Crystal bases

Given a fin-dim. $M \in \mathrm{Rep}\,\mathcal{U}_q(\mathfrak{g})$ for $\mathfrak{g}$ simple

$\Rightarrow$ M has a weight decomposition

and hence a character

$$\mathrm{ch}_q M = \sum_{\lambda \in P} (\dim_{\mathbb{C}(q)} M_\lambda) e^\lambda$$

But how can we determine $\mathrm{ch}_q M$?

Thm (Lusztig): For q generic, we have

$$\mathrm{ch}_q V_q(\lambda) = \mathrm{ch}\, V(\lambda)$$

$\mathrm{ch}\, V(\lambda)$ $\leftarrow$ simple $\mathcal{U}(\mathfrak{g})$-module of h.w. $\lambda \in P^+$

$V_q(\lambda)$: simple $\mathcal{U}_q(\mathfrak{g})$-module h.w. $\lambda \in P$

$\Rightarrow$ What value of q might be good then?

Kashiwara: "Just use $q=0$" (Intuitively speaking)

$\hookrightarrow$ Crystal bases

Lemma: Let $M \in \mathrm{Rep}\, \mathcal{U}_q(\mathfrak{g})$ fin-dim., consider

$u \in M_\lambda$ $(\lambda \in P)$ and fix $\alpha_i \in \Pi$ (simple root)

$\Rightarrow \exists!\, u_1, \dots, u_N \in M$:

$$(*) \qquad u = \sum_{k=0}^{N} F_i^{(k)} u_k \quad \wedge \quad u_k \in M_{\lambda + k\alpha_i}$$

$\rightarrow$ Kashiwara operators:

Def: Given $\alpha_i \in \Pi_i$, define operators

$$\tilde{E}_i u = \sum_{k=1}^{N} F_i^{(k-1)} u_k, \qquad \tilde{F}_i u = \sum_{k=0}^{N} F_i^{(k+1)} u_k$$

where u_k are from $(*)$.

Def: Let $A_0 = \mathbb{Q}[q]_q = \{f/h \mid h(0) \neq 0\}$.

A free submodule $\mathcal{L} \subseteq M$ is a

crystal lattice if

(i) $\mathcal{L} \otimes_{A_0} \mathbb{Q}(q) \cong M$

(ii) $\mathcal{L} = \bigoplus_{\lambda \in P} \mathcal{L}_\lambda \, ; \, \mathcal{L}_\lambda := \mathcal{L} \cap M_\lambda$

(iii) $\tilde{E}_i \mathcal{L} \subset \mathcal{L} \, , \, \tilde{F}_i \mathcal{L} \subset \mathcal{L} \quad \forall \alpha_i \in \Gamma$

Def: A crystal basis for M is

a tuple $(\mathcal{L}, \mathcal{B})$ s.t.

(i) $\mathcal{L}$ is a crystal lattice of M

(ii) $\mathcal{B}$ is a $\mathbb{Q}$-basis of $\mathcal{L}/q\mathcal{L}$

(iii) $\mathcal{B} = \bigsqcup_{\lambda \in P} \mathcal{B}_\lambda \, ; \, \mathcal{B}_\lambda := \mathcal{B} \cap (\mathcal{L}_\lambda / q\mathcal{L}_\lambda)$

(iv) $\tilde{E}_i \mathcal{B} \subset \mathcal{B} \cup \{0\} ; \; \tilde{F}_i \mathcal{B} \subset \mathcal{B} \cup \{0\} \quad \forall \alpha_i \in \Gamma$

(v) $\forall b, b' \in \mathcal{B}, \alpha_i \in \Gamma$: $\quad \tilde{F}_i b = b' \iff b = \tilde{E}_i b'$

Thm: Crystal bases always exist and are unique.

Thm: Let $M \in \operatorname{Rep} \mathcal{U}_q(\mathfrak{g})$ be fin-dim. and $(\mathcal{L}, \mathcal{B})$ be the crystal basis of M. Then

$$ch_q M = \sum_{\lambda \in P} (\# \mathcal{B}_\lambda) e^\lambda$$

Def: Let $(\mathcal{L}, \mathcal{B})$ be the crystal basis of M.

The crystal graph of M has vertices $\mathcal{B}$ and an edge $b \xrightarrow{F_i} b'$ iff $\tilde{F}_i b = b'$

Example 1: Let $m \in \mathbb{N}_0$. Consider the $\mathcal{U}_q(\mathfrak{sl}_2)$-module $V(m)$ w/ basis
$B = \{u, Fu, \dots, F^{(m)}u\}$ s.t.

$$Eu = 0, \quad Ku = q^m u$$

Define $\mathcal{L}(m) := \bigoplus_{k=0}^{m} A_0 F^{(k)} u$, $\mathcal{B}(m) := \{ \overline{F^{(k)} u} \mid 0 \le k \le m \}$,
where $\bar{\ }$ is the image in $\mathcal{L}(m)/q\mathcal{L}(m)$

Then $(\mathcal{L}(m), \mathcal{B}(m))$ is the crystal basis.

The crystal graph is then

$$\bar{u} \xrightarrow{F} \overline{Fu} \xrightarrow{F} \overline{F^{(2)}u} \xrightarrow{F} \dots \xrightarrow{F} \overline{F^{(m)}u}$$

Example 2: Consider $\mathcal{U}_q(\mathfrak{sl}_n)$ and $M = V(1,0,\dots,0)$, i.e.

$M = \langle v_1, \dots, v_n \rangle_{\mathbb{C}(q)}$, s.t.

$$E_i v_j = \begin{cases} v_i & ; \ j = i+1 \\ 0 & ; \ \text{otherwise} \end{cases} ; \quad F_i v_j = \begin{cases} v_{i+1} & ; \ j = i \\ 0 & ; \ \text{otherwise} \end{cases}$$

$$K_i^{\pm 1} v_j = \begin{cases} q^{\pm 1} v_i & ; \ j = i \\ q^{\mp 1} v_{i+1} & ; \ j = i+1 \\ v_j & ; \ \text{otherwise} \end{cases}$$

Then $\left(\bigoplus_{j=1}^{n} A_0 v_j, \{ \overline{v_1}, \dots, \overline{v_n} \} \right)$ is the crystal basis of M.

The crystal graph is

$$\begin{array}{ccccccccc} \overline{v_1} & \xrightarrow{F_1} & \overline{v_2} & \xrightarrow{F_2} & \overline{v_3} & \xrightarrow{F_3} & \cdots & \xrightarrow{F_{n-1}} & \overline{v_n} \\ \| & & \| & & \| & & & & \| \\ \boxed{1} & & \boxed{2} & & \boxed{3} & & & & \boxed{n} \end{array}$$

The tensor product rule

Thm: Given $M_1, M_2 \in \mathrm{Rep}\,\mathcal{U}_q(\mathfrak{g})$ fin-dim.,

let $(\mathcal{L}_i, B_i)$ be the crystal bases.

Then $(\mathcal{L}_1 \otimes_{A_0} \mathcal{L}_2, B_1 \times B_2)$ is the crystal basis

of $M_1 \otimes M_2$. We denote it by $B_1 \otimes B_2$.

The Kashiwara operators act as follows:

$$\tilde{F}_i(b \otimes b') = \begin{cases} \tilde{F}_i b \otimes b' & \varphi_i(b) > \varepsilon_i(b') \\ b \otimes \tilde{F}_i b' & \varphi_i(b) \leq \varepsilon_i(b') \end{cases}$$

$$\tilde{E}_i(b \otimes b') = \begin{cases} \tilde{E}_i b \otimes b' & \varphi_i(b) \geq \varepsilon_i(b') \\ b \otimes \tilde{E}_i b' & \varphi_i(b) < \varepsilon_i(b') \end{cases}$$

where

$$\varphi_i(b) := \max\{k \mid \tilde{F}_i^k b \neq 0\},$$

$$\varepsilon_i(b) := \max\{k \mid \tilde{E}_i^k b \neq 0\}$$

Rmk: If $b\in(B_1)_\lambda$, $b'\in(B_2)_\mu$, then $(b_1,b_2)\in(B_1\otimes B_2)_{\lambda+\mu}$.

Example 3: Consider $\mathcal{U}_q(\mathfrak{sl}_2)$ and the crystal basis $(\mathcal{L}(2),\mathcal{B}(2))$ for $V(2)\in\mathrm{Rep}(\mathcal{U}_q(\mathfrak{sl}_2))$.

The crystal graph of $\mathcal{B}(2)\otimes\mathcal{B}(2)$ is

highest weight 4

$$
\begin{tikzcd}
\bar{u}\otimes\bar{u} \arrow[r, "F"] & \overline{Fu}\otimes\bar{u} \arrow[r, "F"] & \overline{F^{(2)}u}\otimes\bar{u} \arrow[d, "F"] \\
\bar{u}\otimes\overline{Fu} \arrow[r, "F"] & \overline{Fu}\otimes\overline{Fu} \arrow[d] & \overline{F^{(2)}u}\otimes\overline{Fu} \arrow[d, "F"] \\
\bar{u}\otimes\overline{F^{(2)}u} & \overline{Fu}\otimes\overline{F^{(2)}u} & \overline{F^{(2)}u}\otimes\overline{F^{(2)}u}
\end{tikzcd}
$$

highest weight 2

highest weight 0

$$\Longrightarrow \quad V(2)\otimes V(2)\cong V(4)\oplus V(2)\oplus V(0)$$

Example 4: Consider $\mathcal{U}_q(\mathfrak{sl}_3)$ and $V = V(1,0,0)$.

The crystal graph of $V \otimes V$ is

highest weight $(2,0,0)$ highest weight $(1,1,0)$

$$\begin{tikzcd}
\bar{v}_1 \otimes \bar{v}_1 \arrow[d, "F_1"'] & \bar{v}_1 \otimes \bar{v}_2 \arrow[r, "F_2"] & \bar{v}_1 \otimes \bar{v}_3 \arrow[d, "F_1"] \\
\bar{v}_2 \otimes \bar{v}_1 \arrow[r, "F_1"] \arrow[d, "F_2"'] & \bar{v}_2 \otimes \bar{v}_2 \arrow[d, "F_2"] & \bar{v}_2 \otimes \bar{v}_3 \\
\bar{v}_3 \otimes \bar{v}_1 \arrow[r, "F_1"] & \bar{v}_3 \otimes \bar{v}_2 \arrow[r, "F_2"] & \bar{v}_3 \otimes \bar{v}_3
\end{tikzcd}$$

$$\Rightarrow \quad V \otimes V \cong V(2,0,0) \oplus V(1,1,0)$$

$$\cong \operatorname{Sym}^2 V \oplus \Lambda^2 V$$

Rmk: For tensors of form $b = b_1 \otimes \cdots \otimes b_N \in B_1 \otimes \cdots \otimes B_N$ and $i \in I$, do as follows: Write $(\underbrace{----\dots}_{\varepsilon_i(b_1)}, \underbrace{+++\dots}_{\varphi_i(b_1)}, \dots, \underbrace{---\dots}_{\varepsilon_i(b_N)}, \underbrace{+++\dots}_{\varphi_i(b_N)})$ and cancel all $(+-)$ pairs. This yields $i\text{-sgn}(b) := (----\dots, +++\dots)$

Then $\tilde{F}_i$ acts on b_k corresponding to left-most $+$ in $i\text{-sgn}(b)$ and $\tilde{E}_i$ acts on b_j corresponding to right-most $-$ in $i\text{-sgn}(b)$.

Examples: Consider $\mathcal{U}_q(\mathfrak{sl}_2)$, $V(3)^{\otimes 4}$ and

$$b = \overline{F u} \otimes \overline{F^{(3)} u} \otimes \overline{F^{(2)} u} \otimes \overline{F u} \Rightarrow (-++, ---, --+, -++)$$

$$i\text{-sgn}(b) = (-, -, -\overset{}{\underset{E}{-}}, \overset{F}{+}+)$$

$$\Rightarrow \tilde{F} b = \overline{F u} \otimes \overline{F^{(3)} u} \otimes \overline{F^{(2)} u} \otimes \overline{F^{(2)} u}$$

$$\tilde{E} b = \overline{F u} \otimes \overline{F^{(3)} u} \otimes \overline{F\ \ u} \otimes \overline{F\ \ u}$$

For further details on this, see Bump-Schilling section 2.4
[Disclaimer: Their tensor product rule is flipped, section 2.3]

Alternatively: Remark 2.1.2 in 'Crystal graphs for representations of the q-analogue of classical Lie algebras.' by Kashiwara-Nakashima for type A

$\mathcal{U}_q(\mathfrak{gl}_n)$ - crystals and Tableaux

Given $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ how do we get the crystal graph of $V(\lambda)$?

<u>Idea:</u> Embed SSYT into $\mathcal{B}^{\otimes|\lambda|}$, where $\mathcal{B}$ is the crystal corresponding to natural rep. (See example 2) and apply tensor product rule

<u>Def:</u> The Far Eastern Reading of a tableau T of shape λ goes column-wise top-bottom, right-left.

<u>Example 6:</u> $$FER\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}\right) = (2, 4, 1, 3)$$

<u>Thm/Def:</u> Let $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n \geq 0)$ be a partition.

The set $\mathcal{B}(\lambda)$ of SSYT of shape λ forms the crystal graph of $V_q(\lambda)$ with the Kashiwara operators acting as follows for $T \in \mathcal{B}(\lambda)$

$$\tilde{F}_i T = \tilde{F}_i\big(\overbrace{\boxed{b_1} \otimes \ldots \otimes \boxed{b_{|\lambda|}}}^{\in \mathcal{B}^{\otimes|\lambda|}}\big); \quad \text{where } (b_1, \ldots, b_{|\lambda|}) = FER(T)$$

$$\tilde{E}_i T = \tilde{E}_i\big(\boxed{b_1} \otimes \ldots \otimes \boxed{b_{|\lambda|}}\big);$$

Example 7:

$\lambda = (2,2) \quad \mathfrak{g} = \mathfrak{gl}_3$

$$\begin{tikzcd}[column sep=small]
 & \begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array} \arrow[dr, "F_2"] & & \\
 & & \begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array} \arrow[dl, "F_1"'] \arrow[dr, "F_2"] & \\
 & \begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline\end{array} \arrow[dr, "F_2"] & & \begin{array}{|c|c|}\hline 1&1\\\hline 3&3\\\hline\end{array} \arrow[dl, "F_1"'] \\
 & & \begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline\end{array} \arrow[dl, "F_1"'] & \\
 & \begin{array}{|c|c|}\hline 2&2\\\hline 3&3\\\hline\end{array} & &
\end{tikzcd}$$

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array} = \boxed{1}\otimes\boxed{2}\otimes\boxed{1}\otimes\boxed{2}$$

$$\boxed{1}\otimes\boxed{3}\otimes\boxed{1}\otimes\boxed{2} = \begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array}$$

$$\begin{array}{|c|c|}\hline 1&1\\\hline 3&3\\\hline\end{array} = \boxed{1}\otimes\boxed{3}\otimes\boxed{1}\otimes\boxed{3}$$

$$\boxed{2}\otimes\boxed{3}\otimes\boxed{1}\otimes\boxed{2} = \begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline\end{array}$$

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline\end{array} = \boxed{2}\otimes\boxed{3}\otimes\boxed{1}\otimes\boxed{3}$$

$$\begin{array}{|c|c|}\hline 2&2\\\hline 3&3\\\hline\end{array} = \boxed{2}\otimes\boxed{3}\otimes\boxed{2}\otimes\boxed{3}$$

Def: For a partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_n \geq 0)$ let Y be its Young diagram and $j_1, \dots, j_r \in [n]$. We define

$$Y[j_1, \dots, j_r] := \begin{cases} Y[j_1, \dots, j_{r-1}] + \text{box in } j_r\text{-th row} & ; \text{ if it remains a partition} \\ \emptyset & ; \text{ otherwise} \end{cases}$$

Thm: Let $\lambda, \lambda' \in P^+$. Then we have

$$B(\lambda) \otimes B(\lambda') \cong \overbrace{\bigoplus_{\boxed{j_1} \otimes \cdots \otimes \boxed{j_N} \in B(\lambda')}}^{\text{as connected components}} B(\overbrace{Y_\lambda[j_1, \dots, j_r]}^{\text{Diagram corresponding to } \lambda.})$$

$\boxed{\text{This is the Littlewood-Richardson-rule.}}$

Ex: $\mathfrak{g} = \mathfrak{gl}_3$, $\lambda = (2,1,0)$, $\lambda' = (2,0,0)$

$$SSYT(\lambda') = \{ \boxed{1|2} = \boxed{2} \otimes \boxed{1}, \boxed{1|3} = \boxed{3} \otimes \boxed{1}, \boxed{2|3} = \boxed{3} \otimes \boxed{2}, \boxed{1|1} = \boxed{1} \otimes \boxed{1}, \boxed{3|3} = \boxed{3} \otimes \boxed{3}, \boxed{2|2} = \boxed{2} \otimes \boxed{2}, \}$$

$Y_\lambda[2,1] = (3,2,0) =$,

$Y_\lambda[3,1] = (3,1,1) =$

$Y_\lambda[1,1] = (4,1,0) =$

$Y_\lambda[3,3] =$ X

$Y_\lambda[3,2] = (2,2,1) =$

$Y_\lambda[2,2] =$ X

$$\Rightarrow V(\lambda) \otimes V(\lambda') \cong V(3,2) \oplus V(3,1,1) \oplus V(2,2,1) \oplus V(4,1,0)$$

References:

Introduction to quantum groups and crystal bases.; Hong, Jin, Kang, Seok-Jin; 2002

Crystal bases. Representations and combinatorics.; Bump, Daniel; Schilling, Anne; 2017