NOTES ON THE SUPERSPHERE

Let k be an algebraically closed field of characteristic not equal to 2, and let

$$X = \operatorname{Spec} k[x_1, x_2, x_3, \xi_1, \xi_2, \xi_3]/(x_1^2 + x_2^2 + x_3^2 - 1, x_1\xi_1 + x_2\xi_2 + x_3\xi_3).$$

We present two elementary arguments for why X is the "right" notion of a supersphere. The astute reader will notice that both arguments are essentially the same.

0.1. **As a naïve generalization of the classical sphere.** Consider the classical sphere

$$S^2 = \operatorname{Spec} k[x_1, x_2, x_3]/(x_1^2 + x_2^2 + x_3^2 - 1).$$

Using the functor of points approach, the A-points (for A an ordinary commutative k-algebra) are

$$h_{S^2}(A) = \{(a_1, a_2, a_3) \in A^3 \mid a_1^2 + a_2^2 + a_3^2 = 1\}.$$

It is often the case in supergeometry that we may make a super version of a classical object by haphazardly extending its functor of points to work on superalgebras. Let us see what happens if A is a superalgebra, and we plug the heterogeneous elements $a_i + \alpha_i$ to the above definition (for $a_i \in A_{\bar{0}}$ and $\alpha_i \in A_{\bar{1}}$):

$$h_{S^2}(A) = \{(a_1 + \alpha_1, a_2 + \alpha_2, a_3 + \alpha_3) \in A^3 \mid a_1^2 + a_2^2 + a_3^2 + 2a_1\alpha_1 + 2a_2\alpha_2 + 2a_3\alpha_3 = 1\}.$$

Since the relation involves both even and odd variables, we may split it into its homogeneous parts:

$$a_1^2 + a_2^2 + a_3^2 = 1$$
$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = 0$$

These expressions make it clear that h_{S^2}, when its domain is enlarged to the category of commutative superalgebras, is representable by X.

0.2. **As a sphere in a Q-vector space.** A *Q-vector space* is a super vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ equipped with an odd involution $\Pi : V \to V$. Using $V = k^{3|3}$ equipped with Π the "identity" on k^3, we may consider $k^{3|3}$ as an affine supervariety

$$\begin{aligned}\mathbb{A}^{3|3} &= \operatorname{Spec}(\operatorname{Sym}(k^{3|3})^*) \\ &\cong \operatorname{Spec} k[x_1, x_2, x_3, \xi_1, \xi_2, \xi_3]\end{aligned}$$

in which X is a closed subvariety. Note that while Π is NOT a morphism of superschemes, it DOES define an odd k-linear derivation on $k[x_1, x_2, x_3, \xi_1, \xi_2, \xi_3]$ via $x_i \mapsto \xi_i$ and $\xi_i \mapsto x_i$, since $\Pi^* : V^* \to V^*$ is exactly this map for $V^* = \operatorname{Span}_k\{x_1, x_2, x_3, \xi_1, \xi_2, \xi_3\}$. This is the same thing as an odd vector field, i.e. an odd global section of the tangent bundle. In other words,

$$\Pi^* = \sum_{i=1}^{3} \left(\xi_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial \xi_i} \right) \in \Gamma(\mathbb{A}^{3|3}, \mathcal{T}_{\mathbb{A}^{3|3}}).$$

So, if we want a closed subvariety Y of $\mathbb{A}^{3|3}$ which is compatible with Π and whose reduction Y_{red} is the ordinary sphere, then the relation $\Pi^*(x_1^2 + x_2^2 + x_3^2 - 1)$ must also be present in its defining ideal. Up to scale, this is exactly $x_1\xi_1 + x_2\xi_2 + x_3\xi_3$! Hence $Y = X$.

The upshot of this viewpoint is that Π (respectively, Π^*) realizes the tangent bundle (respectively, the cotangent bundle) of X as a "Q-vector bundle", i.e. a $(2|2)$-dimensional vector bundle equipped with an odd involution. In particular, this means that the tangent spaces at closed points will all be Q-vector spaces.