Infinitesimal theory

Ground setting

$X = (|X|, \mathcal{O}_X)$ is a superscheme

Def: We call X **algebraic** $:\Leftrightarrow$

$\exists$ open affine finite cover $\{X_i\}_{i \in I}$

$\forall i \in I$: $\mathcal{O}_X(X_i)$ fin. generated superalgebra

Assumption: All superschemes considered are **algebraic**.

Def: Let $X = (|X|, \mathcal{O}_X)$ superscheme $x \in X$

(i) x is **rational** if $\mathcal{O}_{X,x}/m_{X,x} \cong k$

(ii) x is **closed** if $x = M_x \in \operatorname{Spec} A$
for affine open neighbourhood $\operatorname{Spec} A$ of x
and M_x maximal

Rem: If $k = \bar{k} \Rightarrow$ (closed $\Rightarrow$ rational)

$\mathcal{O}_{X,x}/m_{X,x}$ is finite extension of

$$k \underset{k=\bar{k}}{\Rightarrow} \mathcal{O}_{X,x}/m_{X,x} \cong k$$

Functorial interpretation of rational points

rational points $\longleftrightarrow$ k-points in h_X
$\mathrm{Hom}_{\text{(superschemes)}}(\operatorname{Spec} k, X)$

"$\longleftarrow$" $\operatorname{Spec} k \to X \rightsquigarrow f|_{(x)} : \mathcal{O}_{X,x} \to k$

"$\longrightarrow$" $\mathcal{O}_X(U) \xrightarrow{\text{can.}} \mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_{X,x} \cong k$

<u>Def:</u>
$D : A \to M$; $A \in \mathrm{(salg)}$; $M \in A\text{-mod}$
with:
$D(a+b) = D(a) + D(b)$
$D(k) = 0 \qquad \forall k \in k$
is called a superderivation if
$D(fg) = D(f)\,g + (-1)^{|D||f|} f\,D(g)$

$\rightsquigarrow$ $T_x X = \operatorname{Der}(\mathcal{O}_{X,x}, k)$; for $x \in$ rational
$k \cong \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$

<u>Prop:</u> (1) $T_x X \cong \underline{\mathrm{Hom}}_{\text{(smod)}}(\mathfrak{m}_{X,x}/(\mathfrak{m}_{X,x})^2, k)$

(2) $x \in U = \operatorname{Spec} A \subseteq X$; U open x rational closed
$x = \mathfrak{m}_x \trianglelefteq A$
$\rightsquigarrow T_x X \cong \mathrm{Hom}_{\text{(smod)}}(\mathfrak{m}_{X,x}/(\mathfrak{m}_{X,x})^2, k)$

Differentials: $X = (|X|, \mathcal{O}_X)$ scheme,

$x \in X$ rational

$\pi: \mathcal{O}_{X,x} \xrightarrow{\text{k-algebra map}} \mathcal{O}_{X,x}/m_{X,x} \cong k$

$p: m_{X,x} \longrightarrow m_{X,x}/(m_{X,x})^2$

(1) $f \in \mathcal{O}_{X,x} \rightsquigarrow$ value $f(x) = \pi(f) \in k$

$\rightsquigarrow \quad f - f(x) \in m_{X,x}$

$(df)_x := p(f - f(x))$

(2) $f \in \mathcal{O}_X(U)$, $x \in U$ $\quad f|_x$ element at $\mathcal{O}_{X,x}$

$\rightsquigarrow \quad (df)_x := (d f|_x)_x$

Remark: Differentials satisfy Leibniz rule

Example: $P = (x_1^0, \ldots, x_m^0, 0, \ldots, 0)$ closed rational in $\mathbb{A}^{m|n}$, $m_P = \langle x_i - x_i^0, \xi_j \rangle \trianglelefteq k[x_1, \ldots, x_m, \xi_1, \ldots, \xi_n]$

$m_P/m_P^2 = \mathrm{span}_k\{\underbrace{x_i - x_i^0, \xi_j}_{\text{basis}} \mid i = 1, \ldots, m, \; j = 1, \ldots, n\}$

$(dx_i)_P = [x_i - x_i^0]_{m_{X,P}}$

$(d\xi_j)_P = [\xi_j]_{m_{X,P}}$

Def: $\alpha = (|\alpha|, \alpha^\#) : X \to Y$, $\alpha \in \mathrm{Hom}_{(\text{schemes})}(X,Y)$

$x \in X$ rational $\rightsquigarrow$ $|\alpha|(x) \in Y$ rational

$$\rightsquigarrow d\alpha_x : T_x X \longrightarrow T_{|\alpha|(x)} Y$$
$$D \longmapsto D \circ \alpha^*_x$$
$$\mathrm{Der}(\mathcal{O}_{X,x}, k)$$

where $\alpha^\#_x : \mathcal{O}_{Y,|\alpha|(x)} \longrightarrow \mathcal{O}_{X,x}$

Def: $\alpha : X \to Y$ is a <u>closed embedding</u> iff

(a) $|\alpha|$ homeomorphism onto closed subset of Y

(b) $\alpha^\#: \mathcal{O}_Y \longrightarrow \mathcal{O}_X$ surjective, i.e.

$\alpha^\#_x : \mathcal{O}_{Y,|\alpha|(x)} \longrightarrow \mathcal{O}_{X,x}$ surjective

Prop: $\alpha : X \to Y$ closed embedding

$\Rightarrow (d\alpha)_x$ is injective

Rem: $\alpha : \mathrm{Spec}\, A \to \mathrm{Spec}\, B$ is closed embedding $\Leftarrow$ $\alpha^\#: B \to A$ surjective

$X = \operatorname{Spec} k[x_1, \dots, x_m, \varepsilon_1, \dots, \varepsilon_n] / I \overset{\text{closed}}{\hookrightarrow} \mathbb{A}^{m|n}$

$x \in X$ rational closed $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$

$\rightsquigarrow \quad T_x X \subset T_x \mathbb{A}^{m|n}$

$\subset$ subspace

$\Rightarrow$ can we find equations?

evaluation/scalar product?

Prop: $T_x X = \{ v \in k^{m|n} \mid (df)_x (v) = 0 \ \forall f \in I \}$

$= \{ v \in T_{x(\text{?})}(\mathbb{A}^{m|n}) \mid v((df)_x) = 0 \ \forall f \in I \}$

$\subset \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$

Rem: the generators

Example: $\operatorname{Spec} \mathbb{C}[x, y, \xi, \eta] / (x\xi + y\eta)$

$p = (1, 1, 0, 0)$

$d(x\xi + y\eta)_p$

$= 1 \cdot d(\xi)_p + 0 \cdot (dx)_p + 1 \cdot (d\eta)_p + 0 \cdot (dy)_p$

$= (d\xi)_p + (d\eta)_p$

$= \xi + \eta$

$\rightsquigarrow \quad T_{(1,1,0,0)} \left(\operatorname{Spec} \mathbb{C}[x, y, \xi, \eta] / (x\xi + y\eta) \right)$

$\cong \langle v_1, v_2, v_3, v_4 \in k^{2|2} \mid v_3 - v_4 = 0 \rangle$

Regularity ~ Smoothness via dimension?

Krull superdimension, Masuoka and Zubkov [2020]

$$\mathrm{Ksdim} = \mathrm{sdim}\, m_x/m_x^2$$

$$\mathrm{Spec}\left(\mathbb{C}[x_1, x_2, x_3, \xi_1, \xi_2, \xi_3] \Big/ (x_1^2 + x_2^2 + x_3^2 - 1,\ x_1\xi_1 + x_2\xi_2 + x_3\xi_3)\right)$$

$$p = (1, 0, 0, 0, 0, 0)$$

$$d(x_1^2 + x_2^2 + x_3^2 - 1)_p$$

$$= 2\, d(x_1) = 0 \quad \Leftarrow \quad x_1 - 1 = 0$$

$$d(x_1\xi_1 + x_2\xi_2 + x_3\xi_3)$$

$$= d(\xi_1) = 0 \qquad \xi_1 = 0$$

$$u_1, \ldots, u_6 \qquad u_1 = 0$$

$$u_4 = 0$$

Recommendation: Fioresi et al.: Smoothness of algebraic supergroups (~ points functor)