

Quantum Groups

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Disclaimer: Nothing presented is claimed to be original or accurate.

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1 Background

1.1 Preliminaries on Hopf algebras

We begin by defining coalgebras, the dual to unital associative algebras:

Definition 1.1. A *coalgebra* (C, Δ, ε) over a field k is a k -vector space C equipped with two morphisms: **coproduct** $\Delta : C \rightarrow C \otimes C$ and the **counit** $\varepsilon : C \rightarrow k$ such that the following diagrams commute:

$$\begin{array}{ccc}
 C & \xrightarrow{\Delta} & C \otimes C \\
 \Delta \downarrow & & \downarrow id_C \otimes \Delta \\
 C \otimes C & \xrightarrow{\Delta \otimes id_C} & C \otimes C \otimes C
 \end{array}$$

$$\begin{array}{ccccc}
 & & C & & \\
 & \Delta \swarrow & \downarrow & \searrow \Delta & \\
 C \otimes C & \xrightarrow{id_C \otimes \varepsilon} & C \otimes k \cong C & \cong k \otimes C & \xleftarrow{\varepsilon \otimes id_C} C \otimes C
 \end{array}$$

The property characterized by the commutativity of the first of the above diagrams is referred to as **coassociativity** since it is dual to the diagram that defines the associative property of an associative algebra. The second diagram is dual to the diagram that asserts the existence of a multiplicative identity in a unital algebra. Let's look at some quick examples of coalgebras:

Example 1.2. Let S be a nonempty finite set and fix a field k . Define kS to be the k -vector space with basis S . Define $\Delta : kS \rightarrow kS$ by $\Delta(s) = s \otimes s$ and define $\varepsilon : kS \rightarrow k$ by $\varepsilon(s) = 1$ for all $s \in S$. Then $(kS, \Delta, \varepsilon)$ is a coalgebra.

Remark 1.3. This shows that even without specifying a bilinear product, every vector space can be equipped with a coalgebra structure.

Example 1.4. An important example of a coalgebra is the **divided power coalgebra**. Let H be a k -vector space with basis $\{c_n : n \in \mathbb{Z}_{\geq 0}\}$. One might note this resembles the ring $k[x]$. One can define a coalgebra structure on H by:

$$\Delta(c_n) = \sum_{i=0}^n c_i \otimes c_{n-i} \quad \text{and} \quad \varepsilon(c_n) = \delta_{0,n}$$

where $\delta_{i,j}$ is the Kronecker delta function.

We are now equipped to define a bialgebra:

Definition 1.5. A *bialgebra* is a tuple $(B, \nabla, \eta, \Delta, \varepsilon)$ such that (B, Δ, ε) is a coalgebra over a field k and (B, ∇, η) is a unital associative algebra with multiplication given by $\nabla : B \otimes B \rightarrow B$ and unit given by $\eta : k \rightarrow B$ such that either of the following equivalent conditions hold:

1. Δ and ε are algebra morphisms.
2. ∇ and η are coalgebra morphisms (such a morphism has the expected definition).

Remark 1.6. The definition of a bialgebra is traditionally given in terms of four commutative diagrams that express the necessary compatibility between the algebra and coalgebra structures. However, either of the two equivalent conditions above fully describe these diagrams and provide a more transparent view of their meaning. Each of these diagrams would describe the compatibility of one of the following pairs: ∇ and Δ , ∇ and ε , Δ and η , and η and ε .

Remark 1.7. One can check easily that the divided powers coalgebra can be made into a bialgebra. Another important example of a bialgebra is the **tensor algebra**, although its bialgebra structure is more complicated to describe and will not be mentioned here.

Definition 1.8. Given a coalgebra (C, Δ, ε) , an algebra (A, ∇, η) , and two k -linear maps $f, g : C \rightarrow A$, the **convolution** of f and g is the k -linear map $f \star g : C \rightarrow A$ defined by $c \mapsto (\nabla \circ (f \otimes g) \circ \Delta)(c)$.

With this, we are finally ready to define a Hopf algebra:

Definition 1.9. A **Hopf algebra** $(H, \nabla, \eta, \Delta, \varepsilon, S)$ is a bialgebra $(H, \nabla, \eta, \Delta, \varepsilon)$ equipped with a k -linear map $S : H \rightarrow H$ called an **antipode** such that $\text{id}_H \star S = S \star \text{id}_H = \eta \circ \varepsilon$.

Theorem 1.10. If a Hopf algebra H is cocommutative or commutative (so that $\Delta = \Delta^{op}$), S is an inversion (though it is not in general invertible).

Remark 1.11. As is the case with a bialgebra, the definition of a Hopf algebra is self-dual, so the dual of a Hopf algebra is always a Hopf algebra.

Example 1.12. The universal enveloping algebra $U(\mathfrak{sl}_2)$ has a Hopf algebra structure defined by $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$, and $S(x) = -x$. In fact, the universal enveloping algebra of a Lie algebra is always a Hopf algebra. Without constructing this explicitly, this can be seen by viewing $U(\mathfrak{g})$ as a quotient of the tensor algebra.

Definition 1.13. A Hopf algebra representation is a representation of its underlying associative algebra.

1.2 Monoidal categories

Definition 1.14. A category C is called monoidal if it is equipped with the following:

1. A bifunctor $(-) \otimes (-) : C \times C \rightarrow C$ called the tensor product
2. A left and right identity $1 \in C$ for \otimes
3. A functorial isomorphism $\alpha : \bullet \otimes (\bullet \otimes \bullet) \rightarrow (\bullet \otimes \bullet) \otimes \bullet$ called the associativity isomorphism such that the following diagram commutes (called the pentagon axiom)

$$\begin{array}{ccc}
 & ((W \otimes X) \otimes Y) \otimes Z & \\
 \alpha_{W \otimes X, Y, Z} \swarrow & & \searrow \alpha_{W, X, Y \otimes 1_Z} \\
 (W \otimes X) \otimes (Y \otimes Z) & & ((W \otimes (X \otimes Y)) \otimes Z) \\
 \alpha_{W, X, Y \otimes Z} \downarrow & & \downarrow \alpha_{W, X \otimes Y,} \\
 W \otimes (X \otimes (Y \otimes Z)) & \xleftarrow{1_W \otimes \alpha_{X, Y, Z}} & W \otimes ((X \otimes Y) \otimes Z)
 \end{array}$$

Going forward, we will identify $(X \otimes Y) \otimes Z$ with $X \otimes (Y \otimes Z)$ and omit the associativity isomorphism so that we can simply write $X \otimes Y \otimes Z$.

The importance of the coproduct and the antipode come from how we define dual representations and tensor products of representations. We would like a notion of these that specializes to the familiar duals and tensor products in the cases of Lie groups and universal enveloping algebras.

Recall that the action of G on the dual of a (Lie) group representation V is given by $ad(x^{-1})$. Similarly, the dual of a Lie algebra representation is given by $ad(-x)$. This suggests that the antipode gives us a way to categorify the notion of a dual representation, and this is indeed the case. However, S^2 is not in general the identity, so we may use S or S^{-1} , and get two different notions of a dual when doing so. We briefly recall the definition of a dual object in a monoidal category.

Definition 1.15. *Let C be a monoidal category. An object L is a left dual of an object $V \in C$ (written $L = V^*$) if there exist morphisms:*

- *Coevaluation $\eta : I \rightarrow V \otimes L$*
- *Evaluation $\epsilon : L \otimes V \rightarrow I$*

and it satisfies the triangle identities:

$$(1_V \otimes \epsilon) \circ (\eta \otimes 1_V) = 1_V$$

$$(\epsilon \otimes 1_L) \circ (1_L \otimes \eta) = 1_L$$

*An object R is similarly a right dual of V (written $R = {}^*V$) if it satisfies the identities:*

$$(\epsilon \otimes 1_V) \circ (1_V \otimes \eta) = 1_V$$

$$(1_R \otimes \epsilon) \circ (\eta \otimes 1_R) = 1_R$$

Example 1.16. *Let (V, ρ_V) be a module over a Hopf algebra A . The left dual ρ_{V^*} of ρ_V is given by $\rho_{V^*}(a) = \rho_V(S(a))^*$ and the right dual is given by $\rho_{{}^*V}(a) = \rho_V(S^{-1}(a))^*$.*

Similarly, we can see that of the actions of a Lie group and a Lie algebra on a tensor product of representations can be defined using the coproduct. We quickly introduce some notation:

Given a Hopf algebra A and an element $a \in A$, the coproduct of a is of the form

$$\Delta(a) = \sum_i a_{(1)}^i \otimes a_{(2)}^i$$

for some $a_{(1)}^i, a_{(2)}^i \in A$. This is tedious to write, so we will often just write

$$\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$$

or even just

$$\Delta(a) = a_{(1)} \otimes a_{(2)}$$

where the summation is implied. This is called *Sweedler notation*.

Now, Let X and Y be modules over A . Let $a \in A$. The A -module structure of $X \otimes Y$ is given by

$$a.(x \otimes y) = \Delta(a)(x \otimes y) = a_{(1)}.x \otimes a_{(2)}.y$$

One may wonder when a left dual is also a right dual. It turns out that this is not always the case, but it is the case when our category is *braided*.

Definition 1.17. A monoidal category \mathcal{C} is called *braided* if the bifunctor \otimes is equipped with a natural isomorphism $c : \otimes \rightarrow \otimes^{op}$ such that the following diagrams commute:

$$\begin{array}{ccc} X \otimes Y \otimes Z & & \\ c_{X,Y} \otimes 1_Z \downarrow & \searrow^{c_{X,Y \otimes Z}} & \\ Y \otimes X \otimes Z & \xrightarrow{1_Y \otimes c_{X,Z}} & Y \otimes Z \otimes X \end{array}$$

$$\begin{array}{ccc} X \otimes Y \otimes Z & & \\ 1_X \otimes c_{Y,Z} \downarrow & \searrow^{c_{X \otimes Y,Z}} & \\ X \otimes Z \otimes Y & \xrightarrow{c_{X,Z} \otimes 1_Y} & Z \otimes X \otimes Y \end{array}$$

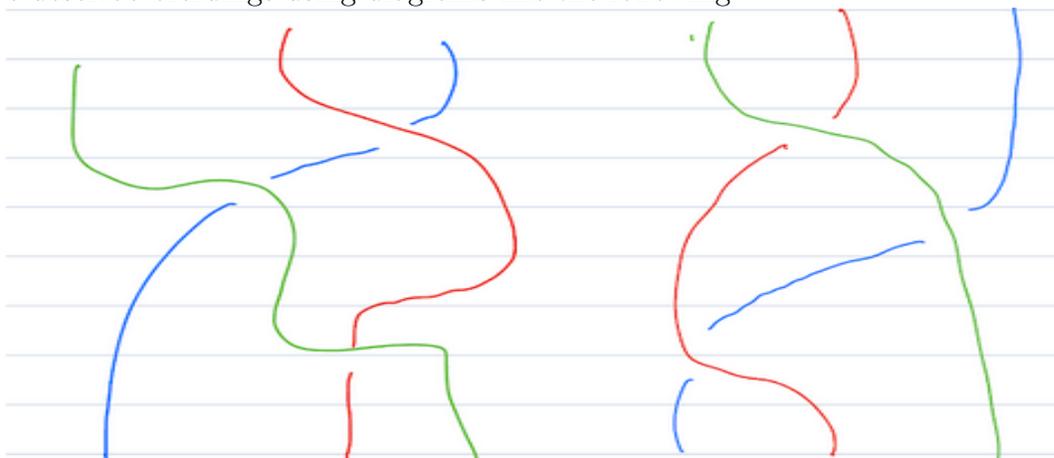
where \otimes^{op} is defined by $X \otimes^{op} Y = Y \otimes X$.

\mathcal{C} is called *symmetric* if $c = c^{-1}$.

2 The Yang-Baxter equation

2.1 The braid group B_n

We describe braidings using diagrams like the following:



where the diagram on the left corresponds to

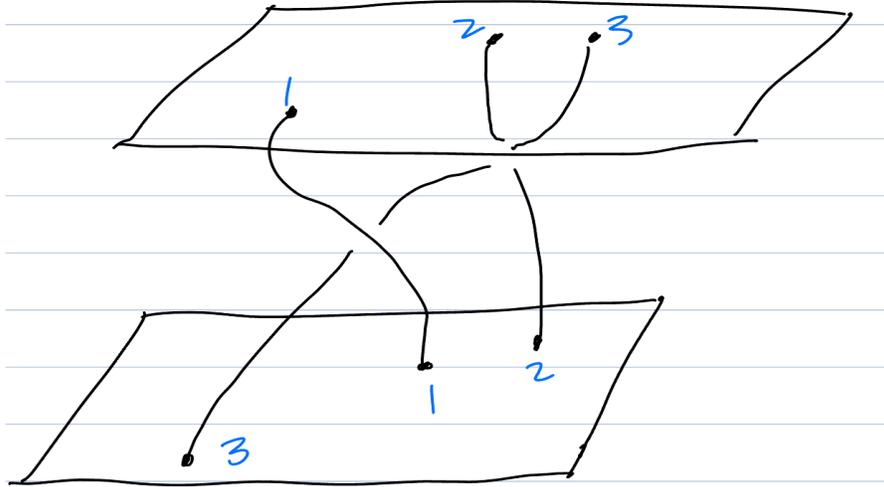
$$X \otimes Y \otimes Z \rightarrow X \otimes Z \otimes Y \rightarrow Z \otimes X \otimes Y \rightarrow Z \otimes Y \otimes X$$

and the diagram on the right corresponds to

$$X \otimes Y \otimes Z \rightarrow Y \otimes X \otimes Z \rightarrow Y \otimes Z \otimes X \rightarrow Z \otimes Y \otimes X$$

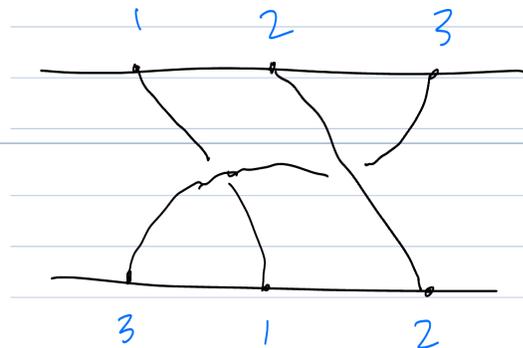
These two diagrams should be equivalent. We formalize this using the *braid group*

Definition 2.1. The braid group on n strings B_n is $\pi_1(\text{Sym}^n \mathbb{R}^2)$ where $\text{Sym}^n X$ denotes the n^{th} symmetric power of X , $X^{\otimes n}/S_n$.

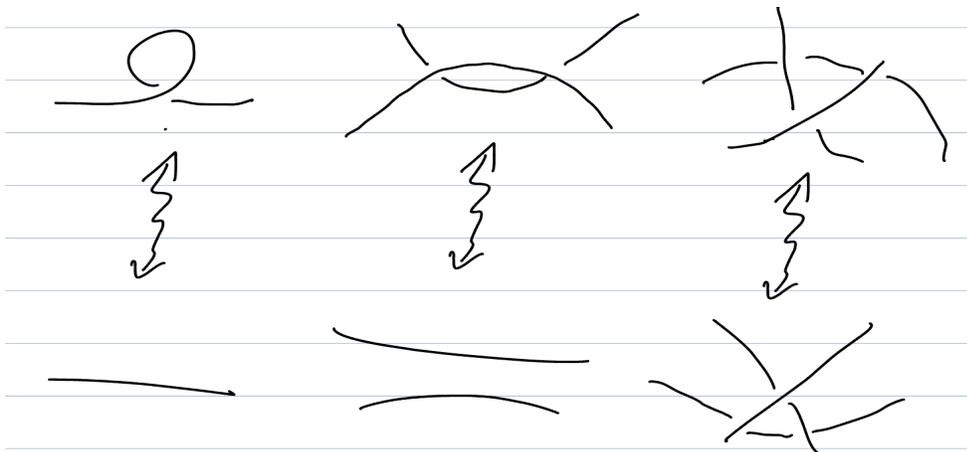


A representative of an element of B_3

We can draw this as if the points in \mathbb{R}^2 lie on a line and project it so that it looks like:



Definition 2.2. The Reidemeister moves of type I, II, and III are the following local moves on a link diagram:



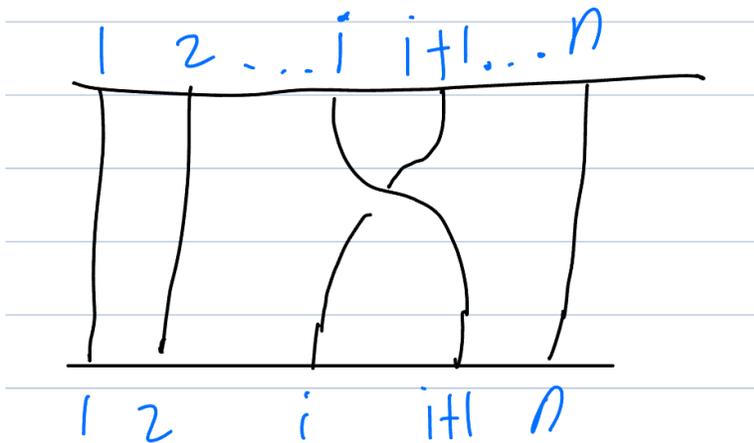
In the case of braid groups, we don't allow type I moves, since they never appear anyways.

Proposition 2.3. Any equivalent representatives of B_n are related by a sequence of Reidemeister moves II and III and ambient isotopies in \mathbb{R}^2 .

Theorem 2.4. B_n has the following presentation

$$\langle s_1, \dots, s_{n-1} \mid s_i s_j = s_j s_i \text{ if } |i - j| > 1, s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \rangle$$

where s_i corresponds to



Let V be a (finite dimensional) vector space and fix $R \in \text{Aut}(V \otimes V)$. Assign to each generator s_i of B_n the element

$$1 \otimes 1 \otimes \dots \otimes R \otimes \dots \otimes 1 \in V^{\otimes n}$$

where R appears in the i^{th} position.

Theorem 2.5. This extends to a representation of B_n on $V^{\otimes n}$ if and only if R satisfies the Yang-Baxter equation

$$(R \otimes 1)(1 \otimes R)(R \otimes 1) = (1 \otimes R)(R \otimes 1)(1 \otimes R)$$

which is often written as

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

where $R_{ij} = \phi_{ij}(R)$ is defined as

$$\phi_{12}(R(a \otimes b)) = a \otimes b \otimes 1, \phi_{13}(R(a \otimes b)) = a \otimes a \otimes b, \text{ and } \phi_{23}(R(a \otimes b)) = 1 \otimes a \otimes b$$

2.2 The Quantum group $U_q(\mathfrak{sl}_2)$

Definition 2.6. Let $q \in \mathbb{C} \setminus \{0, \pm 1\}$. We define **quantum \mathfrak{sl}_2** , denoted $U_q(\mathfrak{sl}_2)$, as the algebra generated by the symbols E, F, K, K^{-1} subject to the relations:

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \quad [E, F] = EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

Remark 2.7. The resemblance of this definition to that of $U(\mathfrak{sl}_2)$ somewhat illustrates how $U_q(\mathfrak{sl}_2)$ can be thought of as a deformation of $U(\mathfrak{sl}_2)$. In fact, setting $K = q^h$ we have that

$$\lim_{q \rightarrow 1} \frac{K - K^{-1}}{q - q^{-1}} = h$$

so as $q \rightarrow 1$ the relations of $U_q(\mathfrak{sl}_2)$ "deform" into those of $U(\mathfrak{sl}_2)$.

Remark 2.8. The set of monomials $\{F^k K^\ell E^m : k, m \in \mathbb{Z}_{\geq 0}, \ell \in \mathbb{Z}\}$ is a Poincaré-Birkhoff-Witt style basis of $U_q(\mathfrak{sl}_2)$.

Let's define a Hopf algebra structure on $U_q(\mathfrak{sl}_2)$ by defining the necessary morphisms on the generators:

$$\begin{aligned} \Delta(E) &= E \otimes 1 + K \otimes E, & \Delta(F) &= F \otimes K^{-1} + 1 \otimes F, & \Delta(K) &= K \otimes K \\ \varepsilon(E) &= \varepsilon(F) = 0, & \varepsilon(K) &= 1 \\ S(E) &= -K^{-1}E, & S(F) &= -FK, & S(K) &= K^{-1} \end{aligned}$$

Remark 2.9. Let $X, Y \in \text{Rep}(U_q\mathfrak{sl}_2)$. We notice that Δ is not cocommutative, so we shouldn't expect that $X \otimes Y$ and $Y \otimes X$ be isomorphic (as $U_q\mathfrak{sl}_2$ representations; they are clearly isomorphic as vector spaces).

Definition 2.10. Let $V \in \text{Rep}(U_q\mathfrak{sl}_2)$ and suppose q is not a root of unity. We say V is a type I representation if all the weights of K are integer powers of q . We say V is type II if not.

Theorem 2.11. If V is type II, there exists a type I representation V_+ such that $V \cong V_+ \otimes \chi$ where $\chi : U_q\mathfrak{sl}_2 \rightarrow \mathbb{C}$ is the character given by $\chi(E) = \chi(F) = 0$, $\chi(K) = -1$

Definition 2.12. Let $X, Y \in \text{Rep}(U_q\mathfrak{sl}_2)$ where q is not a root of unity (we will always assume this going forward). The **universal R matrix** $R \in \text{Hom}_{U_q\mathfrak{sl}_2}(X \otimes Y, X \otimes Y)$ is

$$R = q^{\frac{h \otimes h}{2}} \sum_{j=0}^{\infty} q^{\frac{j(j-1)}{2}} \frac{(q - q^{-1})^j}{[j]_q!} E^j \otimes F^j$$

where $[j]_q!$ is the q -factorial and $q^{\frac{h \otimes h}{2}}$ is an operator defined by $q^{\frac{h \otimes h}{2}}(x \otimes y) = q^{\lambda\mu/2}x \otimes y$ where x has weight λ and y has weight μ .

Remark 2.13. Since E and F are nilpotent, all terms in the summation above are 0 for sufficiently large j , so this is properly defined.

Theorem 2.14. Let $X, Y \in \text{Rep}(U_q(\mathfrak{sl}_2))$ and let $P : X \otimes Y \rightarrow Y \otimes X$ be the map given by $P(x \otimes y) = y \otimes x$. The map $c = P \circ R : X \otimes Y \rightarrow Y \otimes X$ is an isomorphism of $U_q(\mathfrak{sl}_2)$ representations. Moreover, R satisfies the Yang-Baxter equation.

Remark 2.15. Even though the category $\text{Rep}(U_q(\mathfrak{sl}_2))$ isn't symmetric, taking the tensor product of two representations is commutative. There is hope after all!

2.3 The Drinfeld Center

Definition 2.16. Fix a monoidal category C . The Drinfeld center $\mathcal{Z}(C)$ of C is the braided monoidal category of pairs (Z, φ^Z) where Z is an object in C and φ^Z is a functorial isomorphism parameterized by objects $X \in C$

$$\varphi_X^Z : Z \otimes X \xrightarrow{\sim} X \otimes Z$$

such that the following diagram commutes (we omit the associativity isomorphism and parentheses for simplicity)

$$\begin{array}{ccc} Z \otimes X_1 \otimes X_2 & \xrightarrow{\varphi_{X_1}^Z \otimes id} & X_1 \otimes Z \otimes X_2 \\ & \searrow \varphi_{X_1 \otimes X_2}^Z & \downarrow id \otimes \varphi_{X_2}^Z \\ & & X_1 \otimes X_2 \otimes Z \end{array}$$

The monoidal structure of $\mathcal{Z}(C)$ is given by $(Y, \varphi^Y) \otimes (Z, \varphi^Z) = (Y \otimes Z, \varphi^{Y \otimes Z})$ where $\varphi^{Y \otimes Z} = (\varphi^Y \otimes id_Z) \circ (id_Y \otimes \varphi^Z)$. The braiding $c_{(X, \varphi^X), (Y, \varphi^Y)}$ is given by φ_Y^X .

Despite what the notation suggests, there can be (and often are) multiple objects in $\mathcal{Z}(C)$ corresponding the same object of C . In particular, if C is braided we have a functor $i : C \rightarrow \mathcal{Z}(C)$ given by $Z \mapsto (Z, c_{Z, -})$ where $c_{z, -}$ is the half-braiding of Z in C . We also have a forgetful functor, which is a left adjoint to $i : \mathcal{Z}(C) \rightarrow C$ given by $(Z, \varphi^Z) \mapsto Z$ that composes to the identity. This implies that C is a full braided subcategory of $\mathcal{Z}(C)$, though as we will see, $\mathcal{Z}(C)$ is usually much larger.

Example 2.17. The Drinfeld center of $Rep(S_n)$ To do

We are mostly interested in the case where C is the category of (finite-dimensional) representations of a Hopf algebra H over \mathbb{C} . We note that $Z \otimes H \cong \text{Ind}_{\mathbb{C}}^H Z$. Then we have that the map φ_H^Z can be considered as a map

$$\text{Ind}_{\mathbb{C}}^H Z \rightarrow H \otimes Y$$

If we identify Z with the subalgebra $Z \otimes 1$ of $Z \otimes H$, by Frobenius reciprocity, φ_H^Z restricts to a map

$$\tau : Y \cong Y \otimes 1 \rightarrow H \otimes Y$$

Proposition 2.18. τ defines an H -comodule structure on Z (this is just the dual to an H -module structure on Z) where the coaction $Z \rightarrow H \otimes Z$ is given by

$$\tau(z) = z_{(-1)} \otimes z_{(0)}$$

where we have used the convention that negative indices in the Sweedler notation for an H -comodule Z indicate belonging to H and non-negative indices indicate belonging to Z .

Moreover, for $a \in H$ and $z \in Z$, the action and coaction satisfy the Yetter-Drinfeld compatibility condition

$$\tau(a.z) = a_{(1)} z_{(-1)} S(a_{(3)}) \otimes a_{(2)} z_{(0)}$$

where $a_{(1)} \otimes a_{(2)} \otimes a_{(3)} := (id \otimes \Delta) \circ \Delta(a) = ((\Delta \otimes id) \circ \Delta(a))$ and the last equality follows from associativity of H .

Definition 2.19. A Yetter-Drinfeld module Y is a module and comodule over a Hopf algebra H satisfying the Yetter-Drinfeld condition.

Theorem 2.20. $\mathcal{Z}(\text{Rep}(H))$ is equivalent to the category $YD(H)$ of Yetter-Drinfeld modules over H .

The idea of this proof will be that a half-braiding contains all the information needed to define a co-module structure on H and that a half-braiding defined by a co-module structure is a morphism in $\text{Rep}(H)$ if and only if it satisfies the Yetter-Drinfeld condition. We saw above how a half-braiding defines a coaction. Without going into the details, we will see quickly how a coaction defines the specified half-braiding. Let $Y \in \text{Rep}(H)$ be an H -comodule (and module) with coaction given by $\tau(y) = y_{(-1)} \otimes y_{(0)}$. Define for each $X \in \text{Rep}(H)$, $\varphi_X^Y(y \otimes x) = y_{(-1)}.x \otimes y_{(0)}$.

For φ to be a morphism of H -modules, we must have

$$\varphi_X^Y(h.(y \otimes x)) = h.(\varphi(y \otimes x))$$

which one checks is equivalent to the Yetter-Drinfeld condition.

By now, you are surely wondering, “given H , does there exist a Hopf algebra A such that the categories $\mathcal{Z}(\text{Rep}(H))$ and $\text{Rep}(A)$ are equivalent?” Since H is finite-dimensional, we can identify it with its dual H^* (we recall that in general, Hopf algebras are not self-dual). This means that an H -comodule structure uniquely specifies an H^* -module structure. Therefore, one might hope that $\mathcal{Z}(\text{Rep}(H))$ can be realized as a category of representations of an algebra over H and/or H^* .

Definition 2.21. The Drinfeld double (or quantum double) of a Hopf algebra H is the unique Hopf algebra $D(H)$ such that $YD(H) \cong \text{Rep}(D(H))$.

Proposition 2.22. The Drinfeld double exists.

As a vector space, we can identify $D(H)$ with $H \otimes H^*$. However, the algebra structure, called the *bicrossed product*, is more involved and will require a bit of explanation. We will start with the case of a finite group, where this generalizes the notion of a semidirect product, and see how this induces a product on group algebras.

2.4 The Drinfeld Double

What follows is likely full of careless indexing errors and various other mistakes. Please contact me if you find any.

Definition 2.23. A pair of groups (H, K) is called *matched* if there exists a right action $H \times K \rightarrow H$, $(h, k) \mapsto h \triangleleft k$ and a left ‘action’ $H \times K \rightarrow K$, $(h, k) \mapsto h \triangleright k$ such that for all $h, h' \in H$ and $k, k' \in K$ the two actions satisfy the identity axioms:

$$1a. h \triangleright 1_K = 1_K$$

$$1b. 1_H \triangleleft k = 1_H$$

$$2a. 1_H \triangleright k = k$$

$$2b. h \triangleleft 1_k = h$$

and the associativity axioms:

$$3. h \triangleright (kk') = (h \triangleright k) \cdot ((h \triangleleft k) \triangleright k')$$

$$4. (hh') \triangleleft k = (h \triangleleft (h' \triangleright k)) \cdot (h' \triangleleft k)$$

Proposition 2.24. *Let (H, K) be a matched pair of groups as above.*

1. *There is a unique group structure on $H \times K$, called the bicrossed product $H \bowtie K$ such that for all $h, h' \in H$ and $k, k' \in K$,*

$$(h, k)(h', k') = h(h' \triangleleft k), (h' \triangleright k)k'$$

2. *H and K can be identified with subgroups $H \bowtie 1$ and $1 \bowtie K$ respectively, and every element of $H \bowtie K$ can then be written uniquely as a product of an element of H and an element of K .*

3. *If $G = HK$ is an exact factorization of G , then (H, K) is a matched pair and the bijection $G \rightarrow H \times K$ induces an isomorphism $G \cong H \bowtie K$.*

Remark 2.25. *If one of the actions \triangleleft or \triangleright is trivial, this is a semidirect product of H and K .*

Definition 2.26. *Let H be a Hopf algebra and C a coalgebra. C is called an H -module coalgebra if there is a coalgebra morphism $H \otimes C \rightarrow C$ that makes C an H -module.*

Proposition 2.27. *Let G be a group acting on a set X . The action $G \times X \rightarrow X$ induces a morphism of coalgebras $\mathbb{C}[G] \otimes \mathbb{C}[X] \rightarrow \mathbb{C}[X]$ that equips $\mathbb{C}[X]$ with a $\mathbb{C}[G]$ -module coalgebra structure.*

Definition 2.28. *A pair of bialgebras (Y, X) is called matched if the following holds: There exist maps $Y \otimes X \rightarrow X$, $y \otimes x \mapsto y \triangleright x$ and $Y \otimes X \rightarrow Y$, $y \otimes x \mapsto y \triangleleft x$ that make X a Y -module coalgebra and Y a (right) X -module coalgebra such that for $x, x' \in X$ and $y, y' \in Y$, we have:*

$$1. y \triangleright (xx') = (y_{(1)} \triangleright x_{(1)})(y_{(2)} \triangleleft x_{(2)}) \triangleright x'$$

$$2. (yy') \triangleleft x = (y \triangleleft (y'_{(1)} \triangleright x_{(1)}))(y_{(2)} \triangleleft x_{(2)})$$

(one may see this definition presented with additional axioms, but our requirement that X and Y be module coalgebras over each other automatically satisfies those).

Definition 2.29. *Given a matched pair of bialgebras (Y, X) , define the bicrossed product $X \bowtie Y$ to be the bialgebra with underlying set $Y \otimes X$ and with product and coproduct given by:*

$$(y \otimes x) \cdot (y' \otimes x') = y(x_{(1)} \triangleright y'_{(1)}) \otimes (x_{(2)} \triangleleft y'_{(2)})x'$$

$$\Delta(x \otimes y) = (x_{(1)} \otimes y_{(1)}) \otimes (x_{(2)} \otimes y_{(2)})$$

Proposition 2.30. *This is indeed a bialgebra structure on $Y \otimes X$.*

Definition 2.31. The Drinfeld double $D(H)$ of a Hopf algebra H with (invertible) antipode S is the bicrossed product $H \bowtie H^*$ when equipped with the left and right coadjoint actions.

Theorem 2.32. 1. $D(H)$ is a Hopf algebra with the antipode $S_{D(H)} = S_H \otimes S_{H^*}$
 2. H and H^* are Hopf subalgebras of $D(H)$
 3. The category $\text{Rep}(D(H))$ has a braiding given by $c_{XY} = P \circ \mathcal{R}|_{X \otimes Y}$ where $R \in H \otimes H^* \subset D(H) \otimes D(H)$ is given by

$$\mathcal{R} = \sum_i h_i \otimes h_i^*$$

where h_i is a basis of H and h_i^* is the dual basis of H^* with respect to the Hopf pairing (see next section). This is independent of the choice of basis.

4. \mathcal{R} satisfies the (quantum) Yang-Baxter equation

$$\mathcal{R}^{12}\mathcal{R}^{13}\mathcal{R}^{23} = \mathcal{R}^{23}\mathcal{R}^{13}\mathcal{R}^{12}$$

where

$$\begin{aligned} \mathcal{R}^{12} &= \sum h_i \otimes h_i^* \otimes 1 \\ \mathcal{R}^{13} &= \sum h_i \otimes 1 \otimes h_i^* \\ \mathcal{R}^{23} &= \sum 1 \otimes h_i \otimes h_i^* \end{aligned}$$

5. The given product on $D(H)$ is the only one for which \mathcal{R} satisfies the Yang-Baxter equation.
 6. \mathcal{R} is invertible with inverse $\mathcal{R}^{-1} = (S \otimes 1)\mathcal{R}$

2.5 Computing the Universal R matrix

Definition 2.33. Let X and Y be Hopf algebras. A Hopf pairing is a bilinear form $\langle -, - \rangle : X \times Y \rightarrow \mathbb{C}$ such that for all $x, x' \in X$ and $y, y' \in Y$

$$1a. \langle xx', y \rangle = \langle x \otimes x', \Delta(y) \rangle := \langle x, y_{(1)} \rangle \langle x', y_{(2)} \rangle$$

$$1b. \langle x, yy' \rangle = \langle \Delta(x), y \otimes y' \rangle := \langle x_{(1)}, y \rangle \langle x_{(2)}, y' \rangle$$

$$2a. \langle x, 1_Y \rangle = \epsilon_X(x)$$

$$2b. \langle 1_X, y \rangle = \epsilon_Y(y)$$

$$3. \langle S_X(X), y \rangle = \langle x, S_Y(y) \rangle$$

Proposition 2.34. A Hopf pairing of Y and X defines actions of X on Y and Y on X that satisfy the axioms of a bicrossed product. These actions are given by:

$$y \triangleright x = x_{(1)} \langle y, x_{(2)} \rangle \text{ and } y \triangleleft x = \langle y_{(1)}, x \rangle y_{(2)}$$

Proposition 2.35. Let $\langle -, - \rangle : X \times Y \rightarrow \mathbb{C}$ be a nondegenerate Hopf pairing. The map $Y \rightarrow X^{\text{cop}}$ given by $y \mapsto \langle -, y \rangle$ is an isomorphism of Hopf algebras (where X^{cop} is X with opposite coproduct). This means that a Hopf pairing contains the data necessary to construct the Drinfeld double $D(X)$.

Lemma 2.36. *The Yang-Baxter equation is equivalent to the following system of equations*

$$(\Delta \otimes 1)\mathcal{R} = \mathcal{R}^{13}\mathcal{R}^{23}$$

$$(1 \otimes \Delta)\mathcal{R} = \mathcal{R}^{13}\mathcal{R}^{12}$$

Proposition 2.37. *The axioms of a Hopf pairing imply that $\mathcal{R} = \sum h_i \otimes h_i^*$ satisfy the above system.*

Let B^+ be the subalgebra of $U_q(\mathfrak{sl}_2)$ generated by E, K , and K^{-1} and let B^- be the subalgebra generated by F, K , and K^{-1} . One checks that these are Hopf subalgebras. We will find a Hopf pairing between B^+ and B^- . This allows us to define a bicrossed product $B^+ \bowtie B^-$, the Drinfeld double of B^+ . It will then be straightforward to compute \mathcal{R} . We will take $\{E^i K^j\}$, $i \geq 0$ as our basis of B^+ .

Proposition 2.38. *The following defines a Hopf pairing of B^+ and B^- (up to sign?):*

$$\langle E, F \rangle = \frac{1}{q - q^{-1}}, \quad \langle K, K \rangle = q^{-2}$$

and all pairings $\langle E, E \rangle$, $\langle F, F \rangle$, $\langle E, K \rangle$, and $\langle K, F \rangle$ are 0.

Moreover, one checks (using 1a and 1b in the definition) that this implies

$$\langle E^m, F^n \rangle = q^{-\frac{n(n-1)}{2}} \delta_{mn} \frac{(q - q^{-1})^n}{[n]_q!} \quad \text{and} \quad \langle K^m, K^n \rangle = q^{-2mn}$$

where δ_{mn} is the Kronecker delta and $[n]_q! = [1]_q [2]_q \dots [n]_q$ is the q -factorial where $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$.

Corollary 2.39. *The basis*

$$f_n = q^{-\frac{n(n-1)}{2}} \frac{[n]_q!}{(q - q^{-1})^n} F^n$$

is Hopf dual to $e_n = E^n$.

Remark 2.40. *Since $U_q(\mathfrak{sl}_2)$ is infinite dimensional, \mathcal{R} doesn't actually live in $U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ but rather an appropriate completion of it. There is a natural way to define the completion of $H \otimes H^*$ for a Hopf algebra H , but we won't cover it here.*

K doesn't have a Hopf dual since it acts diagonally via powers of q . This means computing \mathcal{R}_{root} the way we did above is not so easy. To finish computing \mathcal{R} , we will go back to the reason we cared about it in the first place:

$$\Delta^{op}\mathcal{R} = \mathcal{R}\Delta$$

Writing

$$\mathcal{R}_{root} = \sum q^{-\frac{n(n-1)}{2}} \frac{[n]_q!}{(q - q^{-1})^n} E^n \otimes F^n$$

Then it is straightforward to check that $q^{\frac{H \otimes H}{2}} \mathcal{R}_{root}$ satisfies this condition.