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Quantum SL_2

Recall: $U_q sl_2$ is the associative alg. over $\mathbb{C}(q)$.
with generators $E, F, K^{\pm}$
s.t.

$$KK^{-1} = K^{-1}K = 1$$
$$KEK^{-1} = q^2 E$$
$$KFK^{-1} = q^{-2} F$$
$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

with some comult.

Q: Where does this come from?

Historically, the q-version of SL_2 was first defined

Classically: $SL_2 \subset Mat_{2\times 2} \circlearrowright \mathbb{A}^2$

$$Mat_{2\times 2} \cong \mathbb{A}^4$$

$$\mathbb{C}[SL_2] \overset{\text{Hopf dual}}{\longleftrightarrow} Usl_2$$

Thus, $Mat_{2\times 2} \times \mathbb{A}^2 \longrightarrow \mathbb{A}^2$

This gives a coaction

$$\mathbb{C}[\mathbb{A}^2] \longrightarrow \mathbb{C}[Mat_{2\times 2}] \otimes \mathbb{C}[\mathbb{A}^2]$$

Picking generators $\mathbb{C}[\mathbb{A}^2] \simeq \mathbb{C}[x, y]$

$$\mathbb{C}[Mat_{2\times 2}] \simeq \mathbb{C}[a, b, c, d]$$

$$x \longmapsto a \otimes x + b \otimes y$$
$$y \longmapsto c \otimes x + d \otimes y$$

coming from matrix mult. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$

<u>What we are looking for:</u>

"Quantum $\mathrm{Mat}_{2\times 2}$" should act on the "quantum plane"

non commutative space with coordinate ring $\mathbb{C}(q)\langle x,y\rangle / (yx - qxy)$

$\mathbb{C}_q[x,y] :=$

More precisely, we look for some non-commutative alg. $\mathbb{C}_q[\mathrm{Mat}_{2\times 2}]$ which "coacts" on $\mathbb{C}_q[x,y]$

Leaving matrix multiplication as is, if $\mathbb{C}_q[\mathrm{Mat}_{2\times 2}]$ is generated by a,b,c,d as before, we want

$$x \longmapsto a\otimes x + b \otimes y$$
$$y \longmapsto c \otimes x + d \otimes y$$

$yx = qxy \implies (c\otimes x + d \otimes y)(a \otimes x + b \otimes y) = q(a\otimes x + b\otimes y)(c\otimes x + d \otimes y)$

x^2: $\therefore$ $ca = qac$

y^2: $db = qbd$

xy: $cb + qda = qad + q^2 bc$ — ③

We also want quantum matrices to act from the right on row vectors, i.e,

$$\mathbb{C}[\mathbb{A}^2] \longrightarrow \mathbb{C}[Mat_{2\times 2}] \otimes \mathbb{C}[\mathbb{A}^2]$$

$$(x, y) \longmapsto (x \ y)\begin{pmatrix} a & b \\ c & d \end{pmatrix} := (a\otimes x + c\otimes y \quad b\otimes x + d\otimes y)$$

should be an alg. hom.

$$yx = qxy \Rightarrow (b\otimes x + d\otimes y)(a\otimes x + c\otimes y) = q(a\otimes x + c\otimes y)(b\otimes x + d\otimes y)$$

x^2: $ba = qab$

y^2: $dc = qcd$

xy: $bc + qda = qad + q^2 cb$ — ⑥

From ③ & ⑥, $bc = cb$

$$ad - da = (q^{-1} - q) bc$$

$$\therefore \mathbb{C}_q[Mat_{2\times 2}] := \mathbb{C}(q)\langle a, b, c, d\rangle \Big/ \begin{pmatrix} ba = qab,\ ca = qac \\ db = qbd,\ dc = qcd \\ bc = cb \\ ad - da = (q^{-1}-q) bc \end{pmatrix}$$

X monoid $\Rightarrow \mathbb{C}[X]$ comm. bialgebra

X group $\Rightarrow \mathbb{C}[X]$ comm. Hopf alg.

We make $\mathbb{C}_q[\mathrm{Mat}_{2\times 2}]$ a bialgebra via

$$\Delta : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a\otimes a + b\otimes c & a\otimes b + b\otimes d \\ c\otimes a + d\otimes c & c\otimes b + d\otimes d \end{pmatrix}$$

$$\varepsilon : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

<u>Quantum determinant:</u>

$$\begin{aligned} \det_q &:= ad - q^{-1}bc \quad \text{is } \underline{\text{central}} \ \& \ \underline{\text{group like}} \\ &= da - qbc \end{aligned}$$

$$\begin{aligned} \Delta(da - qbc) &= (c\otimes b + d\otimes d)(a\otimes a + b\otimes c) - q(a\otimes b + b\otimes d)(c\otimes a + d\otimes c) \\ &= (\cancel{ca\otimes ba} + da\otimes da + cb\otimes bc + \cancel{db\otimes dc}) \\ &\quad - q(\cancel{ac\otimes ba} + ad\otimes bc + bc\otimes da + \cancel{bd\otimes dc}) \\ &= ad\otimes da + (q - q^{-1})bc\otimes da + bc\otimes bc \\ &\quad - q\,ad\otimes bc - q\,bc\otimes da \\ &= ad\otimes da - q^{-1}bc\otimes da + bc\otimes bc - q\,ad\otimes bc \\ &= (ad - q^{-1}bc)\otimes da + (bc - qad)\otimes bc \\ &= \det_q \otimes da - q\det_q\otimes bc \\ &= (\det_q)\otimes(da - qbc) = \det_q\otimes\det_q \end{aligned}$$

Def^n: $\mathbb{C}_q[SL_2] := \mathbb{C}_q[\mathrm{Mat}_{2\times 2}] / \langle \det_q - 1 \rangle$

This is a Hopf alg. with antipode

$$S: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}$$

Rk: This is unique!

$\leadsto$ f.g Noeth. domain

"quantized coordinate ring"

Recall: A Hopf pairing $\langle\ ,\ \rangle : A \otimes B \to \mathbb{C}$

is st

$$\langle \Delta a, b_1 \otimes b_2 \rangle = \langle a, b_1 b_2 \rangle \ ; \ \langle a_1 \otimes a_2, \Delta b \rangle = \langle a_1 a_2, b \rangle$$

$$\langle a, 1_B \rangle = \varepsilon(a) \ ; \ \langle 1_A, b \rangle = \varepsilon(b)$$

$U_q sl_2$ rep theory:

Natural representation $V_{1,+} \simeq \mathbb{C}^2$ with action

$$E \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad K \mapsto \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

Rk: If $u \in U_q sl_2$ acts on $V_{1,+}$ as $\begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$,

then
$\langle a, u \rangle = A(u)$
$\langle b, u \rangle = B(u)$ $\quad \langle d, u \rangle = D(u)$ gives a
$\langle c, u \rangle = C(u)$ $\quad$ non deg. Hopf pairing

Thus $\mathbb{C}_q[SL_2]$ is Hopf dual to $U_q sl_2$

<u>Other reps</u>: For $\lambda \in \mathbb{C}(q)^\times$, we define the <u>Verma module</u>

$$M(\lambda) = \mathrm{span}\{v_i\}_{i\geq 0}$$

$$F \cdot v_n = v_{n+1}$$

$$K \cdot v_n = q^{-2n} \lambda v_i$$

$$E \cdot v_n = [n]_q \frac{q^{-n+1}\lambda - q^{n-1}\lambda^{-1}}{q - q^{-1}} v_{n-1}$$

If $\lambda \notin \{\pm q^n \mid n \in \mathbb{N}\}$, then $M(\lambda)$ is <u>simple</u>

If $\lambda = \pm q^n$, then $E \cdot v_{n+1} = 0$, so $U_q sl_2 \cdot v_{n+1}$ is a submodule of $M(\lambda)$. The corresponding quotient $V(\lambda)$ is <u>fin. dim.</u> & <u>simple</u>

$$V_{n,+} := V(q^n) \quad ; \quad V_{n,-} := V(-q^n)$$

$\rightsquigarrow V_n$ as $q \to 1$ $\qquad\qquad$ $\rightsquigarrow$ NOT a deformation of V_n

<u>weights:</u> $\pm q^{-n}$ . . . . $\pm q^{n-1}$ $\pm q^n$

<u>Rk</u>: $U_q sl_2$ acts as diff. op.s on the quantum plane, i.e, on $\mathbb{C}_q[x,y]$

$$\delta f(x) := \frac{f(qx) - f(q^{-1}x)}{qx - q^{-1}x}$$

Then

$E \mapsto x\delta_y$

$F \mapsto y\delta_x$

$K \mapsto \begin{pmatrix} x \mapsto qx \\ y \mapsto q^{-1}y \end{pmatrix}$

$$\delta_x(x^m y^n) = \frac{q^m x^m y^n - q^{-m} x^m y^n}{qx - q^{-1}x} = [m]_q x^{m-1} y^n$$

$$\delta_y(x^m y^n) = [n]_q x^m y^{n-1}$$

$$\mathbb{C}_q[x,y] \simeq \bigoplus_{n \in \mathbb{N}} V_{n,+}$$

Rk: $\mathrm{Rep}(U_q sl_2)$ is semisimple for generic q.

$$V_{n,-} = V_{n,+} \otimes V_{0,-}$$

Rk: Specialize q to some root of unity in $\mathbb{C}^\times$
Let d be the order of q^2.

- $K^d, E^d, F^d \in Z(U_q sl_2)$
  so they act as scalars on all irreps.

- If V is an irrep, $\dim V \leq d$
  Equality holds when
  $F^d|_V \neq 0$ or $E^d|_V \neq 0$ or $K^d|_V \neq \pm 1$

- The <u>small quantum group</u> is

$$U_q sl_2 / \langle K^d = 1,\ E^d = F^d = 0 \rangle$$

NOT semisimple!

Rk: (Casimir)

$$C = EF + \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2} = FE + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}$$

is central. In fact

$$Z(U_q sl_2) = \mathbb{C}[C]$$

Generalization to higher rank alg-s

$U_q\mathfrak{g}$ gen by $\{E_i, F_i, K_i\}_{i=1}^{r}$

$d_i := \frac{(\alpha_i, \alpha_i)}{2}$

$b_{ij} := d_i a_{ij}$

$$K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j$$

$$K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

+ Serre rel.s

$$\Delta K_i = K_i \otimes K_i$$

$$\Delta E_i = E_i \otimes K_i + 1 \otimes E_i$$

$$\Delta F_i = F_i \otimes 1 + K_i^{-1} \otimes F_i$$

Rep thry: Pick $\lambda \in P$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r) \in \{\pm 1\}^r$

Let $\mathbb{C}_{\lambda,\varepsilon}$ be the 1-dim $U_q\mathfrak{b}$ rep where K_i acts as $\varepsilon_i q^{(\lambda, \alpha_i)}$

Then
$$M_{\lambda,\varepsilon} := U_q\mathfrak{g} \otimes_{U_q\mathfrak{b}} \mathbb{C}_{\lambda,\varepsilon}$$

This has a unique simple quotient $L_{\lambda,\varepsilon}$

Thm: $L_{\lambda,\varepsilon}$ is fin. dim. $\Leftrightarrow \lambda \in P^+$

Facts
- $L_{\lambda,\varepsilon} = L_{\lambda,+} \otimes \mathbb{C}_{0,\varepsilon}$
- $\mathrm{ch}\, L_{\lambda,\varepsilon} = \mathrm{ch}\, L_\lambda$
- $\mathrm{Rep}(U_q\mathfrak{g})$ is semisimple, and

$$\mathrm{Rep}(U_q\mathfrak{g}) = \bigoplus_{\varepsilon \in \{\pm 1\}^r} \mathrm{Rep}_\varepsilon(U_q\mathfrak{g})$$

Schur-Weyl duality:

Classical: $GL_n \mathbb{C} \curvearrowright (\mathbb{C}^n)^{\otimes N} \circlearrowleft \mathbb{C}[S_n]$

Quantum: $U_q\mathfrak{sl}_n \curvearrowright (\mathbb{C}^n)^{\otimes N} \circlearrowleft \mathcal{H}_N(q)$

$$\mathcal{H}_N(q) = \mathbb{C}[\text{Braid group}] / \langle \underbrace{(s_i - q)(s_i + 1)}_{=0} \rangle$$