{0\} : \eta(x, x) = 0\}$ . In this setting it is more natural to consider the sphere in projective space so let  $\mathbb{P}^{n+1}$  denote real projective space of dimension  $n + 1$  defined by  $\mathbb{P}^{n+1} = \{l = [x] : x \in \mathbb{R}^{n+2} \setminus \{0\}\}$  where the equivalence class is the set of points which differ by a non-zero real number. Observe that the quadric

$$Q = \{l = [x] : x \in \mathcal{N}\} \subset \mathbb{P}^{n+1}$$

is isomorphic to  $\mathbb{S}^n$ . One may realise this by considering  $y \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$  and the bijection  $y \mapsto [(1, y)] \in Q$ . Finally let  $\pi : \mathcal{N} \rightarrow Q$  be the natural projection. Although restricting  $\eta$  to the tangent bundle of the null cone does not induce a metric (the bilinear form is degenerate), this restriction does, at each  $x \in \mathcal{N}$ , induce an inner product  $g_x$  on  $T_l Q$  where  $l = \pi(x)$ . To see this let  $X$  be the Euler vector field on  $\mathbb{R}^{n+2}$ , then  $X \in T_x \mathcal{N}$  is perpendicular to  $T_x \mathcal{N}$  (hence the degeneracy mentioned above) and  $\pi_*(X) = 0 \in T_l Q$ . The first isomorphism theorem then ensures  $T_x \mathcal{N} / \text{span} X \cong T_l Q$ . This allows  $g_x(u, v)$ , for  $u, v \in T_l \mathcal{N}$ , to be defined by  $g_x(u, v) = \eta(U, V)$  where  $U, V \in T_x \mathcal{N}$  with  $\pi_* U = u$ ,  $\pi_* V = v$  (note the freedom in this choice as  $\pi_* U = \pi_*(U + cX)$  and  $\pi_* V = \pi_*(V + c'X)$  for  $c, c' \in \mathbb{R}$ ). It is easy to see that  $g_x$  is an inner product on  $T_l Q$  however it is more important to consider the dependence of this inner product on the choice of  $x \in l$ . In particular, a short calculation shows that under a dilation  $x \mapsto \lambda x$ ,  $\lambda \in \mathbb{R}_+$ , the two inner products on  $T_l Q$  are conformally related:  $g_{\lambda x} = \lambda^2 g_x$ . Therefore, as claimed, the Minkowski metric on  $\mathbb{R}^{n+2}$  naturally induces a conformal structure on  $Q \cong \mathbb{S}^n$ ; the metrics in the conformal class on  $Q$  correspond to sections (up to sign) of  $\pi : \mathcal{N} \rightarrow Q$ . For further reading, the flat model is presented in [1, 8, 15] emphasising the groups involved in this construction. Also see [5] for the generalisation to arbitrary conformal manifolds as well as the explicit method of constructing the conformal tractor bundle as an associated vector bundle.

Finally, we mention two observations for conformal transformations which further suggest the extra two dimensions of the tractor bundle. First, if  $A \in O(n + 1, 1)$  then  $A$  restricts to a map from the light cone to itself and induces a transformation from  $Q$  to itself which is easily checked to be a conformal transformation. Second, Liouville's theorem for conformal mappings states that for  $n \geq 3$ , all local conformal transformations of the sphere to itself are realised as the restriction of some  $L \in O(n + 1, 1)$  in this manner.

## 3.2 Weight Bundles

We will often encounter objects in conformal geometry which simply scale by a power of  $\Omega$  when a new choice of metric  $\hat{g} = \Omega^2 g$  is considered. We would like to view these objects as conformally invariant; this is the motivation for weight bundles. As before, let  $\mathcal{Q}$  denote the ray subbundle whose sections are the metrics within the conformal class. The conformal density bundle of weight  $w \in \mathbb{R}$ , denoted  $\mathcal{E}[w]$ , may be succinctly defined to be the quotient of the bundle  $\mathcal{Q} \times \mathbb{R}$  by the equivalence relation  $(g_x, f) \sim (\lambda^2 g_x, \lambda^w f)$ . This definition gives the following picture. Given a choice of  $g \in c$ , the weight bundles trivialise so a section  $\tau \in \mathcal{E}[w]$  is naturally viewed as a function  $f : M \rightarrow \mathbb{R}$ . On choosing a different metric  $\hat{g} = \Omega^2 g$  from the conformal class,  $\tau$  is again viewed as a function  $\hat{f} : M \rightarrow \mathbb{R}$ . These two functions are related by  $\hat{f} = \Omega^w f$ . We follow the convention that, for any vector bundle  $\mathcal{V}$ , we write  $\mathcal{V}[w]$  to denote the tensor product  $\mathcal{V} \otimes \mathcal{E}[w]$ . Considering the conformal metrics and  $\mathcal{E}_{ab}[2]$  we have the following.

**Definition 3.1.** *For a conformal manifold  $(M, c)$ , the conformal metric  $\mathbf{g}_{ab}$  is the tautological section of  $\mathcal{E}_{ab}[2]$  such that in any scale  $g$ , the conformal metric is represented by  $g$  itself. Similarly,  $\mathbf{g}^{ab}$  is the section of  $\mathcal{E}^{ab}[-2]$  such that  $\mathbf{g}_{ab}\mathbf{g}^{bc} = \delta_a^c$ .*

It is also appropriate to mention the conformal volume form. Working locally, so issues of orientability may be ignored, consider an  $n$ -dimensional conformal manifold. Each metric  $g$  determines a Riemann volume form which simply scales by  $\Omega^n$  under a conformal change  $\hat{g} = \Omega^2 g$ . This defines a section of  $\mathcal{E}_{[a_1 \dots a_n]}[n]$ , called the canonical volume form which is compatible with the conformal metric. Importantly, this allows densities of weight  $-n$  to be invariantly integrated.

It is worthwhile to consider the geometry of these weight bundles, even though the definition above gives a sufficient working definition. Recall the fact that a conformal structure on a manifold may equally be defined by a ray subbundle  $\mathcal{Q} \subset \mathcal{E}_{ab}$  whose fibre above  $x \in M$  consists of conformally related metrics at the point. This is illustrated by

$$\begin{array}{ccc} \mathbb{R}_+ & \hookrightarrow & \mathcal{Q} \\ & & \downarrow \pi \\ & & M \end{array}$$

Equivalently  $\pi : \mathcal{Q} \rightarrow M$  is a principal bundle with structure group  $\mathbb{R}_+$ . (The projection is naturally induced by the projection from  $\mathcal{E}_{ab}$  onto  $M$ , and the left action of  $\mathbb{R}_+$  on  $\mathcal{Q}$  is  $\lambda \cdot g_x = \lambda^2 g_x$ .) With principal bundles, one may construct associated bundles via representations. In the case at hand, the representation  $\rho_w : \mathbb{R}_+ \rightarrow \text{End}(\mathbb{R})$ , defined by  $\mathbb{R}_+ \ni \lambda \mapsto \lambda^{-w/2} \in \text{End}(\mathbb{R})$  induces the line bundle  $\mathcal{Q} \times_{\rho_w} \mathbb{R}$  on the conformal manifold. With associated bundles, there is a correspondence between sections of an associated bundle and equivariant functions on the total space. Here, a section  $\tau$  of  $\mathcal{Q} \times_{\rho_w} \mathbb{R}$  is the same as a function  $f : \mathcal{Q} \rightarrow \mathbb{R}$  satisfying the homogeneity property that

$$f(\lambda^2 g_x) = \lambda^w f(g_x).$$

Evidently,  $\mathcal{Q} \times_{\rho_w} \mathbb{R} = \mathcal{E}[w]$ . Finally, we remark that although this shows weight bundles are natural constructions in the category of smooth conformal manifolds (with conformal diffeomorphisms as maps), an alternative construction (detailed in [8]) shows that they exist as natural bundles in the category of smooth manifolds (with diffeomorphisms as maps).

The final task in this section is to extend the Levi-Civita connections to act on  $\mathcal{E}[w]$ . To this end it is appropriate to first see the correspondence between metrics in the conformal class and positive sections of  $\mathcal{E}[1]$ . Specifically for each  $g \in c$  there exists a unique positive  $\sigma \in \mathcal{E}[1]$  such that  $g = \sigma^{-2}\mathbf{g}$ . For this reason, a (positive) section of  $\mathcal{E}[1]$  will often be called a conformal scale (corresponding to a specific metric). Now, if  $\nabla$  denotes the Levi-Civita connection of  $g$ , and  $\tau \in \mathcal{E}[w]$ , we define

$$\nabla_a \tau = \sigma^w \nabla_a (\sigma^{-w} \tau)$$

where  $\nabla_a$  on the right is the usual Levi-Civita connection acting on the unweighted function  $\sigma^{-w} \tau$  (that is, just the exterior derivative  $d$ ). We extend this definition to weighted tensors in the obvious fashion

$$\nabla_e T^{a\dots b}_{c\dots d} = \sigma^w \nabla_e (\sigma^{-w} T^{a\dots b}_{c\dots d})$$

for  $T^{a\dots b}_{c\dots d} \in \mathcal{E}^{a\dots b}_{c\dots d}[w]$ .

Two observations are immediate. First, with  $\sigma$  as the scale of  $g$ , we have  $\nabla_a \sigma = 0$ . Second, since  $\nabla_a g_{bc} = 0$ , the conformal metric is compatible with every Levi-Civita connection

$$\nabla_a \mathbf{g}_{bc} = \sigma^2 \nabla_a (\sigma^{-2} \mathbf{g}_{bc}) = \sigma^2 \nabla_a g_{bc} = 0.$$

Let  $\hat{\nabla}$  be the Levi-Civita connection of  $\hat{g} = \Omega^2 g$  and  $\hat{\sigma}$  the scale corresponding to  $\hat{g}$ . Then  $g = \sigma^{-2} \mathbf{g}$  and  $\hat{g} = \hat{\sigma}^{-2} \mathbf{g}$  so  $\hat{\sigma} = \Omega^{-1} \sigma$ . A simple application of the Leibniz rule shows, for  $\tau \in \mathcal{E}[w]$ , that

$$\begin{aligned} \hat{\nabla}_a \tau &= \hat{\sigma}^w \nabla_a (\hat{\sigma}^{-w} \tau) \\ &= \Omega^{-w} \sigma^w \nabla_a (\Omega^w \sigma^{-w} \tau) \\ &= \sigma^w \nabla_a (\sigma^{-w} \tau) + w (\Omega^{-1} \nabla_a \Omega) \tau \\ &= \nabla_a \tau + w \Upsilon_a \tau \end{aligned} \tag{3.2.1}$$

where  $\Upsilon_a = \Omega^{-1} \nabla_a \Omega$ .

### 3.3 Conformal Transformations

We consider, in some detail, the effect a conformal rescaling of the metric. The transformations of the Levi-Civita connection as well as the various curvature tensors are derived culminating in the transformation of the Ricci scalar which will be particularly important in the subsequent section where we consider the conformal generalisation of the Laplacian. Throughout this section we will consider the conformally related metrics  $\hat{g} = \Omega^2 g$  where  $\Omega$  is a smooth positive function on the manifold. We define the transformation 1-form

$\Upsilon_a = \Omega^{-1}\nabla_a\Omega$  which frequently appears in conformal transformations. We also denote objects associated with  $\hat{g}$  with a circumflex so, for example,  $R_{ab}{}^c{}_d$  and  $\hat{R}_{ab}{}^c{}_d$  will refer to the Riemann tensors of  $g$  and  $\hat{g}$  respectively (more precisely, of their respective Levi-Civita connections).

### Levi-Civita Connection

Consider the two Levi-Civita connections  $\nabla$  and  $\hat{\nabla}$  associated with the respective metrics  $g$  and  $\hat{g}$ . Beginning with the Koszul formula (which uniquely characterises the Levi-Civita connection):

$$2g(\nabla_\chi\xi, \mu) = \chi g(\xi, \mu) + \xi g(\mu, \chi) - \mu g(\chi, \xi) - g(\chi, [\xi, \mu]) + g(\xi, [\mu, \chi]) + g(\mu, [\chi, \xi])$$

(and the corresponding formula for  $\hat{\nabla}$ ) one attains a formula for  $g(\hat{\nabla}_\chi\xi - \nabla_\chi\xi, \mu)$ . This calculation proceeds by treating the vectors as derivations so that

$$\chi\hat{g}(\xi, \mu) = 2\Omega^2g(\xi, \mu)\Upsilon(\chi) + \Omega^2\chi g(\xi, \mu)$$

where  $\Upsilon$  is the (index-free) transformation 1-form  $\Upsilon = \Omega^{-1}d\Omega$ . The result follows by considering the difference between the Koszul formula displayed above and the corresponding formula for  $\hat{\nabla}$  (up to a factor of  $\Omega^2$ ). The result, converting to abstract indices, is

$$\chi^a[(\hat{\nabla}_a - \nabla_a)\xi^b]\mu_b = \chi^a\Upsilon_a\xi^b\mu_b - \chi^a\Upsilon^b\xi_a\mu_b + \chi^a\Upsilon_c\xi^c\mu_a$$

so the desired transformation of the Levi-Civita connection on vectors is

$$\hat{\nabla}_a\xi^b = \nabla_a\xi^b + \Upsilon_a\xi^b - \Upsilon^b\xi_a + \Upsilon_c\xi^c\delta_a^b.$$

To derive the transformation for  $\omega_a \in \mathcal{E}_a$  we use the property that  $\nabla_a f = \hat{\nabla}_a f$  for  $f \in \mathcal{E}$  and that a connection commutes with contraction, in particular if we consider  $\nabla_a(\omega_b\xi^b) = \hat{\nabla}_a(\omega_b\xi^b)$  it is easy to show

$$\hat{\nabla}_a\omega_b = \nabla_a\omega_b - \Upsilon_a\omega_b - \Upsilon_b\omega_a + \Upsilon^c\omega_c g_{ab}. \quad (3.3.1)$$

Finally we require the transformations for the (coupled) connection acting on weighted tensors. These follow from applying the Leibniz law to the transformations for unweighted tensors and (3.2.1). If  $T^{a\dots b}{}_{c\dots d} \in \mathcal{E}^{a\dots b}{}_{c\dots d}[w]$  and  $\sigma, \hat{\sigma}$  are the scales corresponding to  $g, \hat{g}$  respectively so that  $\hat{\sigma} = \Omega^{-1}\sigma$ , then

$$\begin{aligned} \hat{\nabla}_e T^{a\dots b}{}_{c\dots d} &= \Omega^{-w}\sigma^w\hat{\nabla}_e(\Omega^w\sigma^{-w}T^{a\dots b}{}_{c\dots d}) \\ &= \sigma^w\hat{\nabla}_e(\sigma^{-w}T^{a\dots b}{}_{c\dots d}) + w\Upsilon_e T^{a\dots b}{}_{c\dots d}. \end{aligned}$$

One should be careful in reading the above displays as the symbol  $\nabla$  is used in two contexts; it stands for either the Levi-Civita connection, or the Levi-Civita connection coupled to

the connection on the weight bundles. The weighted transformations laws for  $\tau \in \mathcal{E}[w]$ ,  $\xi^b \in \mathcal{E}^b[w]$ , and  $\omega_b \in \mathcal{E}_b[w]$  are

$$\hat{\nabla}_a \tau = \nabla_a \tau + w \Upsilon_a \tau, \quad (3.3.2)$$

$$\hat{\nabla}_a \xi^b = \nabla_a \xi^b + (w+1) \Upsilon_a \xi^b - \Upsilon^b \xi_a + \Upsilon_c \xi^c \delta_a^b, \quad (3.3.3)$$

$$\hat{\nabla}_a \omega_b = \nabla_a \omega_b + (w-1) \Upsilon_a \omega_b - \Upsilon_b \omega_a + \Upsilon^c \omega_c g_{ab}. \quad (3.3.4)$$

## Curvature Tensors

The transformation of the Riemann tensor, defined by  $R_{ab}{}^c{}_d \xi^d = 2\nabla_{[a} \nabla_{b]} \xi^c$ , may be calculated from the transformations of the Levi-Civita connection on vectors and 1-forms. A tedious calculation (see [8]) gives the transformation of the Riemann curvature tensor

$$\hat{R}_{abcd} = \Omega^2 (R_{abcd} + (\Lambda \otimes g)_{abcd}) \quad (3.3.5)$$

where  $\Lambda_{ab} = -\nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \frac{1}{2} \Upsilon_c \Upsilon^c g_{ab}$ . Comparing this with the decomposition of the Riemann tensor (2.2.3), in order for the Riemann tensor of  $\hat{g}$  to decompose correctly into Weyl, Schouten, and metric components (associated with  $\hat{g}$ )  $\hat{R}_{abcd} = \hat{C}_{abcd} + (\hat{P} \otimes \hat{g})_{abcd}$ , we conclude that

$$\hat{C}_{abcd} = \Omega^2 C_{abcd}$$

and  $\hat{P}_{ab} = P_{ab} + \Lambda_{ab}$ :

$$\hat{P}_{ab} = P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \frac{1}{2} \Upsilon_c \Upsilon^c g_{ab}. \quad (3.3.6)$$

The Weyl tensor, as presented, is clearly conformally invariant if considered as a weight 2 tensor, in particular using the conformal metric to raise an index, the Weyl tensor is an unweighted conformally invariant tensor:  $\hat{C}_{ab}{}^c{}_d = C_{ab}{}^c{}_d$ . Contracting (3.3.6) gives the transformation of the Schouten scalar

$$\hat{P} = \Omega^{-2} (P - \nabla_a \Upsilon^a + (1 - \frac{n}{2}) \Upsilon_a \Upsilon^a). \quad (3.3.7)$$

The Ricci tensor and Ricci scalar transformations are now easy to attain. From the Riemann tensor transformation (3.3.5),

$$\begin{aligned} \hat{R}_{ab} &= g^{cd} (R_{cadb} + (\Lambda \otimes g)_{cadb}) \\ &= R_{ab} + \Lambda_c{}^c g_{ab} - \Lambda_{cb} \delta_a^c - \Lambda_a{}^c g_{bc} + \Lambda_{ab} \delta_c^a \\ &= R_{ab} + \Lambda_c{}^c g_{ab} + (n-2) \Lambda_{ab} \\ &= R_{ab} + (-\nabla_c \Upsilon^c + \Upsilon_c \Upsilon^c - \frac{n}{2} \Upsilon_c \Upsilon^c) g_{ab} + (n-2) (-\nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \frac{1}{2} \Upsilon_c \Upsilon^c g_{ab}) \\ &= R_{ab} + (n-2) (\Upsilon_a \Upsilon_b - \nabla_a \Upsilon_b) - (\nabla_c \Upsilon^c + (n-2) \Upsilon_c \Upsilon^c) g_{ab}. \end{aligned}$$



Contracting this with  $\hat{g}^{ab}$  gives the Ricci scalar transformation

$$\begin{aligned}
\hat{R} &= \Omega^{-2} g^{ab} \hat{R}_{ab} \\
&= \Omega^{-2} (R + (n-2)(\Upsilon_a \Upsilon^a - \nabla_a \Upsilon^a) - (\nabla_a \Upsilon^a + (n-2)\Upsilon_a \Upsilon^a)n) \\
&= \Omega^{-2} (R - (n-1)(n-2)\Upsilon_a \Upsilon^a - 2(n-1)\nabla_a \Upsilon^a).
\end{aligned} \tag{3.3.8}$$

In the following chapter on tractor calculus, the curvature tensors will play a central role. It is clear from the previous four calculations that the transformations for the Schouten tensor and scalar are considerably simpler than the respective Ricci tensor and scalar. For this reason, the Schouten tensor is more prominent in both the tractor calculus and conformal geometry in general.

### 3.4 Yamabe Operator

A central question in conformal geometry is how to generalise differential operators on Riemannian manifolds to conformally invariant differential operators. Such operators are known to play an important role in physics, for example, the classical field equations of massless particles depend only on conformal structure [9]. We present here a derivation of the Yamabe operator, the conformally invariant generalisation of the Laplacian. The derivation given below is naive and it ultimately comes as a surprise how little work we need to do in order to attain this conformally invariant generalisation. We will see in the next section how this operator may be attained more elegantly with Thomas'  $D$ -operator and the tractor calculus.

On a pseudo-Riemannian manifold of dimension  $n \geq 3$ , we define the Laplacian  $\Delta : \mathcal{E} \rightarrow \mathcal{E}$  to be the divergence of the function's gradient, that is,  $\Delta f = \nabla^a \nabla_a f$  where  $\nabla$  is the Levi-Civita connection. Under a conformal change  $\hat{g} = \Omega^2 g$  it is clear that the Laplacian is not conformally invariant. Remembering  $\hat{\nabla}_a f = \nabla_a f$ , the transformation law for the Levi-Civita connection on 1-forms (3.3.1) gives

$$\begin{aligned}
\hat{\Delta} f &= \Omega^{-2} g^{ab} \hat{\nabla}_b \nabla_a f \\
&= \Omega^{-2} g^{ab} (\nabla_b \nabla_a f - \Upsilon_b \nabla_a f - \Upsilon_a \nabla_b f + \Upsilon^c \nabla_c f g_{ab}) \\
&= \Omega^{-2} (\Delta f + (n-2)\Upsilon^a \nabla_a f).
\end{aligned}$$

Having previously developed the calculus for weight bundles, it is natural to investigate whether the preceding term involving  $\Upsilon^a \nabla_a f$  may be removed via considering the Laplacian acting on weighted functions  $\Delta : \mathcal{E}[w] \rightarrow \mathcal{E}[w']$ . Under a conformal change from  $g$  to  $\hat{g}$ , the function  $f$  becomes  $\hat{f} = \Omega^w f$  and similar to a previous calculation for weighted functions

$$\hat{\nabla}_a \hat{f} = \Omega^w (\nabla_a f + w \Upsilon_a f).$$

The Laplacian applied to the function in the scale determined by  $\hat{g}$  can now be directly

calculated.

$$\begin{aligned}
\hat{\Delta} \hat{f} &= \Omega^{-2} g^{ab} \hat{\nabla}_b (\Omega^w (\nabla_a f + w \Upsilon_a f)) \\
&= \Omega^{-2} g^{ab} (w \Omega^w \Upsilon_a (\nabla_a f + w \Upsilon_a f) + \Omega^w \hat{\nabla}_b (\nabla_a f + w \Upsilon_a f)) \\
&= \Omega^{w-2} (w \Upsilon^a \nabla_a f + w^2 \Upsilon^a \Upsilon_a f + g^{ab} (\nabla_b \nabla_a f - \Upsilon_b \nabla_a f - \Upsilon_a \nabla_b f + \Upsilon^c \nabla_c f g_{ab}) \\
&\quad + g^{ab} (w \Upsilon_a \nabla_b f + w f (\nabla_b \Upsilon_a - \Upsilon_b \Upsilon_a - \Upsilon_a \Upsilon_b + \Upsilon^c \Upsilon_c g_{ab}))) \\
&= \Omega^{w-2} (w \Upsilon^a \nabla_a f + w^2 \Upsilon^a \Upsilon_a f + \Delta f - \Upsilon^a \nabla_a f - \Upsilon^a \nabla_a f + n \Upsilon^a \nabla_a f \\
&\quad + w \Upsilon^a \nabla_a f + w f \nabla^a \Upsilon_a - 2w \Upsilon^a \Upsilon_a f + n w \Upsilon^a \Upsilon_a f) \\
&= \Omega^{w-2} (\Delta f + (2w + n - 2) \Upsilon^a \nabla_a f + (w + n - 2) w \Upsilon^a \Upsilon_a f + w (\nabla_a \Upsilon^a) f).
\end{aligned}$$

In order to remove the appearance of the first order differential operator  $\Upsilon^a \nabla_a f$  in the preceding display we are forced to set the weight  $w = 1 - \frac{n}{2}$ . Under this condition, the Laplacian appears as

$$\hat{\Delta} \hat{f} = \Omega^{w-2} (\Delta f - \frac{(n-2)^2}{4} \Upsilon^a \Upsilon_a f + (1 - \frac{n}{2}) \nabla^a \Upsilon_a f).$$

The conformal invariance is obtained after introducing a lower order curvature correction via the Ricci scalar (the Schouten scalar could also be used). If we introduce the operator  $\square = \Delta - \alpha R$  with  $\alpha = \frac{n-2}{4(n-1)}$  (which also acts on functions of weight  $w = 1 - \frac{n}{2}$ ) then recalling the transformation of the Ricci scalar (3.3.8), the conformal transformation appears as

$$\begin{aligned}
\hat{\square} \hat{f} &= \hat{\Delta} \hat{f} - \alpha \hat{R} \Omega^w f \\
&= \Omega^{w-2} (\Delta f + (\frac{(n-2)^2}{4} - \frac{(n-2)^2}{4}) \Upsilon^a \Upsilon_a f + (\frac{2-n}{2} - \frac{2-n}{2}) \nabla^a \Upsilon_a f - \alpha R f) \\
&= \Omega^{w-2} (\Delta f - \alpha R f) \\
&= \Omega^{w-2} \square f.
\end{aligned}$$

So this operator is conformally invariant if we consider it as an operator which produces functions of weight  $w - 2 = -1 - \frac{n}{2}$ . Specifically, the Yamabe operator is the conformally invariant differential operator

$$\square : \mathcal{E}[w] \rightarrow \mathcal{E}[w - 2] : f \mapsto (\Delta - \alpha R) f$$

where  $w = 1 - \frac{n}{2}$  and  $\alpha = \frac{n-2}{4(n-1)}$ . Using the Schouten scalar and the identification (2.2.4) it is clear that the Yamabe operator may be equivalently (and slightly more succinctly) defined by

$$\square : \mathcal{E}[w] \rightarrow \mathcal{E}[w - 2] : f \mapsto (\Delta + w P) f$$

where  $w = 1 - \frac{n}{2}$ . (This is the form in which it will appear via the tractor calculus).

It is surprising that such a construction produces a conformally invariant operator. Importantly, it leaves us with little insight into a general method for constructing conformally invariant operators from known operators on pseudo-Riemannian manifolds. As previously

mentioned, the construction of the Yamabe operator is more elegantly introduced using tractor calculus, a calculus which does provide a more general method for constructing conformally invariant operators [19].

Finally, a note on the history of the Yamabe operator. A consequence of the Riemann mapping theorem from complex analysis is that all surfaces are locally conformally flat however for  $n \geq 3$  this is clearly not the case (for example, conformal flatness is determined by the vanishing of the Weyl tensor in dimensions  $n \geq 4$ ). Considering conformal changes (which effectively give the freedom to choose one function), a natural question to ask is whether all Riemannian metrics admit conformally related metrics with constant curvature. This is the Yamabe problem: given a compact Riemannian manifold  $(M, g)$  of dimension  $n \geq 3$ , find a conformally related metric with constant scalar curvature. The problem (as well as its solution) is reviewed in [25]. Ultimately, a conformally related metric  $\hat{g} = \Omega^2 g$  has constant scalar curvature  $S$  if  $\Omega$  satisfies the Yamabe equation  $\square \Omega^{(n-2)/2} = \alpha S \Omega^2$  where  $\alpha = \frac{n-2}{4(n-1)}$ .



# Chapter 4

## Conformal Tractor Calculus

We follow the construction given in [5] to construct the conformal tractor bundle for a conformal manifold  $(M, c)$  of dimension at least 3. This requires basic knowledge of jets which can be found in [22]. In order to agree with the conventions of [1], we will associate the construction with the covariant tractor bundle  $\mathcal{E}_A$  and consider its dual  $\mathcal{E}^A$  as the contravariant tractor bundle. This decision allows the tractor connection to be attained naturally via the consideration of Einstein metrics. It is also notationally more attractive since the placement of lower and upper case Latin indices is consistent.

After introducing other elementary objects associated with the tractor bundle, we construct Thomas'  $D$ -operator which, as previously mentioned, offers the desired method for generalising the Yamabe operator. Finally we consider parallel tractors which play a central role in the construction of conserved quantities in the final chapter. (Although these standard tractors are closely related to Einstein scales and the construction of the tractor connection, it is appropriate to consider them after introducing the  $D$ -operator.)

Throughout this section the conformal metric will be used to raise and lower (lower-case) indices. So the Riemann tensor (for a metric  $g$ ) will be defined as before however  $R_{abcd}$  is defined as  $\mathbf{g}_{cc'}R_{ab}{}^{c'}{}_d \in \mathcal{E}_{abcd}[2]$ . Similarly other tensors, which previously required a metric to contract raise, lower, or contract indices, are now obtained by using the conformal metric and its inverse, in particular the Schouten scalar now has conformal weight  $-2$ .

### 4.1 Contravariant and Covariant Tractor Bundles

The 2-jet prolongation  $J^2(\mathcal{E}[1])$  of the density bundle  $\mathcal{E}[1]$  gives the following jet exact sequences (where the surjective maps are the natural projections)

$$0 \rightarrow \mathcal{E}_{(ab)}[1] \rightarrow J^2(\mathcal{E}[1]) \rightarrow J^1(\mathcal{E}[1]) \rightarrow 0, \quad (4.1.1)$$

$$0 \rightarrow \mathcal{E}_a[1] \rightarrow J^1(\mathcal{E}[1]) \rightarrow \mathcal{E}[1] \rightarrow 0. \quad (4.1.2)$$

These are easy to understand by considering the weightless analogues. The second is low enough order such that we have  $J^1\mathcal{E} \cong \mathcal{E} \oplus \mathcal{E}_a$  given by  $j^1f \mapsto (f, df)$  (and of course  $J^0\mathcal{E} \cong \mathcal{E}$ ). The first is a higher dimension analogue of this argument. It follows by realising

the subbundle of  $J^2\mathcal{E}$  with vanishing 1-jet (the kernel of the projection from  $J^2\mathcal{E}$  onto  $J^1\mathcal{E}$ ) is isomorphic to  $\mathcal{E}_{(ab)}$ . Indeed for  $f \in \mathcal{E}$  with  $j^1f = 0$ , its Hessian defines a symmetric tensor  $T_{ab} \in \mathcal{E}_{ab}$  which transforms correctly since  $df = 0$ , conversely such a symmetric tensor is always realised (locally) by a function with vanishing first derivative and with second derivatives agreeing with the tensor (for details in a more general setting, see section 12.10 of [22]). Tensoring through by  $\mathcal{E}[1]$  gives the two short exact sequences.

The conformal metric  $\mathbf{g}^{ab}$  naturally defines a trace map from  $\mathcal{E}_{(ab)}[w]$  onto  $\mathcal{E}[w-2]$ . For  $w = 1$ , the kernel of this map is the trace-free symmetric tensors  $\mathcal{E}_{(ab)_0}[1]$  and we get the short short exact sequence

$$0 \rightarrow \mathcal{E}_{(ab)_0}[1] \rightarrow \mathcal{E}_{(ab)}[1] \rightarrow \mathcal{E}[-1] \rightarrow 0. \quad (4.1.3)$$

The map  $\mathcal{E}[-1] \ni f \mapsto \frac{1}{n}f\mathbf{g}_{ab} \in \mathcal{E}_{(ab)}[1]$  splits this sequence giving the isomorphism

$$\mathcal{E}_{(ab)}[1] \cong \mathcal{E}[-1] \oplus \mathcal{E}_{(ab)_0}[1]. \quad (4.1.4)$$

Consequently  $\mathcal{E}_{(ab)_0}[1]$  sits inside  $J^2\mathcal{E}[1]$  via the maps in (4.1.1) and (4.1.3). The tractor bundle  $\mathcal{E}_A$  is defined to be the quotient of  $J^2(\mathcal{E}[1])$  by the image of  $\mathcal{E}_{(ab)_0}[1]$ . This gives the short exact sequence

$$0 \rightarrow \mathcal{E}_{(ab)_0}[1] \rightarrow J^2(\mathcal{E}[1]) \rightarrow \mathcal{E}_A \rightarrow 0$$

and as a consequence of (4.1.4) we also get

$$0 \rightarrow \mathcal{E}[-1] \rightarrow \mathcal{E}_A \rightarrow J^1(\mathcal{E}[1]) \rightarrow 0. \quad (4.1.5)$$

This construction states that the tractor bundle  $\mathcal{E}_A$  is the quotient of the 2-jet bundle  $J^2(\mathcal{E}[1])$  by the subbundle of all elements with vanishing 1-jet whose Hessian is trace-free.

This construction is clearly independent of a conformal scale hence the tractor bundle is conformally invariant. A concise way of summarising (4.1.2) and (4.1.5) is to use the semi-direct product notation (details in [3]) which allows us to write

$$\mathcal{E}_A = \mathcal{E}[1] \ltimes \mathcal{E}_a[1] \ltimes \mathcal{E}[-1]. \quad (4.1.6)$$

A benefit of this notation is that, after choosing a conformal scale  $g$ , the short exact sequences (4.1.2) and (4.1.5) split and the tractor bundle trivialises to

$$\mathcal{E}_A \stackrel{g}{=} \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]. \quad (4.1.7)$$

Another benefit is that a short exact sequence  $A = B \ltimes C$  in this notation gives (on taking duals)  $A^* = C^* \ltimes B^*$  so taking the dual of (4.1.6) gives the contravariant tractor bundle  $\mathcal{E}^A$

$$\mathcal{E}^A = \mathcal{E}[1] \ltimes \mathcal{E}^a[-1] \ltimes \mathcal{E}[-1].$$

As before, choosing a conformal scale  $g$  allows the tractor bundle to be written

$$\mathcal{E}^A \stackrel{g}{=} \mathcal{E}[1] \oplus \mathcal{E}^a[-1] \oplus \mathcal{E}[-1]. \quad (4.1.8)$$

In the scale determined by  $g$ , a tractor  $U^A$  (that is, a section of  $\mathcal{E}^A$ ) will be represented in vector notation by

$$U^A \stackrel{g}{=} \begin{pmatrix} \sigma \\ \mu^a \\ \rho \end{pmatrix} \in \begin{matrix} \mathcal{E}[1] \\ \oplus \\ \mathcal{E}^a[-1] \\ \oplus \\ \mathcal{E}[-1] \end{matrix} \stackrel{g}{=} \mathcal{E}^A.$$

To work with this bundle, we need to know how the vector notation of a tractor transforms between conformal scales. We can attain such a transformation law via the map from  $J^2(\mathcal{E}[1])$  to  $\mathcal{E}[1] \oplus \mathcal{E}^a[1] \oplus \mathcal{E}[-1]$  defined by

$$j^2 f \mapsto \begin{pmatrix} f \\ \nabla_a f \\ -\frac{1}{n}(\mathbf{g}^{ab}\nabla_a\nabla_b f + Pf) \end{pmatrix}$$

where the connection is the Levi-Civita connection determined by the conformal scale. This map induces a (metric dependent) isomorphism between  $\mathcal{E}_A$  and  $\mathcal{E}[1] \oplus \mathcal{E}^a[1] \oplus \mathcal{E}[-1]$ . After a short calculation (requiring the Schouten scalar transformation (3.3.7)), we attain

$$\begin{pmatrix} f \\ \hat{\nabla}_a f \\ -\frac{1}{n}(\mathbf{g}^{ab}\hat{\nabla}_a\hat{\nabla}_b f + \hat{P}f) \end{pmatrix} = \begin{pmatrix} f \\ \nabla_a f + \Upsilon_a f \\ -\frac{1}{n}(\mathbf{g}^{ab}\nabla_a\nabla_b f + Pf) - \Upsilon^a\nabla_a f - \frac{1}{2}\Upsilon_a\Upsilon^a f \end{pmatrix}. \quad (4.1.9)$$

This result directly gives the transformation law for covariant tractors however we seek the transformation law for contravariant tractors. Due to (4.1.7) and (4.1.8), there is an obvious isomorphism (in a particular conformal scale) between  $\mathcal{E}_A$  and  $\mathcal{E}^A$  via  $(\sigma, \mu_a, \rho) \mapsto (\sigma, \mu^a, \rho)$ . Postponing the discussion about the tractor metric until after the tractor connection has been introduced, we merely mention this identification so that the transformation law for contravariant tractors may be given. In two conformal scales  $g$  and  $\hat{g}$  (related by  $\hat{g} = \Omega^2 g$  as usual), if a tractor  $U^A \in \mathcal{E}^A$  is represented by  $(\sigma, \mu^a, \rho)$  and  $(\hat{\sigma}, \hat{\mu}^a, \hat{\rho})$  respectively then (4.1.9) gives the appropriate transformation law for tractors

$$\begin{pmatrix} \hat{\sigma} \\ \hat{\mu}^a \\ \hat{\rho} \end{pmatrix} = \begin{pmatrix} \sigma \\ \mu^a + \sigma\Upsilon^a \\ \rho - \mu^b\Upsilon_b - \frac{1}{2}\sigma\Upsilon_b\Upsilon^b \end{pmatrix}. \quad (4.1.10)$$

The previous display leads to two immediate remarks. First, the transformation law implies that the first non-zero entry in the vector representation  $(\sigma, \mu^a, \rho)$  is conformally invariant; this entry is referred to as the projecting part of the tractor. Second, the transformation law indicates another (and considerably more direct) way of defining the tractor

bundle (as done in [10]). Specifically, the tractor bundle may be defined as the quotient of the bundle  $\mathcal{Q} \times \mathcal{E}[1] \oplus \mathcal{E}^a[-1] \oplus \mathcal{E}[-1]$  by the equivalence relation  $(g, (\sigma, \mu^a, \rho)) \sim (\hat{g}, (\hat{\sigma}, \hat{\mu}^a, \hat{\rho}))$  if and only if  $(\sigma, \mu^a, \rho)$  is identified with its counterpart  $(\hat{\sigma}, \hat{\mu}^a, \hat{\rho})$  in the new scale according to (4.1.10).

## 4.2 Tractor Connection

The tractor connection is motivated by considering an equivalent condition for a metric to be conformally Einstein as done in [1]. A metric is called Einstein if  $R_{ab} = \lambda g_{ab}$  for some  $\lambda \in \mathbb{R}$  (so named because any Einstein metric on a 4-dimensional manifold with Lorentzian signature is a solution of the vacuum Einstein field equations with cosmological constant). This is equivalent to requiring that the Schouten tensor be proportional to the metric.

Since a positive conformal scale  $\xi \in \mathcal{E}[1]$  uniquely determines a metric in the conformal class by  $g_{ab} = \xi^{-2} \mathbf{g}_{ab}$ , we consider the conditions under which a new conformal scale  $\sigma \in \mathcal{E}[1]$  determines an Einstein metric. We will write  $\sigma = \Omega^{-1} \xi$  so that  $\hat{g} = \Omega^2 g$ . This is equivalent to saying, in the scale determined by  $\xi$ , the section  $\sigma$  is represented by the function  $\Omega^{-1}$ . Since  $\nabla_a \xi = 0$ ,

$$\begin{aligned} \nabla_a \nabla_b \sigma &= \nabla_a \nabla_b (\Omega^{-1} \xi) \\ &= \xi \nabla_a (-\Omega^{-1} \Upsilon_b) \\ &= \sigma (\Upsilon_a \Upsilon_b - \nabla_a \Upsilon_b). \end{aligned} \tag{4.2.1}$$

Recalling (3.3.6), the new metric is proportional to  $\hat{P}_{ab}$  if and only if

$$P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b$$

be pure trace. From (4.2.1), one sees this is equivalent to requiring the equation

$$\text{Trace-free part of } (\nabla_a \nabla_b + P_{ab}) \sigma = 0. \tag{4.2.2}$$

This conformally invariant equation is known as the conformal almost Einstein equation.

The tractor connection may now be attained by a procedure known as prolongation, specifically, by prolonging the conformal almost Einstein equation above to attain an equivalent first order closed system. Clearly (4.2.2) may be reformulated as requiring that there is some  $\rho \in \mathcal{E}[-1]$  such that

$$(\nabla_a \nabla_b + P_{ab}) \sigma = -\rho \mathbf{g}_{ab}.$$

Introducing  $\mu^a \in \mathcal{E}^a[-1]$  enables the previous equation to be written as the linear system

$$\begin{aligned} \nabla_a \sigma - \mu_a &= 0, \\ \nabla_a \mu^b + \sigma P_a{}^b + \rho \delta_a^b &= 0. \end{aligned}$$

In order to close the system we require an equation for  $\rho$ . Differentiating the second equation



in the system and skewing gives

$$R_{ab}{}^c{}_d \mu^d + 2\sigma \nabla_{[a} P_{b]}{}^c + 2\mu_{[a} P_{b]}{}^c + 2\nabla_{[a} \rho \delta_{b]}^c = 0$$

If we contract on the indices  $a$  and  $c$  in the above display and use the identity  $\nabla_a P_b{}^a = \nabla_b P$  (2.2.6) we attain

$$R_{ab} \mu^b + P_{ab} \mu^b - \mu_a P + (1-n) \nabla_a \rho = 0.$$

The necessary equation follows by substituting  $R_{ab} = (n-2)P_{ab} + P g_{ab}$  (2.2.5) to get

$$(1-n) \nabla_a \rho - (1-n) P_{ab} \mu^b = 0.$$

We conclude that the metric  $\sigma^{-2} \mathbf{g}_{ab}$  is Einstein if and only if there are sections  $\mu^a \in \mathcal{E}^a[-1]$  and  $\rho \in \mathcal{E}[-1]$  such that the following system holds

$$\begin{aligned} \nabla_a \sigma - \mu_a &= 0, \\ \nabla_a \mu^b + \sigma P_a{}^b + \rho \delta_a^b &= 0, \\ \nabla_a \rho - P_{ab} \mu^b &= 0. \end{aligned}$$

The preceding system is conformally invariant linear system with respect to the transformation law for tractors written in vector notation (4.1.10). (see [10] for a direct calculation). It therefore defines an invariant connection on  $\mathcal{E}^A$ . This is the tractor connection and, in the scale determined by  $g$ , the tractor connection  $\nabla_a$  on  $\mathcal{E}^A$  is defined by

$$\nabla_a \begin{pmatrix} \sigma \\ \mu^b \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu^b + \sigma P_a{}^b + \rho \delta_a^b \\ \nabla_a \rho - P_{ab} \mu^b \end{pmatrix}. \quad (4.2.3)$$

### 4.3 Tractor Metric

The tractor bundle carries a natural metric  $h_{AB}$ , the tractor metric, providing the isomorphism between  $\mathcal{E}_A$  and  $\mathcal{E}^A$  mentioned previously. In the scale determined by  $g_{ab}$  it is defined by

$$h_{AB} U^A V^B \stackrel{g}{=} \sigma \gamma + \mathbf{g}_{ab} \mu^a \beta^b + \rho \alpha$$

where

$$U^A \stackrel{g}{=} \begin{pmatrix} \sigma \\ \mu^a \\ \rho \end{pmatrix}, \quad V^B \stackrel{g}{=} \begin{pmatrix} \alpha \\ \beta^b \\ \gamma \end{pmatrix}.$$

The definition implies that if the conformal manifold has signature  $(p, q)$  then the tractor metric has signature  $(p+1, q+1)$ . Conformal invariance is simple to check. Specifically if

$U^A \stackrel{\hat{g}}{=} (\hat{\sigma}, \hat{\mu}^a, \hat{\rho})$  then (4.1.10) gives

$$\begin{aligned} h_{AB}U^AU^B &\stackrel{\hat{g}}{=} 2\sigma(\rho - \mu^b\Upsilon_b - \frac{1}{2}\sigma\Upsilon_b\Upsilon^b) + \mathbf{g}_{ab}(\mu^a + \sigma\Upsilon^a)(\mu^b + \sigma\Upsilon^b) \\ &= 2\sigma\rho + \mathbf{g}_{ab}\mu^a\mu^b \\ &\stackrel{g}{=} h_{AB}U^AU^B. \end{aligned}$$

This ensures, by polarisation, that the metric is well-defined.

The tractor metric is parallel with respect to the tractor connection. The following calculation of  $(\nabla_c h_{AB})U^AV^B$  confirms this. Using  $U^A$  and  $V^B$  as above

$$\begin{aligned} h_{AB}(\nabla_c U^A)V^B + h_{AB}U^A(\nabla_c V^B) &= (\gamma\nabla_c\sigma - \gamma\mu_c + \beta_a\nabla_c\mu^a + \sigma P_c^a\beta_a + \rho\beta_c + \alpha\nabla_c\rho - \alpha P_{ca}\mu^a) \\ &\quad + (\sigma\nabla_c\gamma - \sigma P_{ca}\beta^a + \mu_a\nabla_c\beta^a + \alpha P_c^a\mu_a + \gamma\mu_c + \rho\nabla_c\alpha - \rho\beta_c) \\ &= \nabla_c(\sigma\gamma + \mu^a\beta_a + \rho\alpha) \\ &= \nabla_c(h_{AB}U^AV^B) \end{aligned}$$

hence

$$(\nabla_c h_{AB})U^AV^B = \nabla_c(h_{AB}U^AV^B) - h_{AB}(\nabla_c U^A)V^B - h_{AB}U^A(\nabla_c V^B) = 0.$$

A useful consequence of  $\nabla_c h_{AB} = 0$  is that raising and lowering tractor indices commutes with the tractor connection. With knowledge of how the connection acts on contravariant tractors (4.2.3), it immediately gives the formula for the tractor connection operating on covariant tractors. Specifically if  $U_A \stackrel{g}{=} (\sigma, \mu_a, \rho) \in \mathcal{E}_A$  then

$$\nabla_a \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_a\sigma - \mu_a \\ \nabla_a\mu_b + \sigma P_{ab} + \rho\mathbf{g}_{ab} \\ \nabla_a\rho - P_{ab}\mu^b \end{pmatrix}.$$

## 4.4 Tractor Curvature

The deviation of a conformal manifold from the flat model is measured by the tractor curvature  $\Omega_{ab}^C{}_D$  of the normal tractor connection on  $\mathcal{E}^A$ . The curvature is defined by

$$\Omega_{ab}^C{}_D U^D = 2\nabla_{[a}\nabla_{b]}U^C.$$

In order to take the second derivative we have coupled the tractor connection to the Levi-Civita connection of the metric in which we are working (the result, importantly, remains conformally invariant). Let us work in the scale determined by some metric  $g$  in the conformal

class in order to calculate the action of the curvature on a tractor  $U^A \stackrel{g}{=} (\sigma, \mu^a, \rho)$ .

$$\begin{aligned}
\nabla_a \nabla_b U^C &= \nabla_a \begin{pmatrix} \nabla_b \sigma - \mu_b \\ \nabla_b \mu^c + \sigma P_b^c + \rho \delta_b^c \\ \nabla_b \rho - P_{bd} \mu^d \end{pmatrix} \\
&= \begin{pmatrix} \nabla_a (\nabla_b \sigma - \mu_b) - (\nabla_b \mu_a + \sigma P_{ba} + \rho \mathbf{g}_{ba}) \\ \nabla_a (\nabla_b \mu^c + \sigma P_b^c + \rho \delta_b^c) + (\nabla_b \sigma - \mu_b) P_a^c + (\nabla_b \rho - P_{bd} \mu^d) \delta_a^c \\ \nabla_a (\nabla_b \rho - P_{bd} \mu^d) - P_{ad} (\nabla_b \mu^d + \sigma P_b^d + \rho \delta_b^d) \end{pmatrix} \\
&= \begin{pmatrix} \nabla_a \nabla_b \sigma - 2 \nabla_{(a} \mu_{b)} + \sigma P_{ab} - \rho \mathbf{g}_{ab} \\ \nabla_a \nabla_b \mu^c - (P_a^c \mathbf{g}_{bd} - \delta_a^c P_{bd}) \mu^d + \sigma \nabla_a P_b^c + 2 \delta_{(a}^c \nabla_{b)} \rho + 2 P_{(a}^c \nabla_{b)} \sigma \\ - \mu^d \nabla_a P_{bd} + \nabla_a \nabla_b \rho - P_{(a}^d \nabla_{b)} \mu_d - \sigma P_{ad} P_b^d - \rho P_{ab} \end{pmatrix}
\end{aligned}$$

After skewing over the indices  $a$  and  $b$  the symmetric terms (on the right) vanish. Recalling the decomposition of the Riemann tensor (2.2.3) it is easy to check

$$C_{ab}{}^c{}_d = R_{ab}{}^c{}_d - 2P_{[a}{}^c \mathbf{g}_{b]d} - 2\delta_{[a}^c P_{b]d}$$

so that

$$2\nabla_{[a} \nabla_{b]} U^C = \begin{pmatrix} 0 \\ C_{ab}{}^c{}_d \mu^d + 2\sigma \nabla_{[a} P_{b]}^c \\ -2\mu_d \nabla_{[a} P_{b]d} \end{pmatrix}.$$

Therefore the action of the curvature  $\Omega_{ab}{}^C{}_D$  on a tractor  $U^A \stackrel{g}{=} (\sigma, \mu^a, \rho)$  is given by

$$\Omega_{ab}{}^C{}_D U^D \stackrel{g}{=} \begin{pmatrix} 0 & 0 & 0 \\ 2\nabla_{[a} P_{b]}^c & C_{ab}{}^c{}_d & 0 \\ 0 & -2\nabla_{[a} P_{b]d} & 0 \end{pmatrix} \begin{pmatrix} \sigma \\ \mu^d \\ \rho \end{pmatrix}.$$

This formula implies [1] that the connection is flat if and only if the Weyl tensor vanishes if  $n \geq 4$  or  $2\nabla_{[a} P_{b]c} = 0$  for  $n = 3$  which are exactly the necessary and sufficient conditions for a conformal manifold to be locally equivalent to the flat model.

The curvature also possesses the familiar symmetry  $\Omega_{abCD} = \Omega_{ab[CD]}$  of the Riemann tensor. This is easily seen by realising  $h_{CD} U^C V^D \in \mathcal{E}$  hence  $2\nabla_{[a} \nabla_{b]} h_{CD} U^C V^D = 0$  so the Leibniz rule implies

$$\begin{aligned}
-2(\nabla_{[a} \nabla_{b]} h_{CD}) U^C V^D &= h_{CD} V^D \Omega_{ab}{}^C{}_E U^E + h_{CD} U^C \Omega_{ab}{}^D{}_E V^E \\
&= 2\Omega_{ab(CD)} U^C V^D
\end{aligned}$$

but the tractor metric is preserved by the tractor connection so  $0 = \Omega_{ab(CD)} U^C V^D$ . Therefore  $\Omega_{abCD} = \Omega_{ab[CD]}$ .

## 4.5 $X, Y, Z$ -Calculus

The  $X, Y, Z$ -calculus [19] provides an alternative method to the vector notation for performing tractor calculations. In the scale determined by  $g$  the sections  $X^A \in \mathcal{E}^A[1]$ ,  $Y^A \in \mathcal{E}^A[-1]$ , and  $Z_a^A \in \mathcal{E}_a^A[1]$  are defined by

$$Y^A \stackrel{g}{=} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad Z_b^A \stackrel{g}{=} \begin{pmatrix} 0 \\ \delta_b^a \\ 0 \end{pmatrix}, \quad X^A \stackrel{g}{=} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The weights are chosen so that these tractors act as injections into the tractor bundle. We may write  $U^A = Y^A \sigma + Z_a^A \mu^a + X^A \rho$  for  $\sigma \in \mathcal{E}[1]$ , and  $\mu^a \in \mathcal{E}^a[-1]$ , and  $\rho \in \mathcal{E}[-1]$ .

If a new scale  $\hat{g}$  is chosen and the sections  $\hat{X}^A \in \mathcal{E}^A[1]$ ,  $\hat{Y}^A \in \mathcal{E}^A[-1]$ , and  $\hat{Z}_a^A \in \mathcal{E}_a^A[1]$  are similarly defined by

$$\hat{Y}^A \stackrel{\hat{g}}{=} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{Z}_b^A \stackrel{\hat{g}}{=} \begin{pmatrix} 0 \\ \delta_b^a \\ 0 \end{pmatrix}, \quad \hat{X}^A \stackrel{\hat{g}}{=} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

then we may write the tractor from above as  $U^A = \hat{Y}^A \hat{\sigma} + \hat{Z}_a^A \hat{\mu}^a + \hat{X}^A \hat{\rho}$ . The transformation law (4.1.10) implies

$$\begin{pmatrix} Y^A & Z_a^A & X^A \end{pmatrix} \begin{pmatrix} \sigma \\ \mu^a \\ \rho \end{pmatrix} = \begin{pmatrix} \hat{Y}^A & \hat{Z}_a^A & \hat{X}^A \end{pmatrix} \begin{pmatrix} \sigma \\ \mu^a + \sigma \Upsilon^a \\ \rho - \mu^b \Upsilon_b - \frac{1}{2} \sigma \Upsilon_b \Upsilon^b \end{pmatrix}$$

from which it follows  $X^A = \hat{X}^A$  is invariant and

$$\begin{aligned} Y^A &= \hat{Y}^A + \Upsilon^a \hat{Z}_a^A - \frac{1}{2} \Upsilon_a \Upsilon^a X^A, \\ Z_a^A &= \hat{Z}_a^A - \Upsilon_a X^A. \end{aligned}$$

These are trivially inverted to give

$$\begin{aligned} \hat{Y}^A &= Y^A - \Upsilon^a Z_a^A - \frac{1}{2} \Upsilon_a \Upsilon^a X^A, \\ \hat{Z}_a^A &= Z_a^A + \Upsilon_a X^A. \end{aligned}$$

It is no surprise that  $X^A$  is invariant as it provides the canonical injection  $X^A : \mathcal{E}[-1] \hookrightarrow \mathcal{E}^A$  given by  $\rho \mapsto (0, 0, \rho) \in \mathcal{E}^A$  with  $\rho$  as the projecting (hence conformally invariant) part of the tractor. It also provides the surjection  $X_A : \mathcal{E}^A \rightarrow \mathcal{E}[1]$  given by  $U^A \mapsto X_A U^A$ . The importance of this invariance will become clear upon consideration of Thomas'  $D$ -operator.

We define  $X_A \in \mathcal{E}_A[1]$ ,  $Y_A \in \mathcal{E}_A[-1]$ , and  $Z_{aA} \in \mathcal{E}_{aA}[1]$  by lowering tractor indices with the tractor metric  $h_{AB}$ . We also define  $Z^{aA} \in \mathcal{E}^{aA}[-1]$  and  $Z_A^a \in \mathcal{E}_A^a[-1]$  by raising tensor indices with the conformal metric  $g_{ab}$ . The transformation laws do not change from above since both metrics are conformally invariant. Finally the tractor indices of these sections

may be contracted to obtain

$$\begin{array}{c|ccc} & Y^A & Z_a^A & X^A \\ \hline Y_A & 0 & 0 & 1 \\ Z_{bA} & 0 & \mathbf{g}_{ab} & 0 \\ X_A & 1 & 0 & 0 \end{array}$$

and the tractor metric may be decomposed into a sum of projections

$$h_{AB} = Y_A X_B + Z_{aA} Z_B^a + X_A Y_B. \quad (4.5.1)$$

It is also easy to see how the tractor connection acts on these weighted tractors. Simply applying the Leibniz rule to  $\nabla_a(Y^B \sigma + Z_b^B \mu^b + X^B \rho)$  (where the connection is the coupled connection of the Levi-Civita connection and the tractor connection) gives

$$Y^B \nabla_a \sigma + Z_b^B \nabla_a \mu^b + X^B \nabla_a \rho + \sigma \nabla_a Y^B + \mu^b \nabla_a Z_b^B + \rho \nabla_a X^B$$

Comparing with (4.2.3) implies the action of the connection on these weighted tractors must be

$$\begin{aligned} \nabla_a Y^B &= Z_b^B P_a^b, \\ \nabla_a Z_b^B &= -Y^B \mathbf{g}_{ab} - X^B P_{ab}, \\ \nabla_a X^B &= Z_a^B. \end{aligned}$$

## 4.6 Thomas' $D$ -operator

Thomas'  $D$ -operator is a conformally invariant second-order differential operator. It was originally presented in [30] as a generalisation of the Levi-Civita connection for conformal structures. Although the tractor connection is conformally invariant and possesses many of the features of the Levi-Civita connection on Riemannian manifolds, it does not offer a way to take further derivatives in a conformally invariant manner (it is also only conformally invariant on unweighted tractors). The  $D$ -operator is superior in this respect. It is invariant on weighted tractors and produces a weighted tractor on which one may subsequently reapply the  $D$ -operator (however it does not satisfy a Leibniz rule).

Following [16] we construct the  $D$ -operator in an invariant manner. In a conformal scale  $g$ , define the first-order differential operator

$$\tilde{D}_A : \mathcal{E}^*[w] \rightarrow \mathcal{E}_A \otimes \mathcal{E}^*[w-1] : f \mapsto \tilde{D}f \stackrel{g}{=} \begin{pmatrix} wf \\ \nabla_a f \\ 0 \end{pmatrix}$$

where  $\mathcal{E}^*$  is notation for an arbitrary tractor bundle. This operator is not invariant. Since  $f$  has non-zero weight, we see that under a conformal rescaling  $\hat{\nabla}_a f = \nabla_a f + w \Upsilon_a f$ . In the

scale determined by  $\hat{g}$ , let us define  $\hat{\tilde{D}}_A$  in the natural way

$$\hat{\tilde{D}}_A f \stackrel{\hat{g}}{=} \begin{pmatrix} wf \\ \hat{\nabla}_a f \\ 0 \end{pmatrix} = \begin{pmatrix} wf \\ \nabla_a f + w\Upsilon_a f \\ 0 \end{pmatrix}$$

and also consider the transformation of  $\tilde{D}_A f$ ,

$$\tilde{D}_A f \stackrel{\hat{g}}{=} \begin{pmatrix} wf \\ \nabla_a f + w\Upsilon_a f \\ -\Upsilon_b \nabla^b f - \frac{w}{2} \Upsilon_b \Upsilon^b f \end{pmatrix}.$$

Then

$$\tilde{D}_A f = \hat{\tilde{D}}_A f - X_A (\Upsilon_b \nabla^b f + \frac{w}{2} \Upsilon_b \Upsilon^b f)$$

so the operator is indeed not conformally invariant.

The preceding relation does however immediately imply that the Double- $D$ -operator

$$D_{AB} : \mathcal{E}^*[w] \rightarrow \mathcal{E}_{AB} \otimes \mathcal{E}^*[w] : f \mapsto 2X_{[B} \tilde{D}_{A]} f$$

is conformally invariant. It easy to see that  $\tilde{D}_A$  satisfies a Liebnez property  $\tilde{D}_A(f_1 f_2) = f_1 \tilde{D}_A f_2 + f_2 \tilde{D}_A f_1$  for weighted tractors  $f_1, f_2$  (with respective weights  $w_1, w_2$ ) since  $(w_1 + w_2) f_1 f_2 = (w_1 f_1) f_2 + f_1 (w_2 f_2)$ . Observe that this implies the Double- $D$ -operator also satisfies a Liebnez property.

**Lemma 4.1.** *The operator  $\tilde{D}_A$  defined in a scale  $g_{ab}$  commutes with the tractor metric  $h_{AB}$ . Consequently the tractor metric also commutes with the Double- $D$ -operator.*

*Proof.* Consider (unweighted) tractors  $U^A \stackrel{g}{=} (\sigma, \mu^a, \rho)$  and  $V^B \stackrel{g}{=} (\alpha, \beta^a, \gamma)$ . Using the  $X, Y, Z$ -calculus developed above, we calculate  $\tilde{D}_C U^A$

$$\begin{aligned} \tilde{D}_C U^A &= Z_C^c \nabla_c (Y^A \sigma + Z_a^A \mu^a + X^A \rho) \\ &= Z_C^c (Y^A \nabla_c \sigma + \sigma Z_a^A P_c^a + Z_a^A \nabla_c \mu^a - \mu^a (Y^A \mathbf{g}_{ca} + X^A P_{ca}) + X^A \nabla_c \rho + \rho Z_C^A) \end{aligned}$$

and contracting this with  $V^B$  gives

$$h_{AB} (\tilde{D}_C U^A) V^B = Z_C^c (\gamma \nabla_c \sigma - \gamma \mu_c + \sigma P_c^a \beta_a + \beta_a \nabla_c \mu^a + \rho \beta_c - \alpha P_{ca} \mu^a + \alpha \nabla_c \rho).$$

Adding to this the corresponding expression for  $h_{AB} U^A (\tilde{D}_C V^B)$  cancels four of the six terms above and gives

$$\begin{aligned} h_{AB} (\tilde{D}_C U^A) V^B + h_{AB} U^A (\tilde{D}_C V^B) &= (\nabla_c \sigma \gamma + \nabla_c \mathbf{g}_{ab} \mu^a \beta^b + \nabla_c \rho \alpha) Z_C^c \\ &= \tilde{D}_C (h_{AB} U^A V^B) \end{aligned}$$

from which the first result  $(\tilde{D}_C h_{AB}) U^A V^B = 0$  follows. The second result is now immediate from the first result and the definition of  $D_{AB}$ .  $\square$

We will ultimately associate the  $D$ -operator with (a part of) the symmetric trace-free component of  $h^{AB}D_{AQ}D_{BP}f$  where  $f \in \mathcal{E}^*[w]$ . In order to examine the preceding operator, assume we are working in some scale  $g$  with associated operator  $\tilde{D}_A$ . Some preliminary results will enable a simple calculation of this operator. Recalling (4.5.1), and remembering that  $X^A$  has weight 1,

$$\begin{aligned}\tilde{D}_A X_B &= Y_A X_B + Z_A^a \nabla_a X_B \\ &= Y_A X_B + Z_A^a Z_{aB} \\ &= h_{AB} - X_A Y_B.\end{aligned}\tag{4.6.1}$$

A consequence of the preceding result is

$$\begin{aligned}D_{AB}X_C &= X_B \tilde{D}_A X_C - X_A \tilde{D}_B X_C \\ &= -2X_{[A}g_{B]C} + 2X_{[A}X_{B]}Y_C \\ &= -2X_{[A}g_{B]C}.\end{aligned}\tag{4.6.2}$$

Finally  $X^B \tilde{D}_Q \tilde{D}_B f$  will appear in the calculation and is dealt with in the following way (using (4.6.1) and  $Y^B \tilde{D}_B f = 0$ )

$$\begin{aligned}X^B \tilde{D}_Q \tilde{D}_B f &= \tilde{D}_Q X^B \tilde{D}_B f - (\tilde{D}_Q X^B) \tilde{D}_B f \\ &= w \tilde{D}_Q f - (\delta_Q^B - Y^B X_A) \tilde{D}_B f \\ &= (w - 1) \tilde{D}_Q f.\end{aligned}\tag{4.6.3}$$

We may now investigate  $g^{AB}D_{AQ}D_{BP}f$ . First, consider the first term of  $D_{AQ}(X_P \tilde{D}_B - X_B \tilde{D}_P)f$ ; the Leibniz rule and (4.6.2) give

$$\begin{aligned}D_{AQ}X_P \tilde{D}_B f &= (D_{AQ}X_P) \tilde{D}_B f + X_P D_{AQ} \tilde{D}_B f \\ &= -2X_{[A}h_{Q]P} \tilde{D}_B f + X_P D_{AQ} \tilde{D}_B f.\end{aligned}$$

The second term  $-D_{AQ}X_B \tilde{D}_P f$  is handled similarly. Combining these results and then using the definition of  $D_{AQ}$  gives

$$\begin{aligned}D_{AQ}D_{BP}f &= -2X_{[A}h_{Q]P} \tilde{D}_B f + X_P D_{AQ} \tilde{D}_B f + 2X_{[A}h_{Q]B} \tilde{D}_P f - X_B D_{AQ} \tilde{D}_P f \\ &= -2X_{[A}h_{Q]P} \tilde{D}_B f + 2X_P X_{[Q} \tilde{D}_A] \tilde{D}_B f + 2X_{[A}h_{Q]B} \tilde{D}_P f - 2X_B X_{[Q} \tilde{D}_A] \tilde{D}_P f.\end{aligned}$$

Contracting with  $h^{AB}$  gives (after collecting terms)

$$\begin{aligned}h^{AB}D_{AQ}D_{BP}f &= (2 - h^{AB}h_{AB})X_Q \tilde{D}_P f - h_{QP}X^B \tilde{D}_B f + X_P X_Q \tilde{D}^B \tilde{D}_B f \\ &\quad - X_P X^B \tilde{D}_Q \tilde{D}_B f - X_Q X^A \tilde{D}_A \tilde{D}_P f + X^A X_A \tilde{D}_Q \tilde{D}_P f \\ &= (1 - w - n)X_Q \tilde{D}_P f + (1 - w)X_P \tilde{D}_Q f + X_P X_Q \tilde{D}^B \tilde{D}_B f - wh_{QP}f\end{aligned}$$

where the second equality uses (4.6.3) (and  $h_{AB}h^{AB} = n + 2$ ). The symmetric part of this

operator is therefore

$$h^{AB}D_{A(Q}D_{|B|P)}f = (2 - 2w - n)X_{(Q}\tilde{D}_{P)}f + X_QX_P\tilde{D}^B\tilde{D}_Bf - wh_{QP}f. \quad (4.6.4)$$

Passing to the trace-free part of this implies the existence of a conformally invariant differential operator satisfying

$$h^{AB}D_{A(Q}D_{|B|P)_0}f = -X_{(Q}D_{P)_0}f. \quad (4.6.5)$$

An explicit formula for this operator is immediate from (4.6.4).

**Definition 4.2.** *The (Thomas)  $D$ -operator is the conformally invariant differential operator acting on weighted tensor bundles  $D_A : \mathcal{E}^*[w] \rightarrow \mathcal{E}_A \otimes \mathcal{E}^*[w - 1]$  given by (4.6.5). Explicitly, in a scale with associated operator  $\tilde{D}_A$ ,*

$$D_Af = (n + 2w - 2)\tilde{D}_Af - X_A\Box f$$

where  $\Box : \mathcal{E}^*[w] \mapsto \mathcal{E}^*[w - 2]$  is the box operator given by

$$\Box f = \tilde{D}^B\tilde{D}_Bf = \nabla^b\nabla_bf + wPf.$$

The box-operator is, in general, not invariant. However, immediately from the definition of the  $D$ -operator, if  $n + 2w - 2 = 0$  (hence  $w = 1 - \frac{n}{2}$ ) then  $D_Af = X_A\Box f$  so  $\Box f$  is the projecting (hence conformally invariant) part of the tractor  $D_Af$ . This clearly recovers the Yamabe operator as promised. Importantly, the  $D$ -operator enables us to generalise the Yamabe operator. The box operator is the conformally invariant differential operator acting on sections of tractor bundles with weight  $1 - \frac{n}{2}$  which generalises the Yamabe operator.

## 4.7 Parallel Tractors

The conserved quantities presented in the following chapter require parallel tractors to be present. It is thus appropriate to discuss some basic properties of these tractors. Suppose  $I_A \in \mathcal{E}_A$  is a parallel tractor. If  $I_A \stackrel{g}{=} (\sigma, \mu_a, \rho)$  for some scale  $g \in [g]$  then the condition  $\nabla_a I_B = 0$ , that is,

$$\begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + \sigma P_{ab} + \rho g_{ab} \\ \nabla_a \rho - P_{ab} \mu^b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (4.7.1)$$

implies immediately that

$$I_A \stackrel{g}{=} \begin{pmatrix} \sigma \\ \nabla_a \sigma \\ \frac{-1}{n}(\Delta + P)\sigma \end{pmatrix}.$$



Recalling the formula for the  $D$ -operator, we see that  $I_A = \frac{1}{n}D_A\sigma$  where  $\sigma = X^A I_A$ . Moreover, since  $\sigma \in \mathcal{E}[1]$  we may (supposing the zero set of  $\sigma$  is empty) investigate the metric  $\sigma^{-2}\mathbf{g}_{ab}$ . The equation,  $Z_b^B \nabla_a I_B = \nabla_a \mu_b + \sigma P_{ab} + \rho \mathbf{g}_{ab} = 0$ , contained in (4.7.1) is precisely the information that implies  $\sigma$  solves the conformal Einstein equation (4.2.2). We will refer to  $\sigma$  as an Einstein scale. In [17], emphasis is placed on the possible structure of the zero set  $X_A I^A$  for an arbitrary parallel tractor, however, for this dissertation, the symmetric divergence-free tensors will be defined assuming the existence of an Einstein scale. Summarising we have the following.

**Proposition 4.3.** *A conformal manifold  $(M, c)$  (of dimension  $n \geq 3$ ) contains an Einstein metric in the conformal class if and only if the tractor bundle admits a parallel section  $I_A$  with  $\sigma = X^A I_A$  nowhere vanishing.*



# Chapter 5

## Conserved Quantities

The final chapter in this dissertation investigates a novel application of the tractor calculus in dimension 4. We begin by introducing a gauge for the electromagnetic 1-form which is conformally invariant. Using this gauge we construct an electromagnetic tractor  $\Phi_A \in \mathcal{E}_A[-1]$  which satisfies the equation  $\square\Phi_A = 0$ . This equation is precisely a tractor generalisation of the field equation  $\square\psi = 0$  where  $\psi$  is a massless scalar field in the scale of conformal coupling. The initial suggestion is that this observation may lead to a construction of symmetric divergence-free (rank 2) tensors which have conserved quantities that are related to an electromagnetic field. There are two constructions which produce divergence-free tensors. The first construction works in conformally flat settings while the second works in conformally Einstein settings.

An extension of this observation is to see, parallel to the ideas of the curved translation principle, whether such a construction works in reverse. In particular we investigate the energy-momentum tensor of an electromagnetic field. As before, two constructions are possible however in this case, an interpretation of the corresponding tractor is not available.

Throughout this chapter we will assume  $(M, c)$  is a conformal manifold of dimension 4. Moreover, we will assume that  $c$  is an equivalence class of pseudo-Riemannian metrics  $g \in c$  where each metric has Lorentzian signature. Any manifold  $(M, g)$  is also assumed to be 4-dimensional and  $g$  is assumed to have Lorentzian signature. Finally we say that  $(M, g)$  descends to  $(M, c)$  if  $g \in c$ .

### 5.1 A Conformally Invariant Maxwell Gauge

In the classical theory of electromagnetism, the Maxwell equations are conformally invariant [7, 9], however the commonly chosen Lorenz gauge condition does not possess this invariance. Using ideas related to the twistor theory of Penrose, a gauge (which is weaker than the Lorenz gauge) is given in [12] however conformal invariance of the gauge equation is conditional upon the Maxwell equations being satisfied. This gauge is attained in [3] via tractor calculus. This method is self-contained and introduces a conformally invariant splitting operator for the canonical bundle projection  $P_{\bar{A}}^A : \mathcal{E}_A[w] \rightarrow \mathcal{E}_{\bar{A}}[w]$ . Here  $\mathcal{E}_{\bar{A}}$  denotes the 1-jet prolongation of  $J^1(\mathcal{E}[1])$  (4.1.2) so we have the composition series  $\mathcal{E}_{\bar{A}} = \mathcal{E}[1] \oplus \mathcal{E}_a[1]$  (this is just the quotient

of the tractor bundle under the image of  $X_A$ ). The presentation given below will not follow this construction, we will simply act the middle operator on the electromagnetic potential to achieve the same tractor.

On a 4-dimensional manifold with Lorentzian metric, the (vacuum) Maxwell equations on a 2-form  $\omega$  are  $d\omega = 0$  and  $\delta\omega = 0$  where  $\delta$  is the formal adjoint of the exterior derivative. (It is these equations which are classically known to be conformally invariant.) Provided the topology of the manifold is not involved, the Poincaré lemma and the equation  $d\omega = 0$  imply  $\omega = d\varphi$  for a 1-form  $\varphi$  and so the Maxwell equations reduce to  $\delta d\varphi = 0$ . In abstract index notation, the Maxwell equations

$$\begin{aligned} 3\nabla_{[c}\omega_{ab]} &= 0, \\ \nabla^a\omega_{ab} &= 0, \end{aligned}$$

for  $\omega_{ab} \in \mathcal{E}_{[ab]}$  reduce to

$$2\nabla^b\nabla_{[b}\varphi_a] = 0 \tag{5.1.1}$$

for  $\varphi_a \in \mathcal{E}_a$ . Considering Lorenz condition  $\nabla^a\varphi_a = 0$  and the transformation formula (3.3.1), we get

$$\hat{\nabla}^a\varphi_a = \nabla^a\varphi_a + (n + w - 2)\Upsilon^a\varphi_a.$$

So the gauge condition is invariant on 1-forms with weight  $-2$  however the Maxwell potential has weight 0 hence the condition is not invariant.

We will now construct the tractor of [3] which encodes both the electromagnetic 1-form and the Eastwood-Singer gauge (and we will subsequently refer to this as the electromagnetic tractor). In order to do this, we will make use of the Middle-operator [31]. This is the first order operator  $M_A^a : \mathcal{E}_a[w] \rightarrow \mathcal{E}_A[w - 1]$  defined by

$$M_A^a\mu_a \stackrel{g}{=} \begin{pmatrix} 0 \\ (n + w - 2)\mu_a \\ -\nabla^b\mu_b \end{pmatrix}$$

where  $n$  is the dimension of the conformal manifold.

**Definition 5.1.** *Let  $\varphi_a \in \mathcal{E}_a$  be a 1-form defined on a conformal manifold of dimension 4 which satisfies the Maxwell equations (5.1.1). Then the electromagnetic tractor (associated with  $\varphi_a$ ) is the tractor  $\Phi_A \in \mathcal{E}_A[-1]$  defined by  $\Phi_A = \frac{1}{2}M_A^a\varphi_a$ . Specifically,*

$$\Phi_A \stackrel{g}{=} \begin{pmatrix} 0 \\ \varphi_a \\ -\frac{1}{2}\nabla^b\varphi_b \end{pmatrix}.$$

Given  $\Phi_A$  has weight  $-1$  (on a manifold of dimension 4) we may invariantly act on it by

the box-operator. The result is

$$\square\Phi_A \stackrel{g}{=} (\Delta - P)\Phi_A = \begin{pmatrix} 0 \\ \nabla^b \nabla_{[b} \varphi_{a]} \\ -\frac{1}{4} \nabla_b (\nabla^b \nabla^c + 4P^{bc} - 2P\mathbf{g}^{bc}) \varphi_c \end{pmatrix}.$$

The Maxwell equations appear in the projecting slot. Supposing  $\varphi_a$  solves these equations, the bottom slot becomes conformally invariant. This is the Eastwood-Singer gauge of [12]. The constructions given below require  $\Phi_A$  solves the equation  $\square\Phi_A = 0$  so for the rest of this dissertation, we will assume this gauge has been chosen.

## 5.2 Lagrangian Formulation

This section states some necessary results from the Lagrangian formulation of field theory. For a more complete discussion the reader is referred to [21, 26]. Our discussion will detail the formulations for a massless scalar field  $\psi \in \mathcal{E}$  and an electromagnetic potential  $\varphi_a \in \mathcal{E}_a$ . (More precisely we should consider  $\psi$  as merely a representative of a section of  $\mathcal{E}[-1]$  however the mathematics is considerably lighter if we avoid dealing with the weight bundle initially.)

Consider a Lagrangian density  $\mathcal{L}$ . This is a scalar function which is dependent on a field  $\Psi^{a\dots b}_{c\dots d}$ , its first covariant derivative  $\nabla_e \Psi^{a\dots b}_{c\dots d}$ , and the metric  $g_{ab}$ . (Higher derivatives may be included [18], however we will not concern ourselves with this generalisation.) The field equations are then obtained by requiring that the action

$$S = \int_D \mathcal{L} dv$$

be stationary under variations of the field in the interior of a compact 4-dimensional region  $D$ . As remarked in 3.2, the volume form  $dv$  carries weight 4 in the conformal setting. This does not pose any problems except that, in such a case, the Lagrangian density must carry weight  $-4$ . The Lagrangian densities given below are not written in a conformal manifold setting however it is clear that, if the objects (metric, curvature quantities, and fields themselves) which are used to construct the Lagrangian density are interpreted as their weighted analogues, then the Lagrangian density does indeed have weight  $-4$ .

Minimising the actions over a variation of the fields leads to the equivalent condition that the Euler-Lagrange equations (that is, the field equations)

$$\frac{\partial \mathcal{L}}{\partial \Psi^{a\dots b}_{c\dots d}} - \nabla_e \left( \frac{\partial \mathcal{L}}{\partial \nabla_e \Psi^{a\dots b}_{c\dots d}} \right) = 0$$

hold (with the above notation denoting functional derivatives).

The energy-momentum tensor is obtained similarly, however now the action is required to be stationary under a variation of the metric. The rank 2 tensor constructed in this manner is symmetric and divergence-free (conditioned upon the field satisfying the field equations).

## Scalar Field

Commonly the Lagrangian density for a massless scalar field  $\psi \in \mathcal{E}$  is taken to be

$$\mathcal{L} = -\frac{1}{2}g^{ab}\nabla_a\psi \cdot \nabla_b\psi \quad (5.2.1)$$

from which the Euler-Lagrange equation is simply  $\Delta\psi = 0$  and the energy-momentum tensor is

$$T_{ab} = \nabla_a\psi \cdot \nabla_b\psi - \frac{1}{2}g_{ab}\nabla_c\psi \cdot \nabla^c\psi. \quad (5.2.2)$$

The constructions for conserved quantities given in the following section will not work for this tensor (in particular the tensor will not be divergence-free). This is because the action for this Lagrangian density is not conformally invariant. It is, in fact, the Minkowski space version (where  $P = 0$ ) of a Lagrangian density which produces a conformally invariant action. Within Quantum Field Theory in curved settings this is known as the scale of minimal coupling. It is easy to see that the associated action is not conformally invariant and neither is the resulting field equation. The more natural Lagrangian density (which is required in Quantum Field Theory for the theory to be renormalisable) is [4, 20]

$$\mathcal{L} = -\frac{1}{2}(g^{ab}\nabla_a\psi \cdot \nabla_b\psi + P\psi^2) \quad (5.2.3)$$

from which the Euler-Lagrange equation is now conformally invariant

$$\Delta\psi - P\psi = 0.$$

The calculation of the energy-momentum tensor from (5.2.3) is considerably more involved than in the first case due to the coupling of the curvature  $P$  (which is dependent on the metric) with the field  $\psi^2$ . The result, which is calculated in detail in [26], is

$$T_{ab} = \nabla_a\psi \cdot \nabla_b\psi - \frac{1}{2}g_{ab}\nabla_c\psi \cdot \nabla^c\psi + \frac{1}{3}(P_{ab} - P g_{ab})\psi^2 + \frac{1}{6}(g_{ab}\Delta\psi^2 - \nabla_a\nabla_b\psi^2). \quad (5.2.4)$$

Investigating the Lagrangian density, it is clear why this field equation is conformally invariant. Specifically, if we consider its natural extension to a conformal setting where  $\psi$  now takes on the role of a section of  $\mathcal{E}[-1]$  then, under a conformal rescaling to  $\hat{g} = \Omega^2g$ , the Lagrangian density becomes (3.3.2), (3.3.7)

$$\begin{aligned} -2\hat{\mathcal{L}} &= \mathbf{g}^{ab}(\hat{\nabla}_a\psi \cdot \hat{\nabla}_b\psi + \hat{P}\psi^2) \\ &= \mathbf{g}^{ab}((\nabla_a\psi - \Upsilon_a\psi)(\nabla_b\psi - \Upsilon_b\psi) + (P - \nabla_a\Upsilon^a - \Upsilon^2)\psi^2) \\ &= -2\mathcal{L} + \Upsilon^2\psi - 2\Upsilon^a\psi\nabla_a\psi - \nabla_a\Upsilon^a \cdot \psi^2 - \Upsilon^2\psi^2 \\ &= -2\mathcal{L} - \nabla_a(\Upsilon^a\psi^2). \end{aligned}$$

So the difference between the Lagrangians is a divergence term, hence does not influence the field equation (by Stokes' theorem, this term vanishes on performing a field variation).

## Electromagnetic Field

In the electromagnetic case, the Lagrangian density for  $\varphi_a \in \mathcal{E}_a$  is taken to be

$$\mathcal{L} = -\frac{1}{4}\nabla_{[a}\varphi_{b]} \cdot \nabla^{[a}\varphi^{b]}$$

which is gauge invariant. From this, the Euler-Lagrange equations are the conformally invariant Maxwell equations

$$2\nabla^b\nabla_{[b}\varphi_{a]} = 0$$

and the energy-momentum tensor is

$$T_{ab} = \nabla_{[a}\varphi_{c]} \cdot \nabla_{[b}\varphi_{d]}g^{cd} - \frac{1}{4}g_{ab}\nabla_{[c}\varphi_{d]} \cdot \nabla^{[c}\varphi^{d]}.$$

## 5.3 Conserved Quantities for an Electromagnetic Field

### An Initial Scale-Dependent Construction in Conformally Flat Settings

We begin this section by considering the initial construction observed by my supervisor which initiated this aspect of research. Suppose we take the Lagrangian density for a massless scalar in the scale of minimal coupling (5.2.1) and then consider the associated energy-momentum tensor (5.2.2). If we consider the connection as a coupled connection, replace the appearance of  $\psi$  by the electromagnetic tractor  $\Phi_A$ , and contract the tractor indices, then we obtain a new tensor

$$S_{ab} = \nabla_a\Phi_E \cdot \nabla_b\Phi^E - \frac{1}{2}g_{ab}\nabla_c\Phi_E \cdot \nabla^c\Phi^E.$$

The resulting object is clearly symmetric so checking that it is divergence-free we get

$$\begin{aligned} \nabla^a S_{ab} &= \Delta\Phi_E \cdot \nabla_b\Phi^E + \nabla_a\Phi_E \cdot \nabla^a\nabla_b\Phi^E - \nabla_c\Phi_E \cdot \nabla_b\nabla^c\Phi^E \\ &= \Delta\Phi_E \cdot \nabla_b\Phi^E + 2\nabla^a\Phi_E \cdot \nabla_{[a}\nabla_{b]}\Phi^E \\ &= \Delta\Phi_E \cdot \nabla_b\Phi^E + \nabla^a\Phi_E \cdot \Omega_{ab}{}^E{}_F\Phi^F. \end{aligned}$$

We require two conditions in order to ensure that this tensor is divergence-free. First,  $(M, g)$  must be conformally flat (so that  $\Omega_{ab}{}^C{}_D = 0$ ). Second we need to have  $P = 0$  (so that  $\square\Phi_A = \Delta\Phi_A$ ). This setting is too restrictive. The solution is to use the Lagrangian density (5.2.3). Two constructions are now possible which will annihilate the appearance of the tractor curvature. The first requires  $(M, g)$  to be conformally flat so that when  $(M, g)$  descends to  $(M, c)$  the tractor curvature vanishes. The second requires  $(M, g)$  to be conformally Einstein so that a parallel tractor exists which may be used to annihilate the tractor curvature.

## The General Construction in Conformally Flat Settings

Suppose  $(M, g)$  is conformally flat and descends to the conformal manifold  $(M, c)$ . Then the same method as used in the previous section may be used to construct a divergence-free symmetric tensor from the energy-momentum tensor of (5.2.4). The result is the following.

**Theorem 5.2.** *Suppose  $(M, g)$  is conformally flat and descends to the conformal manifold  $(M, c)$ . Let  $\varphi_a \in \mathcal{E}_a$  be a 1-form which satisfies the Maxwell equations and the Eastwood-Singer gauge. Then the associated electromagnetic tractor  $\Phi_A \in \mathcal{E}_A[-1]$  solves the equation  $\square\Phi_A = 0$  and, for any  $g \in c$ , the tensor*

$$S_{ab} = \nabla_a \Phi_E \cdot \nabla_b \Phi^E - \frac{1}{2} g_{ab} \nabla_c \Phi_E \cdot \nabla^c \Phi^E + \frac{1}{3} (P_{ab} - P g_{ab}) (\Phi_E \Phi^E) \\ + \frac{1}{6} (g_{ab} \Delta (\Phi_E \Phi^E) - \nabla_a \nabla_b (\Phi_E \Phi^E))$$

is symmetric and divergence-free.

*Proof.* That  $\square\Phi_A = 0$  is a result of [3]. It is clear that the tensor is symmetric as the tractor metric commutes with the connection. In order to see why  $\nabla^a S_{ab} = 0$  it is easier, and more appropriate, to consider why the original energy-momentum tensor  $T_{ab}$  of (5.2.4) is divergence-free. As previously stated,

$$T_{ab} = \nabla_a \psi \cdot \nabla_b \psi - \frac{1}{2} g_{ab} \nabla_c \psi \cdot \nabla^c \psi + \frac{1}{3} (P_{ab} - P g_{ab}) \psi^2 + \frac{1}{6} (g_{ab} \Delta \psi^2 - \nabla_a \nabla_b \psi^2)$$

Dealing with the final bracket, we consider  $\nabla^a (g_{ab} \Delta \psi^2 - \nabla_a \nabla_b \psi^2)$ . Calculating this explicitly gives

$$\nabla_b \Delta \psi^2 - \Delta \nabla_b \psi^2 = g^{ac} (\nabla_b \nabla_a \nabla_c - \nabla_a \nabla_c \nabla_b) \psi^2 = g^{ac} (\nabla_b \nabla_a \nabla_c - \nabla_a \nabla_b \nabla_c) \psi^2 \\ = 2g^{ac} \nabla_{[b} \nabla_{a]} \nabla_c \psi^2 = g^{ac} R_{ab}{}^d{}_c \nabla_d \psi^2 = -(2P_b{}^d + P \delta_b^d) \nabla_d \psi^2 \quad (5.3.1)$$

The second equality is of interest, in particular the statement  $\nabla_a \nabla_c \nabla_b \psi^2 = \nabla_a \nabla_b \nabla_c \psi^2$ . Of course, this just follows from the torsion-free property of the Levi-Civita connection and will also hold when  $\psi^2$  is replaced by  $\Phi_E \Phi^E$ . Using (5.3.1) we may give a direct calculation that  $T_{ab}$  is divergence-free. (Note we use the observation that  $P_{ab} - P g_{ab}$  is proportional to the divergence-free Einstein tensor.)

$$\nabla^a T_{ab} = \Delta \psi \cdot \nabla_b \psi + \nabla_a \psi \cdot \nabla^a \nabla_b \psi - \nabla_b \nabla_c \psi \cdot \nabla^c \psi \\ + \frac{1}{3} (P_{ab} - P g_{ab}) \nabla^a \psi^2 - \frac{1}{6} (2P_{ab} + P g_{ab}) \nabla_a \psi^2 \\ = \Delta \psi \cdot \nabla_b \psi + 2\nabla^a \psi \cdot \nabla_{[a} \nabla_{b]} \psi - \frac{1}{2} P \nabla_b \psi^2 \\ = (\Delta \psi - P \psi) \cdot \nabla_b \psi + 2\nabla^a \psi \cdot \nabla_{[a} \nabla_{b]} \psi$$

The calculation for  $S_{ab}$  follows precisely the same argument. However where we concluded  $\nabla_{[a} \nabla_{b]} \psi = 0$  due to the torsion-free property of the connection, we will now conclude  $\nabla_{[a} \nabla_{b]} \Phi_E = 0$  since  $\Omega_{ab}{}^C{}_D = 0$ .  $\square$



## The General Construction in Conformally Einstein Settings

We now assume the general setting that  $(M, g)$  descends to a conformal Einstein manifold  $(M, c)$  with an associated parallel tractor  $I_A$ . The construction is easy to describe. Take the energy-momentum tensor of (5.2.4), replace every appearance of  $\psi$  by the electromagnetic tractor  $\Phi^A$  (all indices up). Consider the connection as a coupled connection and interpret all tensor objects as the associated weighted objects. Finally contract all tractor indices upon two copies of the parallel tractor  $I_A$ . The result is the following.

**Theorem 5.3.** *Suppose  $(M, g)$  is conformally Einstein and descends to the conformal manifold  $(M, c)$  which possesses a parallel tractor  $I_A$ . Let  $\varphi_a \in \mathcal{E}_a$  be a 1-form which satisfies the Maxwell equations and the Eastwood-Singer gauge. Then the associated electromagnetic tractor  $\Phi_A \in \mathcal{E}_A[-1]$  solves the equation  $\square\Phi_A = 0$  and, for any  $g \in c$ , the tensor*

$$S_{ab} = \left( \nabla_a \Phi^E \cdot \nabla_b \Phi^F - \frac{1}{2} \mathbf{g}_{ab} \nabla_c \Phi^E \cdot \nabla^c \Phi^F + \frac{1}{3} (P_{ab} - P \mathbf{g}_{ab}) (\Phi^E \Phi^F) \right. \\ \left. + \frac{1}{6} (\mathbf{g}_{ab} \Delta (\Phi^E \Phi^F) - \nabla_a \nabla_b (\Phi^E \Phi^F)) \right) I_E I_F$$

is symmetric and divergence-free.

*Proof.* As before, that  $\square\Phi_A = 0$  is a result of [3]. It is clear that the tensor is symmetric; the only issue is the term involving  $\nabla_a \nabla_b (\Phi^E \Phi^F)$  however the curvature terms resulting from

$$\nabla_a \nabla_b (\Phi^E \Phi^F) = \nabla_b \nabla_a (\Phi^E \Phi^F) + \Omega_{ab}{}^E{}_G \Phi^G \Phi^F + \Omega_{ab}{}^F{}_G \Phi^E \Phi^G$$

are annihilated by the double appearance of the parallel tractor. The principal result is that  $S_{ab}$  is divergence-free and this may be easily verified. Since  $\nabla_a I_B = 0$  we may pass the parallel tractors through the brackets and contract tractor indices first. If we introduce  $\chi = \Phi^E I_E \in \mathcal{E}[-1]$  then  $S_{ab}$  may be written

$$S_{ab} = \nabla_a \chi \cdot \nabla_b \chi - \frac{1}{2} \mathbf{g}_{ab} \nabla_c \chi \cdot \nabla^c \chi + \frac{1}{3} (P_{ab} - P \mathbf{g}_{ab}) \chi^2 + \frac{1}{6} (\mathbf{g}_{ab} \Delta \chi^2 - \nabla_a \nabla_b \chi^2).$$

Also  $\square\chi = 0$  as a direct consequence of  $\square\Phi_A = 0$  and  $\nabla_a I_B = 0$ . Therefore  $\chi$  solves precisely the field equation required of a massless scalar field such that its associated energy momentum tensor  $T_{ab}$  (5.2.4) is divergence-free. This coincides with  $S_{ab}$  hence  $\nabla^a S_{ab} = 0$ .  $\square$

The proof relies on  $\square\Phi_A = 0$  which only holds in the Eastwood-Singer gauge. This is unlike conventional conserved quantities for an electromagnetic field where conserved quantities are gauge invariant.

Since  $I_A$  is parallel, there exists a conformal scale  $\sigma \in \mathcal{E}[1]$  such that  $I_A = \frac{1}{4} D_A \sigma$ . Let  $\hat{g} = \sigma^{-2} \mathbf{g}_{ab} \in c$ . We may then write  $I_A \stackrel{\hat{g}}{=} (1, 0, -\frac{1}{4} P)$  and

$$\chi = \Phi^E I_E \stackrel{\hat{g}}{=} -\frac{1}{2} \nabla^a \varphi_a.$$

The conserved quantities associated with  $S_{ab}$  are dependent on  $\square(-\frac{1}{2} \nabla^a \varphi_a)$ . That  $\varphi_a$  solves this equation may be seen as a direct consequence of the Eastwood-Singer gauge and the

knowledge that  $\hat{P}_{ab} = \lambda \hat{g}_{ab}$  for some  $\lambda \in \mathbb{R}$  (which holds since  $\hat{g}$  is Einstein).

A natural direction for further investigation is to consider whether this tensor may be obtained using a Lagrangian. As observed in the proof of Theorem 5.3 we may pass copies of  $I_A$  under the brackets and contract the tractor indices immediately. So the scalar  $\Phi^A I_A \in \mathcal{E}[-1]$  solves the field equation associated with a massless scalar. Potentially then, this tensor may be obtained by performing a kind of metric variation with the Lagrangian density of (5.2.3) with  $\psi$  replaced with  $\Phi^A I_A$ . The result however is not immediate as now, one must consider how to correctly handle  $I_A$ . This is defined in terms of a conformal scale (which of course depends on a metric within the conformal class) and thus, depending on interpretation, could change under a metric variation. However in the common Lagrangian framework, the metric variations are required to leave the field unaffected.

## 5.4 Conserved Quantities for a Scalar Field

The final section in this dissertation considers whether such a construction also works if we begin with the energy-momentum tensor associated with the electromagnetic 1-form  $\varphi_a \in \mathcal{E}_a$

$$T_{ab} = \nabla_{[a}\varphi_{c]} \cdot \nabla_{[b}\varphi_{d]} g^{cd} - \frac{1}{4}g_{ab}\nabla_{[c}\varphi_{d]} \cdot \nabla^{[c}\varphi^{d]}.$$

We would like to consider a reverse of the preceding section. Here, we desire a construction for conserved quantities associated with the scalar  $\psi \in \mathcal{E}$  using the electromagnetic energy-momentum tensor  $T_{ab}$ . In a similar vein to before, consider formally replacing  $\varphi_a$  by  $\Psi_a^B \in \mathcal{E}_a^B$  and contracting the resulting tractor indices with a parallel tractor  $I_A$ . We attain

$$S_{ab} = (\nabla_{[a}\Psi_{c]}^E \cdot \nabla_{[b}\Psi_{d]}^F g^{cd} - \frac{1}{4}g_{ab}\nabla_{[c}\Psi_{d]}^E \cdot \nabla^{[c}\Psi^{d]F}) I_E I_F. \quad (5.4.1)$$

Clearly this tensor is symmetric so we investigate under what conditions it is divergence-free. To this end, it is advantageous to consider directly why the usual electromagnetic stress-energy tensor is divergence-free. Consider

$$T_{ab} = \omega_{ac}\omega_{bd}g^{cd} - \frac{1}{4}g_{ab}\omega_{cd}\omega^{cd}$$

with  $\omega_{ab} \in \mathcal{E}_{[ab]}$  solving the Maxwell equations. Then

$$\nabla^a T_{ab} = \nabla^a \omega_{ac} \cdot \omega_b^c + \omega_{ac} \nabla^a \omega_b^c - \frac{1}{2} \omega_{cd} \cdot \nabla_b \omega^{cd}.$$

The first term vanishes since  $\nabla^a \omega_{ab} = 0$ . Splitting the second term and manipulating indices gives

$$\begin{aligned} \nabla^a T_{ab} &= \frac{1}{2} \omega^{ac} (\nabla_a \omega_{bc} + \nabla_a \omega_{bc} - \nabla_b \omega_{ac}) \\ &= \frac{1}{2} \omega^{ac} (\nabla_a \omega_{bc} + \nabla_c \omega_{ab} + \nabla_b \omega_{ca}) \end{aligned} \quad (5.4.2)$$

where the second term is a consequence of  $\omega^{ac}\nabla_a\omega_{bc} = \omega^{ca}\nabla_c\omega_{ba} = -\omega^{ac}\nabla_c\omega_{ba} = \omega^{ac}\nabla_c\omega_{ab}$ . This now vanishes since  $\omega_{ab}$  is a closed 2-form,  $\nabla_{[a}\omega_{bc]} = 0$ . In order to investigate  $S_{ab}$  in (5.4.1) the following result is useful; it is a simple consequence of requiring the Leibniz rule (2.3.1) to hold for coupled connections.

**Lemma 5.4.** *If  $\Psi_c^D \in \mathcal{E}_c^D$  then  $2\nabla_{[a}\nabla_b]\Psi_c^D = \Omega_{ab}^D \Psi_c^E - R_{ab}{}^d{}_c \Psi_d^D$ .*

Consider first  $\nabla_{[a}\nabla_b]\Psi_c^D$ . Using Lemma 5.4, one calculates

$$6\nabla_{[a}\nabla_b]\Psi_c^D = 3\Omega_{[ab]}^D \Psi_{|c]}^E - 3R_{[ab}{}^d{}_c]\Psi_d^D.$$

The Bianchi symmetry removes the second term and we are left with

$$2\nabla_{[a}\nabla_b]\Psi_c^D = \Omega_{[ab]}^D \Psi_{|c]}^E. \quad (5.4.3)$$

Although this does not vanish, it is annihilated by the parallel tractor in (5.4.1). Almost identically to the case of the electromagnetic stress-energy tensor, by calculating the divergence of the tensor of (5.4.1) we get

$$\nabla^a S_{ab} = (\nabla^a \nabla_{[a} \Psi_{c]}^E \cdot \nabla_{[b} \Psi_{d]}^F g^{cd} + \nabla^{[a} \Psi^{c]E} \cdot \nabla_a \nabla_{[b} \Psi_{c]}^F - \frac{1}{2} \nabla_{[c} \Psi_{d]}^E \cdot \nabla_b \nabla^{[c} \Psi^{d]F}) I_E I_F$$

The first term does not necessarily vanish, however, by restricting our attention to the final two terms, we note, precisely as in (5.4.2),

$$\nabla^{[a} \Psi^{c]E} \cdot \nabla_a \nabla_{[b} \Psi_{c]}^F - \frac{1}{2} \nabla_{[c} \Psi_{d]}^E \cdot \nabla_b \nabla^{[c} \Psi^{d]F} = 3 \nabla^{[a} \Psi^{c]E} \cdot \nabla_{[a} \nabla_b \Psi_{c]}^F.$$

Using (5.4.3) the calculation ends with the following result

$$\begin{aligned} \nabla^a T_{ab} &= (\nabla^a \nabla_{[a} \Psi_{c]}^E \cdot \nabla_{[b} \Psi_{d]}^F g^{cd} + 3 \nabla^{[a} \Psi^{c]E} \cdot \Omega_{[ab]}^F \Psi_{|c]}^G) I_E I_F \\ &= (\nabla^a \nabla_{[a} \Psi_{c]}^E \cdot \nabla_{[b} \Psi_{d]}^F g^{cd}) I_E I_F. \end{aligned}$$

For the tensor to be divergence-free (and non-zero) it is necessary that  $\Psi_a^B$  satisfies

$$\nabla^a \nabla_{[a} \Psi_{b]}^E I_E = 0. \quad (5.4.4)$$

There is a natural (but ultimately uninteresting) choice for  $\Psi_a^B \in \mathcal{E}_a^B$  if we believe the conserved quantities should relate to a scalar field. Specifically, for  $\psi \in \mathcal{E}[-1]$  this is

$$\Psi_a^B = \nabla_a (X^B \psi) \stackrel{g}{=} \begin{pmatrix} 0 \\ \psi \delta_a^b \\ \nabla_a \psi \end{pmatrix}.$$

This solves  $\nabla^a \nabla_{[a} \Psi_{b]}^C = 0$  (and thus also (5.4.4) as required) however  $\nabla_a \Psi_b^C$  is symmetric in  $a, b$  and so the tensor (5.4.1) is identically zero.



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