A Brief Introduction to Hopf Algebras and $U_q(\mathfrak{sl}_2)$

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1 Introduction

Within the last 50 years, physicists have developed a sudden interest in a particular algebraic structure called a *quantum group*. The development of the theory of quantum groups is well-motivated, as they occur naturally and often in the study of integrable systems in quantum field theory and statistical mechanics. Although it clearly denotes the object's close relationship with quantum mechanics, the term quantum group is ill-fitting because quantum groups are not actually groups. Rather, they are an instance of *Hopf algebras*, a rich algebraic structure whose axioms appear so naturally that they were studied by mathematicians decades before they found a place in modern physics.

The theory of quantum groups is too intricate to be done justice in such a short setting. For our purposes, one can loosely think of a quantum group as a "deformation" of a universal enveloping algebra. Instead of giving an uninspired introduction to quantum groups, we will introduce Hopf algebras and briefly explore the simplest example of a quantum group.

2 Hopf Algebras

We begin by defining coalgebras, the dual to unital associative algebras:

Definition 2.1. *A* ***coalgebra*** (C, Δ, ε) *over a field k is a k-vector space C equipped with two morphisms:* ***comultiplication*** $\Delta : C \to C \otimes C$ *and the* ***counit*** $\varepsilon : C \to k$ *such that the following diagrams commute:*

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "id_C \otimes \Delta"] \\
C \otimes C \arrow[r, "\Delta \otimes id_C"'] & C \otimes C \otimes C
\end{tikzcd}$$

$$\begin{tikzcd}
& C \arrow[dl, "\Delta"'] \arrow[d] \arrow[dr, "\Delta"] & \\
C \otimes C \arrow[r, "id_C \otimes \varepsilon"'] & C \otimes k \cong C \cong k \otimes C & C \otimes C \arrow[l, "\varepsilon \otimes id_C"]
\end{tikzcd}$$

The property characterized by the commutativity of the first of the above diagrams is referred to as **coassociativity** since it is dual to the diagram that defines the associative property of an associative algebra. The second diagram is dual to the diagram that asserts the existence of a multiplicative identity in a unital algebra. Let's look at some quick examples of coalgebras:

Example 2.2. *Let S be a nonempty finite set and fix a field k. Define kS to be the k-vector space with basis S. Define $\Delta : kS \to kS$ by $\Delta(s) = s \otimes s$ and define $\varepsilon : kS \to k$ by $\varepsilon(s) = 1$ for all $s \in S$. Then $(kS, \Delta, \varepsilon)$ is a coalgebra.*

Remark 2.3. *This shows that even without specifying a bilinear product, every vector space can be equipped with a coalgebra structure.*

Example 2.4. *An important example of a coalgebra is the **divided power coalgebra**. Let H be a k-vector space with basis $\{c_n : n \in \mathbb{Z}_{\geq 0}\}$. One might note this resembles the ring $k[x]$. One can define a coalgebra structure on H by:*

$$\Delta(c_n) = \sum_{i=0}^{n} c_i \otimes c_{n-i} \quad \textit{and} \quad \varepsilon(c_n) = \delta_{0,n}$$

where $\delta_{i,j}$ is the Kronecker delta function.

We are now equipped to define a bialgebra:

Definition 2.5. *A **bialgebra** is a tuple $(B, \nabla, \eta, \Delta, \varepsilon)$ such that (B, Δ, ε) is a coalgebra over a field k and (B, ∇, η) is a unital associative algebra with multiplication given by $\nabla : B \otimes B \to B$ and unit given by $\eta : k \to B$ such that either of the following equivalent conditions hold:*

1. *Δ and ε are algebra morphisms.*

2. *∇ and η are coalgebra morphisms (such a morphism has the expected definition).*

Remark 2.6. *The definition of a bialgebra is traditionally given in terms of four commutative diagrams that express the necessary compatibility between the algebra and coalgebra structures. However, either of the two equivalent conditions above fully describe these diagrams and provide a more transparent view of their meaning. Each of these diagrams would describe the compatibility of one of the following pairs: ∇ and Δ, ∇ and ε, Δ and η, and η and ε.*

Remark 2.7. *One can check easily that the divided powers coalgebra can be made into a bialgebra. Another important example of a bialgebra is the **tensor algebra**, although its bialgebra structure is more complicated to describe and will not be mentioned here.*

Definition 2.8. *Given a coalgebra (C, Δ, ε), an algebra (A, ∇, η), and two k-linear maps $f, g : C \to A$, the **convolution** of f and g is the k-linear map $f \star g : C \to A$ defined by $c \mapsto (\nabla \circ (f \otimes g) \circ \Delta)(c)$.*

With this, we are finally ready to define a Hopf algebra:

Definition 2.9. *A **Hopf algebra** $(H, \nabla, \eta, \Delta, \varepsilon, S)$ is a bialgebra $(H, \nabla, \eta, \Delta, \varepsilon)$ equipped with a k-linear map $S : H \to H$ called an **antipode** such that $id_H \star S = S \star id_H = \eta \circ \varepsilon$.*

Remark 2.10. *As is the case with a bialgebra, the definition of a Hopf algebra is self-dual, so the dual of a Hopf algebra is always a Hopf algebra.*

Example 2.11. *The universal enveloping algebra $U(\mathfrak{sl}_2)$ has a Hopf algebra structure defined by $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$, and $S(x) = -x$ where $x \in \{E, F, H\}$. In fact, the universal enveloping algebra of a Lie algebra is always a Hopf algebra. Without constructing this explicitly, this can be seen by viewing $U(\mathfrak{g})$ as a quotient of the tensor algebra.*

3 The Quantum Group $U_q(\mathfrak{sl}_2)$

We dedicate the rest of this short paper to discussing a fundamental example of a Hopf algebra:

Definition 3.1. *Let $q \in \mathbb{C} \setminus \{0, \pm 1\}$. We define* ***quantum*** *$\mathfrak{sl}_2$, denoted $U_q(\mathfrak{sl}_2)$, as the algebra generated by the symbols E, F, K, K^{-1} subject to the relations:*

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2}F, \quad [E, F] = EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

Remark 3.2. *The resemblance of this definition to that of $U(\mathfrak{sl}_2)$ somewhat illustrates how $U_q(\mathfrak{sl}_2)$ can be thought of as a deformation of $U(\mathfrak{sl}_2)$. In fact, setting $K = q^h$ we have that*

$$\lim_{q \to 1} \frac{K - K^{-1}}{q - q^{-1}} = h$$

so as $q \to 1$ the relations of $U_q(\mathfrak{sl}_2)$ "deform" into those of $U(\mathfrak{sl}_2)$. This statement can be made rigorous, although we will not do so here.

Remark 3.3. *By the Poincaré-Birkhoff-Witt theorem, the set of monomials $\{F^k K^\ell E^m : k, m \in \mathbb{Z}_{\geq 0}, \ell \in \mathbb{Z}\}$ is a basis of $U_q(\mathfrak{sl}_2)$.*

Let's define a (unique) Hopf algebra structure on $U_q(\mathfrak{sl}_2)$ by defining the necessary morphisms on the generators:

$$\Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F, \quad \Delta(K) = K \otimes K$$

$$\varepsilon(E) = \varepsilon(F) = 0, \quad \varepsilon(K) = 1$$

$$S(E) = -K^{-1}E, \quad S(F) = -FK, \quad S(K) = K^{-1}$$

Checking that this is a Hopf algebra is, although not very difficult, actually a bit exhausting. One must check that this satisfies all the axioms of a Hopf algebra and that each morphism respects the generating relations.

We end by noting one reason $U_q(\mathfrak{sl}_2)$ is so important. The relatively simple construction we just gave extends in a slightly more complicated manner to the case of *any* finite-dimensional simple Lie algebra $\mathfrak{g}$ of rank r (even more generally, it extends to symmetrizable Kac-Moody algebras, although we won't discuss or define them in this paper).

Definition 3.4. *Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank r. Let $A = (a_{ij})$ be its Cartan matrix. Recall that we have unique relatively prime positive integers d_i for $i \in \{1, ..., r\}$ such that $d_i a_{ij} = d_j a_{ji}$. As before, let $q \in \mathbb{C} \setminus \{0, \pm 1\}$ and set $q_i = q^{d_i}$. Further, suppose that $q_i \neq \pm 1$ for each i. The quantum group $U_q(\mathfrak{g})$ is the algebra generated by E_i, F_i, K_i, K_i^{-1} subject to the relations:*

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i, \quad K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

and the q-Serre relations:

$$\sum_{\ell=0}^{1-a_{ij}} \frac{(-1)^\ell}{[l]_{q_i}![1 - a_{ij} - \ell]_{q_i}!} E_i^{1-a_{ij}-\ell} E_j E_i^\ell = 0 \quad \text{for } i \neq j$$

$$\sum_{\ell=0}^{1-a_{ij}} \frac{(-1)^\ell}{[l]_{q_i}![1 - a_{ij} - \ell]_{q_i}!} F_i^{1-a_{ij}-\ell} F_j F_i^\ell = 0 \quad \text{for } i \neq j$$

where δ_{ij} is the Kronecker delta function and $[n]_q!$ is the q-factorial, defined as:

$$[n]_q! := \prod_{i=1}^{n} \frac{1 - q^i}{1 - q}$$

Proposition 3.5. *$U_q(\mathfrak{g})$ has a unique Hopf algebra strikingly similar to that of $U_q(\mathfrak{sl}_2)$. It is given by:*

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta(K_i) = K_i \otimes K_i$$

$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1$$

$$S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i, \quad S(K_i) = K_i^{-1}$$

Note that as is the case with $U_q(\mathfrak{sl}_2)$, the limit $q \to 1$ deforms $U_q(\mathfrak{g})$ into $U(\mathfrak{g})$.

Remark 3.6. *The introduction of the q-Serre relations, while necessary, makes verifying that this is indeed a Hopf algebra in the general case even more tedious than doing so in the case of $U_q(\mathfrak{sl}_2)$.*

Works Refrenced

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