Lusztig's Canonical bases

Setup: $\mathfrak{g}$ of type ADE $\rightsquigarrow$ $\mathcal{U}_q(\mathfrak{g})$ quantised UEA

Root data: $\underset{\text{simple}}{\Gamma} \subset \underset{\text{positive}}{\Delta_+} \subset \Delta$; W: Weyl group

We say:

α_i, α_j adjacent $\Leftrightarrow$ their corresponding nodes in the Dynkin diagram are adjacent

$:\Leftrightarrow \alpha_i \longrightarrow \alpha_j$

<u>Construction:</u> (Tingley 2017)

In $\mathcal{U}_q(\mathfrak{g})$ we have generators

$K_i \text{"}\hat{=}\text{"} \alpha_i^\vee \qquad E_i \text{"}\hat{=}\text{"} \alpha_i \qquad F_i \text{"}\hat{=}\text{"} -\alpha_i \qquad ; \alpha_i \in \Gamma$

But what about other roots?

<u>Thm:</u> Fix a reduced expression $w_0 = s_{i_1} \cdot \ldots \cdot s_{i_N}$

$$\Delta_+ = \{ \alpha_{i_1}, s_{i_1}(\alpha_{i_2}), s_{i_1}s_{i_2}(\alpha_{i_3}), \ldots, s_{i_1} \cdots s_{i_{N-1}}(\alpha_{i_N}) \}$$

$\Rightarrow$ Enumerate Δ_+ via reflections "on $\mathcal{U}_q(\mathfrak{g})$"

Braid group action

Def: On $\mathcal{U}_q(\mathfrak{g})$ for $\alpha_i \in \Pi$ define the automorphism T_i of $\mathcal{U}_q(\mathfrak{g})$ via

$$T_i(F_j) := \begin{cases} F_j & ; \alpha_i \not\sim \alpha_j \\ F_j F_i - q F_i F_j & ; \alpha_i \text{—} \alpha_j \\ -K_j^{-1} E_j & ; \alpha_i = \alpha_j \end{cases}$$

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$$T_i(K_j) := \begin{cases} K_j & ; \alpha_i \not\sim \alpha_j \\ K_i K_j & ; \alpha_i \text{—} \alpha_j \\ K_j^{-1} & ; \alpha_i = \alpha_j \end{cases}$$

The T_i satisfy the braid relations

$$T_iT_j = T_jT_i \quad (\alpha_i \not- \alpha_j)$$

$$T_iT_jT_i = T_jT_iT_j \quad (\alpha_i - \alpha_j)$$

Def: Denote $\underline{i} = (i_1, \dots, i_N)$ for the reduced expression $w_0 = s_{i_1} \cdots s_{i_N}$

We define

$$F_{\underline{i},\beta_1} := F_{i_1}$$

$$F_{\underline{i},\beta_2} := T_{i_1} F_{i_2}$$

$$F_{\underline{i},\beta_3} := T_{i_1} T_{i_2} F_{i_3}$$

$$\vdots$$

$$F_{\underline{i},\beta_N} := T_{i_1} \cdots T_{i_{N-1}} F_{i_N}$$

Rmk: In fact $\mathrm{wt}(F_{\underline{i},\beta_k}) = -s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$

Ex: $\mathfrak{g} = \mathfrak{sl}_3$

1. $w_0 = s_1 s_2 s_1$. Then $i = (1,2,1)$. We have

$$(\beta_1, \beta_2, \beta_3) = (\alpha_1, \alpha_1 + \alpha_2, \alpha_2) \quad \text{and}$$

$$(F_{i,\beta_1}, F_{i,\beta_2}, F_{i,\beta_3}) = (F_1, F_2F_1 - qF_1F_2, F_2)$$

2. $w_0 = s_2 s_1 s_2$. Then $i = (2,1,2)$. We have

$$(\beta_1, \beta_2, \beta_3) = (\alpha_2, \alpha_1 + \alpha_2, \alpha_1) \quad \text{and}$$

$$(F_{i,\beta_1}, F_{i,\beta_2}, F_{i,\beta_3}) = (F_2, F_1F_2 - qF_2F_1, F_1)$$

Def: For $a \in \mathbb{N}_0^N$ and $w_0 = s_{i_1} \cdots s_{i_N}$ define

$$F_i^a := F_{i,\beta_1}^{(a_1)} F_{i,\beta_2}^{(a_2)} \cdots F_{i,\beta_N}^{(a_N)} \quad \text{where}$$

$$F_{i,\beta_k}^{(a_k)} := \frac{F_{i,\beta_k}^{a_k}}{[a_k]_q!}$$

Thm: (PBW) Fix $w_0 = s_{i_1} \cdots s_{i_N}$. The set

$$B_i := \{ F_i^a \mid a \in \mathbb{N}_0^N \} \text{ is a } \mathbb{Q}(q)\text{-basis of } \mathcal{U}_q^-(\mathfrak{g})$$

But are different PBW bases linked?

<u>Thm / Def:</u> Fix $w_0 = s_{i_1} \cdots s_{i_N}$. Consider

$$\mathcal{L} = \operatorname{span}_{\mathbb{Z}[q]} B_i$$

<u>Then:</u>

(i) $\mathcal{L}$ is independent of the reduced expression i

(ii) The basis $B_i + q\mathcal{L}$ in $\mathcal{L}/q\mathcal{L}$ is independent of i

<u>Rmk:</u> Suppose i and i' are connected via a braid move at $(\ldots, i_k, i_{k+1}, i_{k+1}, \ldots)$

Then we have $F_i^a \equiv F_{i'}^{a'} \mod q\mathcal{L}$ where

$a'_v = a_v$ for $v \notin \{k, k+1, k+2\}$ and

$$a'_k = \max\{a_{k+1}, a_{k+1} + a_{k+2} - a_k\}$$

$$a'_{k+1} = \min\{a_k, a_{k+2}\}$$

$$a'_{k+2} = \max\{a_{k+1}, a_{k+1} + a_k - a_{k+2}\}$$

Ex: $\mathfrak{g} = \mathfrak{sl}_3$ $\quad i = (1,2,1)$, $i' = (2,1,2)$

$$F_{i,\beta_1}^{(2)} F_{i,\beta_2}^{(3)} F_{i,\beta_3}^{(4)} \equiv F_{i',\beta_1}^{(5)} F_{i',\beta_2}^{(2)} F_{i',\beta_3}^{(3)} \mod q\mathcal{L}$$

Recall: The bar involution $\bar{\cdot}: \mathcal{U}_q(\mathfrak{g}) \to \mathcal{U}_q(\mathfrak{g})$

$$\overline{E_i} = E_i; \quad \overline{F_i} = F_i; \quad \overline{K_i} = K_i^{-1}; \quad \overline{q} = q^{-1}$$

Thm: Fix reduced expression i and $a \in \mathbb{N}_0^N$. Then

$$\overline{F_i^a} = F_i^a + \sum_{a' \prec a} p_{a'}^{a}(q) F_i^{a'} \quad \text{where}$$

$p_{a'}^{a}(q) \in \mathbb{Z}[q, q^{-1}]$

where $a' \prec a \iff \mathrm{wt}(F_i^a) = \mathrm{wt}(F_i^{a'})$

and $a' >_{lex} a \ \wedge \ a' >_{invlex} a$

here, $a' >_{invlex} a$ means, if $a = (a_1, \ldots, a_N)$, $a' = (a'_1, \ldots, a'_N)$,

$$(a'_N, \ldots, a'_1) >_{lex} (a_N, \ldots, a_1)$$

Thm: There is a unique basis B of $U_q^-(g)$ s.t.

(i) $B \subseteq \mathcal{L}$

(ii) $B + q\mathcal{L} = B_i + q\mathcal{L}$ in $\mathcal{L}/q\mathcal{L}$ for some reduced expression i

(iii) $\forall b \in B: \bar{b} = b$

Additionally the base-change $B_i \to B$ is unit-triangular with coefficients in $q\mathbb{Z}[q]$

We call B the <u>canonical basis</u>

Proof: Induction on $\preceq$ for a fixed $i = (i_1, \dots, i_N)$.

1st step: Minimal elements are in B.

2nd step: Assume B is constructed for all $a' < a$.

$$\text{Then} \quad \overline{F_i^a} \overset{\text{Induction}}{\overset{!}{=}} F_i^a + \sum_{a' < a} p_{a'}^a(q)\, b^{a'} \; ; \; p_{a'}^a(q) \in \mathbb{Z}[q, q^{-1}]$$

$$\text{But} \quad \overline{\overline{F_i^a}} = F_i^a$$

$$\Rightarrow \quad F_i^a = F_i^a + \sum_{a'<a} p^a_{a'}(q)\, b^{a'} + \sum_{a'<a} p^a_{a'}(q^{-1})\, b^{a'}$$

$$\Rightarrow \quad p^a_{a'}(q) = -\, p^a_{a'}(q^{-1})$$

$$\Rightarrow \quad p^a_{a'}(q) = q f^a_{a'}(q) - q^{-1} f^a_{a'}(q^{-1}) \text{ ; for some } f^a_{a'}(q) \in \mathbb{Z}[q]$$

Set $b^a := F_i^a + \sum_{a'<a} q f^a_{a'}(q)\, b^{a'}$

Then:

$$\overline{b^a} = F_i^a + \sum_{a'<a} \left(q f^a_{a'}(q) - \underline{q^{-1} f^a_{a'}(q^{-1})}\right) b^{a'} + \sum_{a'<a} \underline{q^{-1} f^a_{a'}(q^{-1})}\, b^{a'}$$
$$= b^a$$

$\square$

<u>Example:</u> For $\mathfrak{g} = \mathfrak{sl}_3$ we have

$$B = \left\{ F_1^{(a)} F_2^{(b)} F_1^{(c)},\ F_2^{(a)} F_1^{(b)} F_2^{(c)} \mid b \geq a + c \right\}$$

where
$$F_1^{(a)} F_2^{(a+c)} F_1^{(c)} = F_2^{(c)} F_1^{(a+c)} F_2^{(a)}$$

is considered once.

Properties of $\mathcal{B}$:

(i) $b, b' \in \mathcal{B} \Rightarrow b \cdot b' = \sum_{b'' \in \mathcal{B}} c_{b,b'}^{b''} b''$ w/ $c_{b,b'}^{b''} \in \mathbb{N}[q, q^{-1}]$

(ii) Given $\lambda \in P^+ \Rightarrow \mathcal{B}(\lambda) := \{ b \in \mathcal{B} \mid b . v_\lambda \neq 0 \}$ is a basis of $V(\lambda)$ $\Rightarrow$ $V(\lambda)$ has structure constants in $\mathbb{N}[q, q^{-1}]$

We call $\mathcal{B}(\lambda)$ the canonical basis of $V(\lambda)$

(iii) Given $\lambda, \mu \in P^+$, and $b \in \mathcal{B}(\lambda)$, $b' \in \mathcal{B}(\mu)$

$\Rightarrow \exists !\, b \diamond b' \in \mathcal{U}_q^-(\mathfrak{g}) \otimes \mathcal{U}_q^-(\mathfrak{g})$

s.t. $b \diamond b' = b \otimes b' + \sum_{\substack{x \in \mathcal{B}(\lambda) \\ y \in \mathcal{B}(\mu) \\ (x,y) < (b,b')}} q f_{x,y}(q)\, b_x \otimes b_y$, $f_{x,y}(q) \in \mathbb{Z}[q]$

for an order $<$ and $\{ b \diamond b' \mid b \in \mathcal{B}(\lambda), b' \in \mathcal{B}(\mu) \}$ satisfies similar properties to the canonical bases.

For details on (iii), see Theorem 24.3.3 (in section 24.3) in Lusztig's book.

References:

P. W. Tingley, Elementary construction of Lusztig's canonical basis (2017); MR3649181

G. Lusztig, Introduction to quantum groups (2010); MR2759715
—> Chapters 14 and Part IV (in particular, Chapters 24,25 and 27)