Introductory Lie Theory Notes

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Disclaimers:
These notes are largely based on the book on the subject by Kirillov. Some material may also be taken from the texts by Serre or Fulton and Harris.
These notes are not comprehensive and no work presented is claimed to be original.
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Contents

1 Basics of Lie Groups

1.1 Definitions, Lie subgroups, and the closed subgroup theorem

Definition 1.1. *A real Lie group is a group G such that G is also a manifold and the group multiplication and inversion maps are smooth. A morphism of Lie groups is a smooth group homomorphism.*

Definition 1.2. *A complex Lie group is a group G such that G is also a complex analytic manifold such that the multiplication and inversion maps are analytic. A morphism of complex Lie groups is an analytic group homomorphism.*

Theorem 1.3 (**Discrete Normal Subgroups**). *Let G be a Lie group. Let G^0 be the connected component of the identity. Then G^0 is a normal Lie subgroup and G/G^0 is discrete.*

Proof. Since the inversion map $i : G \to G$ is continuous, it takes connected components to connected components. The image of G^0 under i must contain 1, so it must be G^0. Similarly, since the multiplication map $m : G \times G \to G$ is smooth, the image of $G^0 \times G^0$ is G. Now, let $g \in G$ and $h \in G^0$. Since conjugation by g is continuous and fixes 1, $ghg^{-1} \in G^0$, so G^0 is normal. The elements of G/G^0 are the connected components of G, so the quotient is discrete. □

Definition 1.4. *A closed Lie subgroup is a subgroup that is also a submanifold.*

Theorem 1.5 (**Closed subgroup theorem**). *Any closed Lie subgroup is closed in G and any closed subgroup of a Lie group is a closed real Lie subgroup.*

Proof. We prove only the first statement. Let H be a closed Lie subgroup of G and let $h \in H$. h acts bijectively on H, so $H \subset h\overline{H}$ which is closed. This implies $\overline{H} \subset h\overline{H}$ since multiplication by h is continuous which gives $h^{-1}\overline{H} \subset \overline{H}$ so $\overline{H}$ is closed under multiplication. A similar argument shows it is closed under inversion, so $\overline{H}$ is a subgroup.

Now, there exists a neighborhood U of 1 such that $U \cap H = U \cap \overline{H}$. Then $hU \cap hH = hU \cap h\overline{H} = hU \cap \overline{H}$ is open in $\overline{H}$. Let $x \in \overline{H}$. Since xH is dense $xH \cap H \neq \varnothing$ so there exists some $y \in H$ such that $xy \in H$ which implies $x \in H$ and $H = \overline{H}$. □

Corollary 1.6. -
1) If G is connected and U is a neighborhood of 1, U generates G.
2) If $f : G_1 \to G_2$ with G_2 connected such that f_* is surjective, f is surjective.

Proof. 1) Let H be the subgroup generated by G. H is open. For all $h \in H$, hU is an open subset of G so it is a submanifold and thus a closed Lie subgroup, so it is closed, so $H = G$. 2)f is surjective in a neighborhood of $1 \in G_2$ so by 1) it is surjective on all of G_2 □

Example 1.7. *Let $f : \mathbb{R} \to T^2 = \mathbb{R}^2/\mathbb{Z}^2$ where $f(t) = (t \bmod \mathbb{Z}, \alpha t \bmod \mathbb{Z}$ with α irrational. The image of f is a Lie subgroup that is not closed (it is everywhere dense). It is an immersed manifold.*

Theorem 1.8 (**Homomorphism theorems**). *Let $f : G_1 \to G_2$. H =$Kerf$ is a closed normal Lie subgroup and gives rise to an immersion $G_1/H \to G_2$. If Imf is embedded, this is a closed lie subgroup and we have $G_1/H \cong G_2$*

1.2 Classical Lie groups

The following subsets of $GL_n\mathbb{K}$ are referred to as the "classical Lie groups"

- $GL_n\mathbb{K}$. *This is a real Lie group when* $\mathbb{K} = \mathbb{R}$ *and complex when* $\mathbb{K} = \mathbb{C}$
- $SL_n\mathbb{K}$, *the subset of* $GL_n\mathbb{K}$ *whose elements all have determinant 1*
- $O_n\mathbb{K}$, *distance (more generally, bilinear form) preserving maps (i.e.* $XX^t = 1$)
- $SO_n\mathbb{K}$. *Note elements of* $O_n\mathbb{K}$ *can have determinant -1*
- $Sp_{2n}\mathbb{K} = \{A : \mathbb{K}^{2n} \to \mathbb{K}^{2n} : \omega(Ax, Ay) = \omega(x, y)\}$ *where* ω *is the skew-symmetric bilinear form*

  $$\sum_{i=1}^{n} x_y y_{n+i} - y_i x_{n+i}$$

  Equivalently, $\omega(Jx, y)$ *where* $(\cdot, \cdot)$ *is the standard symmetric bilinear form and*

  $$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

- U_n, *isometries of* $\mathbb{C}^n$ *(i.e.* $x^* = x^{-1}$). *Note that this is a real Lie group and that* U_n *is the intersection of any two of the following:* $O_{2n}\mathbb{R}, GL_n(\mathbb{C})$ *and,* $Sp_{2n}(\mathbb{R})$
- SU_n
- $Sp(n) = Sp_{2n}(\mathbb{C}) \cap SU_{2n}$, *the compact symplectic group, also known as the unitary quaternionic group*

We recall some properties about the matrix exponential and logarithm*:*
1) $log(\exp(x)) = x, \exp(log(X)) = X$ *when defined*
2) $\exp(0) = 1$ *and* $\mathrm{d}\exp(0) = id$
3) If $xy = yx$, $\exp(x + y) = \exp(x)\exp(y)$. *If* $XY = YX$, $log(XY) = log(X) + log(Y)$ *in some neighborhood of the identity. In particular,* $\exp(x)\exp(-x) = 1$ *so* $\exp(x) \in GL_n\mathbb{K}$ *for any* $x \in \mathfrak{gl}_n\mathbb{K}$ *since* $\exp(x)$ *is invertible.*
4) Fix $x \in \mathfrak{gl}_n(\mathbb{K})$. *The map* $\mathbb{K} \to \mathfrak{gl}_n(\mathbb{K})$ *given by* $t \mapsto \exp(tx)$ *is a Lie group morphism and the image is called a one-parameter subgroup.*
5) $\exp(AxA^{-1}) = A\exp(x)A^{-1}$ *and* $\exp(x^t) = \exp(x)^t$

Theorem 1.9 (**Smoothness of classical groups**)**.** *For each classical group* G *(listed above), there exists a vector space* $\mathfrak{g}$ *such that in a neighborhood of* $1 \in G$ *and* $0 \in \mathfrak{g}$*, log and exp are mutually inverse.*

Corollary 1.10. *Each classical group is a Lie group with* $T_1G = \mathfrak{g}$ *and* $dimG = dim\mathfrak{g}$. U_n, SU_n, *and* Sp_n *are real Lie groups. The groups* $GL_n(\mathbb{K}), SL_n(\mathbb{K}), SO_n(\mathbb{K})$, *and* $Sp_n(\mathbb{K})$ *are real Lie groups for* $\mathbb{K} = \mathbb{R}$ *and complex Lie groups for* $\mathbb{K} = \mathbb{C}$.

1.9 immediately implies the second part of 1.10 since G is a submanifold of $GL_n(\mathbb{K})$ near the identity. Translation by elements in g demonstrates this to be true globally. For the first part, consider the map $exp_* : T_0\mathfrak{g} \to T_1G$. $T_0\mathfrak{g} = \mathfrak{g}$ since $\mathfrak{g}$ is a vector space and $exp_* =$d $\exp(x) = 1 + x + ...$ is the identity at 0. We now prove 1.9 case by case.

Proof. $GL_n\mathbb{K}$: Follows immediately from (3) above.
$SL_n\mathbb{K}$: Let $X \in SL_n$ be close enough to 1. Then $X = \exp(x)$ for some $x \in \mathfrak{gl}_n$. We have $det(X) = 1 \implies det(\exp(x)) = 1 \implies \exp(\mathrm{tr}(x))$ so $\exp(x) \in SL_n$ iff $\mathrm{tr}(x) = 0$. Thus $\mathfrak{sl}_n = \{x \in \mathfrak{gl}_n : \mathrm{tr}(x) = 0\}$
$O_n, SO_n\mathbb{K}$: Recall $X \in O_n$ iff $XX^t = I$ which implies X and X^t commute. Write $X = \exp(x)$ and $X^t = \exp(x^t)$, $\exp(x)\exp(x^t) = \exp(x + x^t) = 1$, so $x + x^t = 0$. The converse holds as well, if $x + x^t = 0$, x and x^t commute which implies $g = \{x : x + x^t = 0\}$, skew-symmetric matrices. No change needs to be made for SO_n, since $x + x^t = 0$ already implies that the diagonal entries are 0.
U_n, SU_n: The same argument as in the case of O_n gives $x + x^* = 0$ but this time, this does not imply the diagonal entries are 0, so we must add the condition that $\mathrm{tr}(x) = 0$ for $SO_n\mathbb{K}$.
$Sp_n\mathbb{K}, Sp(n)$: As above, we find that $\exp(x) \in Sp_n$ iff $x + J^{-1}x^tJ = 0$. We need the additional condition that $x + x^* = 0$ for $Sp(n)$ □

1.3 Actions and orbit-stabilizer

Definition 1.11. *An action of a real/complex Lie group G on a manifold M, is an assignment to each $g \in G$ a diffeomorphism/invertible holomorphic map $\rho(g)$ such that $\rho(1) = id, \rho(gh) = \rho(g)\rho(h)$ and such that the map $G \times M \to M(g, m) \mapsto \rho(g).m$ is smooth/holomorphic.*

Definition 1.12. *A representation of a Lie group G is a vector space V with a group morphism $\rho : G \to GL(V)$. If V is finite-dimensional, ρ is smooth/analytic so that it is a Lie group morphism. A morphism of representations is a linear map $f : V \to W$ which commutes with the group action: $f\rho_V(g).m = \rho_W(f(g)).m$ Note real Lie groups can act on complex vector spaces.*

Example 1.13. -
1) Representations of G on $C^\infty(M)$ given by $(\rho(g)f)(m) = f(g^-1.m)$
2) Representations of G on $Vect(M)$ given by $(\rho(g).v)(m) = g_*(v(g^{-1}.m))$
3)Fix $m \in M$ such that $g.m = m$ for all $g \in G$. G acts canonically on T_mM by $\rho(g) = g_*$. Similarly for $T*_m M$ and exterior powers of it.

Theorem 1.14 (**Orbit-Stabilizer**). *Let $\mathcal{O}_m = \{g. : g \in G\}$ be the orbit of $m \in M$ under G and let $G_m = Stab_G(m) = \{g \in G : g.m = m\}$ be the stabilizer of m with respect to G. Then G_m is a closed Lie subgroup of G and the map $g \mapsto g.m$ is an embedding $G/G_m \hookrightarrow M$ whose image is $\mathcal{O}_m$.*

Theorem 1.15. -
1. The stabilizer is a closed Lie subgroup with Lie algebra $\{x \in \mathfrak{g} : \rho_*(x)(m) = 0\}$ where $\rho_*(x)$ is the vector field on M associated to x. In particular, if V is a representation of G, $v \in V$ with stabilizer G_V. G_V is a Lie subgroup with Lie algebra $\mathfrak{h} = \{x \in \mathfrak{g} : x.v = 0\}$.
2. $G/G_m \to M$ given by $g \mapsto g.m$ is an immersion and the orbit $\mathcal{O}_m$ is an immersed submanifold with tangent space $T_m\mathcal{O} = \mathfrak{g}/\mathfrak{h}$.

Definition 1.16. *A G-homogenous space is a manifold with a transitive G action (when there is only one orbit).*

Remark 1.17. *Any set with a transitive G action has the canonical structure of a manifold isomorphic to G/H.*

Corollary 1.18. *The flag variety $\mathcal{F}_n(\mathbb{R})$ is isomorphic to $GL_n(\mathbb{R})/B_n(\mathbb{R})$ where B_n is the group of invertible upper-triangular matrices.*

1.4 Adjoint action and invariant vector fields

Remark 1.19. *Since the adjoint action $Ad\ g = L_g R_g$ preserves the identity, it induces an action of G on T_1G. This action is also written as $Ad\ g$. In another abuse of notation, the actions induced by L_g and R_g on a vector $v \in T_mG$ will be written as $g.v$ and $v.g$ respectively.*

Definition 1.20. *A vector field is left-invariant if $g.v = v$ for all v and right-invariant if $v.g = v$ for all g. It is called bi-inviariant if it is both left- and right-invariant.*

Theorem 1.21 (Space of invariant vector fields)**.** *The map $Vect(G) \to T_1G$ given by $v \mapsto v(1)$ is an isomorphism of the space of left-invariant (and right-invariant) vector fields with T_1G.*

Proof. It suffices to show every left $x \in T_1G$ extends to a left-invariant vector field on G. Define $v(g) = g.x \in T_gG$. □

Remark 1.22. *Note that the extension of a vector to a left-invariant vector field and a right-invariant vector field may differ.*

Theorem 1.23. *$v \mapsto v(1)$ defines an isomorphism of bi-invariant vector fields with the vector space of invariants of the adjoint action: $(T_1G)^{\mathrm{AdG}}$. Note a similar result holds for other tensor fields.*

1.5 The exponential map

Proposition 1.24. *Let G be a Lie group with $T_1G = \mathfrak{g}$ and let $x \in \mathfrak{g}$. There exists a unique Lie group morphism $\gamma_x : \mathbb{K} \to G$ such that $\dot{\gamma}_x(0) = x$. This map is called the one-parameter subgroup of x.*

Proof. It is easy to see that $\dot{\gamma}_x(t) = \gamma_x(t)\dot{\gamma}_x(0) = \dot{\gamma}_x(0)\gamma_x(t)$. If v_x is a left-invariant vector field on G with $v_x(1) = x$, γ is an integral curve for v_x which proves uniqueness. Now, if Φ^t is the time t flow of v_x, Φ^t is also left invariant. Letting $\gamma_x(t) = \Phi^t(1)$ gives $\gamma_x(t+s) = \Phi^{t+s}(1) = \Phi^s((\Phi^t(1))) = \Phi^s(\gamma_x(t)) = \gamma_x(t)\Phi^s(1) = \gamma(t)\gamma(s)$. This proves existence. □

Definition 1.25. *The exponential map* $\exp : \mathfrak{g} \to G$ *is defined by* $\exp(x) = \gamma_x(1)$ *where* $\dot{\gamma}_x(0) = x$.

Remark 1.26. *By the uniqueness of γ_x, $\gamma_x(\lambda t) = \gamma_{\lambda x}(t)$ for any $\lambda \in \mathbb{K}$.*

Corollary 1.27. *If v is a left- (resp right-) invariant vector field on G, the time t flow of v is given by $g \mapsto g\exp(tx)$ (resp $g \mapsto \exp(tx)g$) where $x = v(1)$.*

Example 1.28. *Let $G = \mathbb{R}$ so $\mathfrak{g} = \mathbb{R}$. Then for $a \in \mathfrak{g}$, $\gamma_a(t) = ta$ so $\exp(a) = a$.*

Example 1.29. *Let $G = S^1 = \mathbb{R}/\mathbb{Z}$ so $\mathfrak{g} = \mathbb{R}$ and for $a \in \mathfrak{g}$, $\gamma_a(t) = at \bmod \mathbb{Z}$ so again $\exp(a) = a \bmod \mathbb{Z}$. If we consider S^1 as $\{z \in \mathbb{C} : |z| = 1\}$ then $\gamma_z(t) = e^{2\pi i a t}$ where $z = e^{2\pi a}$ so $exp(a) = e^{2\pi i a}$.*

Theorem 1.30. *Let G be a Lie group and $\mathfrak{g} = T_1 G$. The exponential map has the following properties:*
1) $\exp(x) = 1 + x + \frac{1}{2}x^2 + ...$ so $\exp(0) = 1$ and $exp_*(0) : \mathfrak{g} \to \mathfrak{g}$ is the identity.
2) $\exp$ is locally a diffeomorphism between a neighborhood of $0 \in \mathfrak{g}$ and $1 \in G$.
3) $\exp((t+s)x) = \exp(tx)\exp(sx)$ for $t, s \in \mathbb{K}$.
4) For any map $\varphi : G_1 \to G_2$ and $x \in \mathfrak{g}_1$, $\exp(\varphi_x(x)) = \varphi(\exp(x))$. That is, $\exp$ "commutes" with Lie group morphisms.
5) For any $X \in G$ and $y \in \mathfrak{g}$, $X\exp(y)X^{-1} = \exp(Ad\ X.y)$ where $Ad\ X$ is the action of X on $\mathfrak{g}$ induced by the adjoint action $Ad\ x : G \to G$.

Corollary 1.31. *If G_1 is connected, a map $\varphi : G_1 \to G_2$ is determined by $\varphi_* : T_1 G_1 \to T_1 G_2$.*

Remark 1.32. *The exponential map is surjective on compact Lie groups.*

1.6 The commutator

Definition 1.33. *The group law in logarithmic coordinates is the map $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that $exp(x)exp(y) = exp(\mu(x, y))$. This exists since $\exp$ locally identifies $\mathfrak{g}$ with G.*

Lemma 1.34. *The Taylor series of μ is $x + y + \lambda(x, y) + ...$ where λ is bilinear and "..." denotes terms of order three or higher.*

Proof. This follows from checking $x = 0$ and $y = 0$. ☐

Definition 1.35. *The commutator $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is defined by $[x, y] = 2\lambda(x, y)$. This map is bilinear and skew-symmetric.*

Proposition 1.36. -
1) Any Lie algebra map induced by a map of Lie groups preserves the commutator
2) The adjoint action of a Lie group preserves the commutator
3) $\exp(x)\exp(y)\exp(-x)\exp(-y) = exp([x, y], ...)$ where "..." denotes terms of order three or higher.

Proof. 1 and 2 follow from the fact that exp commutes with Lie group morphisms. 3 can be checked by direct computation. ☐

Corollary 1.37. *If G is commutative, $[\cdot, \cdot]$ is identitcally 0 on $\mathfrak{g}$.*

Lemma 1.38. *Let $ad = Ad_*$ where $Ad : G \to GL(\mathfrak{g})$ is the adjoint action. ad is a map $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$.*
1) $ad\ x.y = [x, y]$
2) $Ad(\exp(x)) = exp(ad\ x)$ as operators $\mathfrak{g} \to \mathfrak{g}$.

Proof. 1 follows from direct computation using 1.36.3. 2 follows from 1.30.4. □

Theorem 1.39. *The commutator satisfies the following equivalent conditions*
1) $[x,[y,z]] = [[x,y],z] + [y,[x,z]]$
2) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]]$
3) $ad\ x.[y,z] = [ad\ x.y, z] + [y.ad\ x.z]$
4) $ad[x,y] = ad\ xad\ y - ad\ yad\ x.$

Proof. 4) Follows from the fact that the commutator in $\mathfrak{gl}(\mathfrak{g})$ is $[x,y] = xy - yx$. This can be checked directly with 1.36.3. Equivalence of the four follows from the fact that the bracket "commutes" with Lie group morphisms. □

1.7 Cambell-Hausdorff formula

Theorem 1.40. *For $x, y \in \mathfrak{g}$ close enough to 0, we have*

$$\exp(x)\exp(y) = \exp(\mu(x,y))$$

for some $\mathfrak{g}$-valued map μ given by the following universal formula (convergent in a neighborhood of 0):

$$\mu(x,y) = x + y + \sum_{n\geq 2}\mu_n(x,y)$$

where μ_n is a Lie polynomial of degree n .

$$\mu(x,y) = x + y + \frac{1}{2}[x,y] + \frac{1}{12}([x,[x,y]] + [y,[y,x]]) + ...$$

Proof. The proof is hard. The idea is to write $\mu(tx,y)$ as a power series $\sum a_n t^n (adx)^n y$ and solve the differential equation in t. □

Corollary 1.41. *The group law in a connected Lie group is determined by the commutator in its Lie algebra.*

2 Structure Theory of Lie Algebras

2.1 Fundamental theorems of Lie theory

In this section we'll explore some fundamental properties of Lie algebras.

Theorem 2.1. *For any Lie group G, there is a bijection between connected Lie subgroups $H \subset G$ and Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$ given $\mathfrak{h} = T_1 H$*

Theorem 2.2. *For Lie groups G_1 and G_2 with G_1 simply connected, $Hom(G_1, G_2) =$* $\hom(\mathfrak{g}_1, \mathfrak{g}_2)$

Theorem 2.3. (Lie's third theorem) *Any finite-dimensional Lie algebra is isomorphic to the Lie algebra of a Lie group.*

The proofs of these theorems are postponed for now.

Corollary 2.4. *Any finite-dimensional Lie algebra $\mathfrak{g}$ has a unique simply-connected Lie group G such that $Lie(G) = \mathfrak{g}$. Any other connected Lie group G' with $Lie(G') = \mathfrak{g}$ must be of the form G/Z for some discrete central subgroup Z. In other words, the categories of finite-dimensional Lie algebras and simply-connected Lie groups are equivalent.*

Proof. Given a Lie algebra $\mathfrak{g}$, there exists a Lie group G with $Lie(G) = \mathfrak{g}$. The universal cover of G is the simply-connected group desired. If another such group G' exists, there is a map $G \to G'$ which is locally an isomorphism so $G' = G/Z$ for some discrete central subgroup Z. Since $\pi_1(G/Z) = Z$, this proves uniqueness. □

The idea behind the proof of 2.3 is to use Ado's theorem to show all Lie algebras are subalgebras of $\mathfrak{gl}_n\mathbb{K}$. The theorem then follows from 2.1. We now show that 2.2 also follows from 2.1.

Proof. We've shown that if G_1 is connected, any morphism $G_1 \to G_2$ is determined by the corresponding map of Lie algebras, so it remains only to show that every morphism f of Lie algebras can be lifted to a morphism φ of Lie groups such that $f = \varphi_*$.

Let $G = G_1 \times G_2$ so $Lie(G) = \mathfrak{g}_1 \times \mathfrak{g}_2$. Let $\mathfrak{h} = \{x, f(x) : x \in \mathfrak{g}_1\}$. One can show this is a subalgebra. By 2.1, there is a connected Lie subgroup $H \hookrightarrow G_1 \times G_2$. Composing this injection with the projection $p : G \to G_1$ gives a morphism $\pi : H \to G_1$. The induced map $\pi_* : \mathfrak{h} \to \mathfrak{g}_1$ is an isomorphism. This implies π is a covering map but since G_1 is simply connected and H is connected, π is an isomorphism so it has an inverse. Define $\varphi : G_1 \to G_2$ as $G_1 \xrightarrow{\pi^{-1}} H \hookrightarrow G \to G_2$. □

2.2 Universal enveloping algebras and Poincare-Birkhoff-Witt theorem

Definition 2.5. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. The universal enveloping algebra of $\mathfrak{g}$, denoted $U\mathfrak{g}$ or $U(\mathfrak{g})$, is the unital associative algebra over $\mathbb{K}$ generated by all $x \in \mathfrak{g}$ subject to the relation $xy - yx = [x, y]$. Alternatively, $U\mathfrak{g}$ can be viewed as the quotient of the tensor algebra $T\mathfrak{g} = \bigoplus_{n\geq 0} \mathfrak{g}^{\oplus n}$ by the ideal $(xy - yx - [x, y])$.*

Remark 2.6. *If $\mathfrak{g}$ is commutative, $U\mathfrak{g}$ is the symmetric algebra of $\mathfrak{g}$, $S\mathfrak{g} \cong \mathbb{K}[x_1, ...x_n]$ where $\{x_i\}$ is a basis of $\mathfrak{g}$.*

Remark 2.7. *Even when $\mathfrak{g} \subset \mathfrak{gl}_n$, the product in $U\mathfrak{g}$ need not be matrix multiplication and in general is not.*

The following justifies the use of the term "universal." The proof is omitted.

Theorem 2.8. *Let A be a unital associative algebra over $\mathbb{K}$ and let $\rho : \mathfrak{g} \to A$ be a map such that $\rho(x)\rho(y) - \rho(y)\rho(x) = \rho([x, y])$. ρ extends uniquely to a morphism of associative algebras $U\mathfrak{g} \to A$.*

Corollary 2.9. *Any $\mathfrak{g}$-representation has a canonical structure of a $U\mathfrak{g}$-module and vice-versa. That is, the categories of $\mathfrak{g}$-representations and $U\mathfrak{g}$-modules are equivalent.*

Let's define a filtration on $U\mathfrak{g}$. For any $k \geq 0$, let $U_k\mathfrak{g}$ be the subspace spanned by products $x_1...x_m$, $m \leq k$. We then have $U\mathfrak{g} = \bigcup U_k\mathfrak{g}$. This filtration has the following properties

Theorem 2.10. *Let $x \in U_p\mathfrak{g}$ and $y \in U_q\mathfrak{g}$.*
1. $U\mathfrak{g}$ is a filtered algebra. That is, $xy \in U_{p+q}\mathfrak{g}$.
2. $xy - yx \in U_{p+q-1}\mathfrak{g}$.
3. Let $x_1, ..., x_n$ be a basis in $\mathfrak{g}$. Monomials of the form $x_1^{k_1}...x_n^{k_n}$ with $\sum k_i \leq p$ span $U_p\mathfrak{g}$.

Proof. 1 is immediate. 2 is by induction on p. Notice

$$x(y_1...y_q) - (y_1...y_q)x = \sum_i y_1...[x, y_i]...y_q \in U_q\mathfrak{g}$$

so for $x \in \mathfrak{g}, y \in U_q\mathfrak{g}$, $xy \equiv yx \mod U_q\mathfrak{g}$. Then

$$x_1...x_{p+1}y \equiv yx_1...x_{p+1} \mod U_{p+q}\mathfrak{g}$$

3 is also by induction. The base case is clear. Note $U_{p+1}\mathfrak{g}$ is generated by elements of the form xy, $x \in \mathfrak{g}, y \in U_p\mathfrak{g}$. y can be written as a sum of monomials as in the theorem, so we have by part 2

$$x_i(x_1^{k_1}...x_n^{k_n}) - (x_1^{k_1}...x_i^{k_i+1}...x_n^{k_n}) \in U_p\mathfrak{g}$$

Again by the induction hypothesis, $x(x_1^{k_1}...x_n^{k_n})$ can be written as a sum of monomials. □

Corollary 2.11. *Each $U_p\mathfrak{g}$ is finite-dimensional and the associated graded algebra $Gr\ U\mathfrak{g} = \bigoplus_p U_p\mathfrak{g}/U_{p-1}\mathfrak{g}$ is commutative.*

Theorem 2.12. (Poincare-Birkhoff-Witt) *Let $x_1, ...x_n$ be an ordered basis of $\mathfrak{g}$. Monomials of the form $x_1^{k_1}...x_n^{k_n}$ with $\sum_i k_i \leq p$ form a basis of $U_p\mathfrak{g}$.*

It only remains to show linear independence. The idea is to construct a representation V with a basis given by $x_1^{k_1}...x_n^{k_n}$ for any k_i and show that the operators $\rho(x_1^{k_1}...x_n^{k_n})$ are linearly independent. The theorem can also be stated in the following equivalent manner:

Theorem 2.13. (Poincare-Birkhoff-Witt II) *The associated graded $Gr\ U\mathfrak{g}$ is isomorphic to the symmetric algebra $S\mathfrak{g}$. This isomorphism is given by the following maps which are well-defined:*

$$S^p\mathfrak{g} \to Gr^pU\mathfrak{g}, a_1...a_p \mapsto a_1...a_p \mod U_{p-1}\mathfrak{g}$$

$$Gr^pU\mathfrak{g} \to S^p\mathfrak{g}, a_1...a_p \mapsto a_1...a_p \text{ and } a_1...a_q \mapsto 0 \text{ if } l < q$$

Corollary 2.14. *The map $\mathfrak{g} \to U\mathfrak{g}$ is injective. This is immediate.*

Corollary 2.15. *Let $\mathfrak{g}_1, \mathfrak{g}_2 \subset \mathfrak{g}$ be Lie subalgebras such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ as a vector space. The multiplication map $U\mathfrak{g}_1 \otimes U\mathfrak{g}_2 \to U_\mathfrak{g}$ is an isomorphism of vector spaces.*

Proof. The multiplication map is a bijection on basis elements. □

Corollary 2.16. *$U\mathfrak{g}$ has no zero-divisors*

Proof. The product of two basis elements $x_1^{k_1}...x_n^{k_n} \cdot x_1^{j_1}...x_n^{j_n}$ is an element of $U_{2n}\mathfrak{g}$ written as $x_1^{k_1+j_1}...x_n^{k_n+j_n} + y$ where y denotes terms of lower degree. Since $x_1^{k_1+j_1}...x_n^{k_n+j_n} \neq 0$, this concludes the proof. □

2.3 Nilpotent and solvable Lie algebras

Definition 2.17. *The derived (or upper central) series of a Lie algebra* $\mathfrak{g}$ *is the sequence defined by* $D^0\mathfrak{g} = \mathfrak{g}$ *and* $D^{i+1}\mathfrak{g} = [D^i\mathfrak{g}, D^i\mathfrak{g}]$. *Note each* $D^i\mathfrak{g}$ *is an ideal and* $D^i\mathfrak{g}/D^{i+1}\mathfrak{g}$ *is abelian.*

Proposition 2.18. *The following are equivalent:*
1. $D^n\mathfrak{g} = 0$ *for some sufficiently large* n.
2. There is a sequence of subalgebras $\mathfrak{a}^0 = \mathfrak{g} \supset \mathfrak{a}^1 ... \supset \mathfrak{a}^k = 0$ *such that* $\mathfrak{a}^{i+1}$ *is an ideal in* $\mathfrak{a}^i$ *and the quotient* $\mathfrak{a}^i/\mathfrak{a}^{i+1}$ *is abelian.*

Proof. $1 \implies 2$ can be shown by taking $\mathfrak{a}^i = D^i\mathfrak{g}$. For $2 \implies 1$, note that $\mathfrak{a}^{i+1}$ contains $[\mathfrak{a}^i, \mathfrak{a}^i]$ since the quotient $\mathfrak{a}^i/\mathfrak{a}^{i+1}$ is abelian. The result follows from induction on i. □

Definition 2.19. *A Lie algebra is called solvable if it satisfies the equivalent conditions in the above proposition. (Solvable Lie algebras are those that can be obtained from semidirect products of one-dimensional Lie algebras)*

Definition 2.20. *The lower central series of a Lie algebra* $\mathfrak{g}$ *is the sequence defined by* $D_0\mathfrak{g} = \mathfrak{g}$ *and* $D_{i+1}\mathfrak{g} = [\mathfrak{g}, D_i\mathfrak{g}]$.

Proposition 2.21. *The following are equivalent:*
1. $D_n\mathfrak{g} = 0$ *for some sufficiently large* n.
2. There is a sequence of ideals $\mathfrak{a}_0 = \mathfrak{g} \supset \mathfrak{a}_1 ... \supset \mathfrak{a}_k = 0$ *such that* $[\mathfrak{g}, \mathfrak{a}_i] \subset \mathfrak{a}_{i+1}$.

Proof. $1 \implies 2$ can be shown as before by taking $\mathfrak{a}_i = D_i\mathfrak{g}$. For $2 \implies 1$, note that if $[g, \mathfrak{a}_i] \subset \mathfrak{a}_{i+1}$, then $D_i\mathfrak{g} \subset a_i$ since its the smallest algebra satisfying this condition. □

Definition 2.22. *A Lie algebra is called nilpotent if it satisfies the equivalent conditions in the above proposition. (This means that* $\mathfrak{g}$ *can be built from central extensions of abelian groups.)*

Example 2.23. *Let* $\mathfrak{b} \subset \mathfrak{gl}_n\mathbb{K}$ *be the subalgebra of upper triangular matrices and let* $\mathfrak{n}$ *be the subset of strictly upper triangular matrices.* $\mathfrak{b}$ *is solvable and* $\mathfrak{n}$ *is nilpotent.*
Let $\mathcal{F} = (\{0\} \subset V_1 ... \subset V_n = V)$ *be a (partial) flag in a finite dimensional vector space* V.
Let $\mathfrak{b}(\mathcal{F}) = \{x \in \mathfrak{gl}(V) : xV_i \subset V_i \text{ for all } V_i\}$
and let $\mathfrak{n}(\mathcal{F} = \{x \in \mathfrak{gl}(V) : xV_i \subset V_{i-1} \text{ for all } V_i\}$.
When $\mathcal{F}$ *is the standard flag, we recover* $\mathfrak{b}$ *and* $\mathfrak{n}$. *We will show* $\mathfrak{n}(\mathcal{F})$ *is nilpotent.*
Define the algebras $\mathfrak{a}_k(\mathcal{F}) = \{x \in \mathfrak{gl}(V) : xV_i \subset V_{i-k} \text{ for all } i\}$. *Then* $\mathfrak{a}_0 = \mathfrak{b}$ *and* $\mathfrak{a}_1 = \mathfrak{n}$. *Note for* $x \in \mathfrak{a}_k$ *and* $y \in \mathfrak{a}_l$, $xy \in \mathfrak{a}_{k+l}$ *so* $[\mathfrak{a}_k, \mathfrak{a}_l] \subset \mathfrak{a}_{k+l}$. *Then* $D_1\mathfrak{n} = [\mathfrak{a}_1, \mathfrak{a}_1] \subset \mathfrak{a}_2$ *and more generally* $D_i\mathfrak{n} = [\mathfrak{a}_1, D_{i-1}\mathfrak{n}] \subset \mathfrak{a}_{i+1}$ *by induction. Since* $\mathfrak{a}_i$ *eventually vanishes, this proves nilpotence of* $\mathfrak{n}$.
Now, for $x, y \in \mathfrak{b}$, $[x, y] = xy - yx \in \mathfrak{n}$ *since the diagonal entries of* xy *are those of* yx. *Thus* $D^1\mathfrak{b} \subset \mathfrak{a}_1$. *Using the fact that* $[\mathfrak{a}_k, \mathfrak{a}_l] \subset \mathfrak{a}_{k+l}$, *by induction we have* $D^{i+1}\mathfrak{b} \subset \mathfrak{a}_{2^i}$ *so* $\mathfrak{b}$ *is solvable.*
Note that $\mathfrak{b}$ *is not nilpotent: Let* x *be a diagonal matrix with entries* λ_k *and let* e_{ij} *be an elementary matrix in* $\mathfrak{b}$. *Using the fact that* $[x, e_{ij}] = (\lambda_i - \lambda_j)e_{ij}$ *and that elementary matrices generate* $\mathfrak{b}$, *we see that* $D_2\mathfrak{b} = [\mathfrak{b}, D_1\mathfrak{b}] = D_1\mathfrak{b} = \mathfrak{n}$.

We now list some useful properties of nilpotent and solvable Lie algebras:

Theorem 2.24. -
1. A real Lie algebra is solvable (resp nilpotent) if its complexification is solvable (resp nilpotent).
2. If $\mathfrak{g}$ is solvable (resp nilpotent), any subalgebra or quotient of $\mathfrak{g}$ is solvable (resp nilpotent).
3. If $\mathfrak{g}$ is nilpotent it is solvable.
4. If $I \subset \mathfrak{g}$ is an ideal such that I and $\mathfrak{g}/I$ are solvable, $\mathfrak{g}$ is solvable. This may not hold for nilpotence.

Proof. 1 and 2 can be deduced from the definitions. 3 follows from the inclusion $D^i\mathfrak{g} \subset D_i\mathfrak{g}$. To prove 4, let $\phi : \mathfrak{g} \to \mathfrak{g}/I$ be the canonical projection. $\phi(D^n\mathfrak{g}) = D^n(\mathfrak{g}/I) = 0$ for some sufficiently large $\mathfrak{n}$. This implies $D^n\mathfrak{g} \subset I$ which implies $D^{n+k\mathfrak{g}} \subset D^k I$ which is 0 for large enough k. □

2.4 Lie and Engel's theorems

Theorem 2.25. (Lie's theorem) *Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a complex representation of a solvable Lie algebra $\mathfrak{g}$. There exists a basis for V such that all operators $\rho(x)$ with $x \in \mathfrak{g}$ are upper-triangular. In other words, there is a (full) flag of invariant subspaces of V.*

Lemma 2.26. *Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a complex representation of a Lie algebra $\mathfrak{g}$. There exists a common eigenvector of all $\rho(x)$ with $x \in \mathfrak{g}$.*

Proof. Since $\mathfrak{g}$ is solvable, $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$. Then $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is a nonzero abelian Lie algebra with a codimension 1 ideal that corresponds to some codimension 1 ideal $\mathfrak{h}$ of $\mathfrak{g}$. We can write $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{C}x$ for some $x \in \mathfrak{g}$. We now induct on the dimension of $\mathfrak{g}$: There exists $v \in V$ which is a common eigenvector for all $\rho(h)$, $h \in \mathfrak{h}$ that has eigenvalue λ. Let W be the corresponding eigenspace of $\mathfrak{h}$. If $[x, h] = 0$ on W, then W is fixed by $\rho(x)$ and we're finished. Let $v_0 = v \in W, v_1 = \rho(x)v_0, v_2 = \rho(x)^2v_0$,...and let n such that $v_{n+1} \in \langle v_0, ...v_n\rangle$. Let $W_i = \langle v_0, ...v_i\rangle$. We claim W_i is stable under the action of any $h \in \mathfrak{g}'$ and that it is a subset of W:

$$hv_k = \lambda(h)v_k \text{ mod } W_{k-1}$$

This can be checked without pain via induction using the fact that $[h, x] \in \mathfrak{h}$:

$$hv_k = hxv_{k-1} = xhv_{k-1} + [h, x]v_{k-1} = \lambda(h)xv_{k-1} + \lambda([h, x])v_{k-1} + ...$$

where "..." indicates scalars of v_i, $i < k$.

This implies that in the basis $\{v_0, ...v_n\}$, $\rho(h)$ is upper-triangular with $\lambda(h)$ on the diagonal. Then we have $\text{tr}_W(\rho(h)) = (n + 1)(\lambda(h))$. Since $\text{tr}_W([h, x]) = 0$, we have $\lambda([h, x]) = 0$. Since x preserves the weight space W, W contains an eigenvector of x. Take this as the desired common eigenvector. □

This lemma directly implies Lie's theorem

Proof. We induct on the dimension of V. Let v be a common eigenvector of all $x \in \mathfrak{g}$. By the induction hypothesis, there is a basis $\{v_1, ...v_n\}$ in $V/\mathbb{C}v$ so that the action of $\mathfrak{g}$ in this basis is upper-triangular. Let $\tilde{v_i}$ be a preimage of v_i. Then the action of x in the basis $\{v, \tilde{v_1}, ...\tilde{v_n}\}$ is upper-triangular. □

Corollary 2.27. -
1. Any irreducible representation of a solvable Lie algebra is 1-dimensional
2. There is a unique conjugacy class of maximal solvable subalgebras of $\mathfrak{gl}_n\mathbb{C}$ (or $\mathfrak{gl}_n\mathbb{R}$).
3. If $\mathfrak{g}$ is a complex solvable Lie algebra, there exists a sequence of ideals $0 \subset I_1 \subset ... \subset I_n = \mathfrak{g}$ such that I_{k+1}/I_k is 1-dimensional.
4. $\mathfrak{g}$ is solvable iff $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

Proof. 1 and 2 are immediate. 3 follows by applying Lie's theorem to the adjoint representation and noting that subrepresentations correspond to ideals of $\mathfrak{g}$. To prove 4, first note if $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent, it is solvable. Since $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian, it is also solvable so $\mathfrak{g}$ is as well. Now, assume $\mathfrak{g}$ is solvable. By Lie's theorem applied to the adjoint action $\text{ad}\,\mathfrak{g} \subset \mathfrak{b}$ (where $\mathfrak{b}$ is the Borel algebra of upper-triangular matrices) in some basis of $\mathfrak{g}$. We have shown above that $\mathfrak{b}$ is solvable so the algebra $[\text{ad}\,\mathfrak{g}, \text{ad}\,\mathfrak{g}] = \text{ad}[\mathfrak{g}, \mathfrak{g}]$ is nilpotent (since $[x, y]$ always has trivial diagonal). Then for sufficiently large n and any $x_i \in [\mathfrak{g}, \mathfrak{g}], [x_1, ...[x_{n-1}, x_n]...] = 0$. As an operator, $\text{ad}[\mathfrak{g}, \mathfrak{g}] = [[\mathfrak{g}, \mathfrak{g}], \cdot] = -[\cdot, [\mathfrak{g}, \mathfrak{g}]]$ so take any $x \in \mathfrak{g}$ and we have $[x, ...[x_{n-1}, x_n]...] = 0$. □

Theorem 2.28. *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie subalgebra consisting of nilpotent operators. There exists a basis of V such that every $x \in \mathfrak{g}$ is strictly upper-triangular.*

The proof is omitted, but it is similar to the proof of Lie's theorem. It has the following corollary:

Theorem 2.29. (Engel's theorem) *A Lie algebra $\mathfrak{g}$ is nilpotent iff for every $x \in \mathfrak{g}$, $\text{ad}\, x : \mathfrak{g} \to \mathfrak{g}$ is nilpotent.*

Proof. If $\mathfrak{g}$ is nilpotent, then $[x, [x, ...[x, y]...] = \text{ad}^n\, x.y = 0$ for some n. If $\text{ad}\, x$ is nilpotent for all $x \in \mathfrak{g}$, then by 2.28 there exists a sequence of subalgebras $0 \subset \mathfrak{g}_1 ... \mathfrak{g}_n = \mathfrak{g}$ such that $\text{ad}\, x.\mathfrak{g}_i \subset \mathfrak{g}_{i-1}$ so $[\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_{i-1}$. □

2.5 Semisimple and reductive Lie algebras and Levi decomposition

Definition 2.30. *A Lie algebra $\mathfrak{g}$ is semisimple if it has no nonzero solvable ideals.*

Note that this implies $\mathfrak{z}(\mathfrak{g}) = 0$ since $\mathfrak{z}(\mathfrak{g})$ is always solvable. Semisimplie Lie algebras can be thought of as the "opposite" of solvable Lie algebras in that solvable Lie algebras are those which are close to being abelian, and semisimple are as far from abelian as possible.

Definition 2.31. *A Lie algebra $\mathfrak{g}$ is simple if it is not abelian and contains no proper ideals.*

We exclude abelian Lie algebras so the following result holds:

Lemma 2.32. *Any simple Lie algebra is semisimple*

Proof. If $\mathfrak{g}$ is simple and contains a nonzero solvable ideal, it must be $\mathfrak{g}$ itself, so $\mathfrak{g}$ must be solvable. Then $[\mathfrak{g}, \mathfrak{g}]$ is a nonzero ideal strictly smaller than $\mathfrak{g}$, which is a contradiction. □

Proposition 2.33. *Any Lie algebra $\mathfrak{g}$ contains a unique maximal solvable ideal known as the radical of $\mathfrak{g}$ and denoted $\text{rad}(\mathfrak{g})$.*

Proof. For existence, note that if I and J are solvable ideals, $I+J$ is as well: $(I+J)/I = J/(I \cap J)$ which is solvable since it is a quotient of a solvable ideal. Since $(I+J)/I$ and I are solvable, so is $I+J$. By induction, any finite sum of solvable ideals is solvable, so take $\mathrm{rad}(\mathfrak{g}) = \sum_I I$ where I runs over all solvable ideals. Since $\mathfrak{g}$ is finite-dimensional, this sum is finite. Uniqueness is obvious □

Corollary 2.34. *A Lie algebra $\mathfrak{g}$ is semisimple if and only if $\mathrm{rad}(\mathfrak{g}) = 0$.*

Corollary 2.35. *For any Lie algebra $\mathfrak{g}$, $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is semisimple. If $\mathfrak{b}$ is a solvable ideal such that $\mathfrak{g}/\mathfrak{b}$ is semisimple, $\mathfrak{b} = \mathrm{rad}(\mathfrak{g})$.*

Theorem 2.36. (Levi decomposition) *Any Lie algebra $\mathfrak{g}$ can be written as a direct sum $\mathfrak{g} = \mathrm{rad}(\mathfrak{g}) \oplus \mathfrak{g}_{ss}$ where $\mathfrak{g}_{ss}$ denotes a semisimple subalgebra of $\mathfrak{g}$.*

The proof is omitted. While not absurdly difficult, it involves some homological algebra beyond the background assumed in these notes.

Theorem 2.37. *Let V be an irreducible complex representation of $\mathfrak{g}$. Any $h \in \mathrm{rad}(\mathfrak{g})$ acts by scalar operators. That is, $\rho(h) = \lambda(h) Id_V$. Moreover, any $h \in [\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]$ acts by 0.*

Proof. Let V_λ be a weight space of $\mathrm{rad}(\mathfrak{g})$. As in the prove of Lie's theorem, one can show that $[x, h] = 0$ on V_λ for any $x \in \mathfrak{g}$ and $h \in \mathrm{rad}(\mathfrak{g})$. Thus, x preserves V_λ so it is a subrepresentation. Since V is irreducible, $V = V_\lambda$. □

The presence of non-zero elements that act by 0 in any irreducible representation is troubling from the view of representation theory, so we may want to consider algebras from which $[\mathfrak{g}, \mathrm{rad}(\mathfrak{g})] = 0$.

Definition 2.38. *A Lie algebra $\mathfrak{g}$ is reductive if $[\mathfrak{g}, \mathrm{rad}(\mathfrak{g})] = 0$. This is equivalent to saying $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is semisimple.*

Any semisimple Lie algebra is reductive, but a reductive Lie algebra may not be semisimple. From the Levi theorem, we have the following result

Corollary 2.39. *Any reductive Lie algebra $\mathfrak{g}$ is a direct sum of abelian and semisimple algebras $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_{ss}$ with $[\mathfrak{z}, \mathfrak{g}_{ss}] = 0$.*

Definition 2.40. *A bilinear for B on $\mathfrak{g}$ is invariant if $B(\mathrm{ad}x.y, z) = B(y, \mathrm{ad}x.z)$ for all $x, y, z \in \mathfrak{g}$.*

Lemma 2.41. *Let B be an invariant bilinear form on $\mathfrak{g}$ and let I be an ideal. Define $I^\perp = \{x \in \mathfrak{g} : B(x, y) = 0 \text{ for all } y \in I\}$. Then $I^\perp$ is an ideal and in particular $\mathrm{Ker}(B) = \mathfrak{g}^\perp$ is as well.*

Theorem 2.42. (Reductive criterion)
Define the symmetric invariant linear form B_V on a representation V of $\mathfrak{g}$ by $B_V(x, y) = tr_V(\rho(x)\rho(y))$. If B_V is nondengenrate, $\mathfrak{g}$ is reductive.

Lemma 2.43. *Let $\rho : \mathfrak{g} \to V$ be a finite-dimensional representation. If W is a subrepresentation of V, we have $B_V = B_W + B_{V/W}$*

Proof. Write $V = W \oplus V/W$. V has a basis $v_1, ...v_n$ such that $\{v_1, ..., v_k\}$ is a basis of W and the image of $\{v_{k+1}, ..., v_n\}$ under the quotient map $V \to V/W$ is a basis of V/W. Write $W' = \langle v_{k+1}, ..., v_n \rangle$ For $x \in \mathfrak{g}$, we can write $\rho(x) = \rho(y+z) = \rho(y) + \rho(z)$ for some $y \in W, z \in W'$. In this basis $\rho(x)$ has the form

$$\begin{pmatrix} A & * \\ 0 & B \end{pmatrix}$$

where A is the matrix of $\rho(y)$ in the representation W with the chosen basis (we have absued notation slightly here). We can see that B is the matrix of $\rho(z)$ in the quotient representation V/W. This concludes the proof. □

Armed with this lemma we can prove the theorem above.

Proof. Let $x \in [\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]$. It suffices to show $x = 0$. By Lie's theorem, $x \in \mathrm{Ker}(B_{V_i})$ for an irreducible representation $V_i \subset V$. If we have a short exact sequence $0 \to V_1 \to W \to V_2 \to 0$ then $B_W = B_{V_1} + B_{V_2}$ (this will be proven below). By induction, $x \in \mathrm{Ker}(B_V)$. Since B_V is nondegenerate, we're done. □

Corollary 2.44. (Properties of classical Lie algebras) *All classical Lie algebras are reductive.* $\mathfrak{sl}_n\mathbb{K}, \mathfrak{so}_n\mathbb{K}$ *(for $n > 2$),* $\mathfrak{su}_n, \mathfrak{sp}_n\mathbb{K}$ *are semisimple.* $\mathfrak{gl}_n\mathbb{K}$ *and* $\mathfrak{su}_n$ *have one-dimensional center:* $\mathfrak{gl}_n\mathbb{K} = \mathbb{K} \oplus \mathfrak{sl}_n\mathbb{K}$ *and* $mfu_n = i\mathbb{R} \oplus \mathfrak{su}_n$

Proof. For $\mathfrak{gl}_n$ this follows from the definition $B(xy) = \sum x_{ij}y_{ji}$. For $\mathfrak{sl}_n$ it follows from the decomposition $mfgl_n\mathbb{K} = \mathbb{K} \oplus \mathfrak{sl}_n\mathbb{K}$ and the fact that K and $\mathfrak{sl}_n\mathbb{K}$ are orthogonal with respect to B. For $\mathfrak{so}_n, B(x,y) = \sum x_{ij}y_{ji} = -2\sum x_{ij}y_{ij}$ which is nondegenerate. For $\mathfrak{u}_n, B(x,y) = -\mathrm{tr}(x\overline{y}^t) = -\sum x_{ij}\overline{y}_i j$ which is nondegenrate. In this case, $B(x,x) = -\sum |x_{ij}^2$ so B is negative definite. Therefore, its restriction to $\mathfrak{u}_n$ is also negative definite and non-degenerate. Semisimplicity can be shown by using the decomposition theorem of reductive Lie algebras and taking the quotients of $\mathfrak{gl}_n, \mathfrak{o}_n$, and $\mathfrak{u}_n$ by their centers. The only one left to compute is $\mathfrak{z}(\mathfrak{so}_n) = 0$. □

2.6 Killing form and Cartan's criteria

A special case of an invariant bilinear form B_V occurs when V is the adjoint representation.

Definition 2.45. *The Killing form on a Lie algebra $\mathfrak{g}$ is the bilinear form $K(\cdot,\cdot)$ defined by* $K(x,y) = \mathrm{tr}(\mathrm{ad}\ x\ \mathrm{ad}\ y)$.

Proposition 2.46. *The killing form is invariant and symmetric. More generally, if* $\rho : \mathfrak{g} \to \mathrm{Aut}(V)$ *is a representation, the form* $B_V(x,y) = \mathrm{tr}(\rho(x)\rho(y))$ *is invariant and symmetric.*

Proof. This can be checked directly. Symmetry is clear. To show invariance, use Ado's theorem to embed the Lie algebra into $\mathfrak{gl_n}\mathbb{K}$ and note:

$$B_V(\mathrm{ad}\, x.y, z) + B_V(y, \mathrm{ad}\, x.z) = \mathrm{tr}([x,y]z + y[x,z]) = \mathrm{tr}(xyz - yxz + yxz - yzx) = \mathrm{tr}(xyz - yzx) = 0$$

or recall that invariance under the adjoint action of $GL_n\mathbb{K}$ is equivalent to invariance under the adjoint action of the Lie algebra and notice that $\mathrm{tr}(gxg^{-1}gyg^{-1}) = tr(xy)$. □

Theorem 2.47. *Any invariant bilinear form on a simple Lie algebra is a scalar multiple of the Killing form.*

Proof. This follows from Schur's lemma, which will be presented in a later section. The proof is postponed until then. □

Next we will present Cartan's criteria. However, their proofs rely on the following result, which must be stated first:

Theorem 2.48. (Baby Jordan decomposition)
1.Any linear operator A on a finite-dimensional complex vector space V can be written as a sum of commuting semisimple (diagonalizable) and nilpotent operators $A = A_n + A_s$ with $A_n A_s = A_s A_n$.
2.Define ad $A : End(V) \to End(V)$ *by* ad $A.B = AB - BA$. *Then* $(\mathrm{ad}A)_s = \mathrm{ad}A_s$ *and* $\mathrm{ad}A_s$ *can be written as a polynomial $P \in t\mathbb{C}[t]$ in* $\mathrm{ad}A$.
3. If $\overline{A}_s$ is an operator with the same eigenspaces as A but with complex conjugate eignevalues. Then $\overline{A}_s$ can also can be written as a polynomial $Q \in t\mathbb{C}[t]$ in $\mathrm{ad}A$.

We won't present the proof of this theorem right now, but we will quickly recall some useful definitions and lemmas it uses.

Definition 2.49. *An operator A is nilpotent if $A^n = 0$ for some n. A is semisimple if every A-invariant subspace has an A-invariant complement.*

Lemma 2.50. *Let V be a complex finite-dimensional vector space and let $A : V \to V$ be an operator.*
1. A is semisimple iff it is diagonalizable (this doesn't hold if V is a real vector space).
2. The restriction of a semisimple operator A to an invariant subspace W is semisimple and so is its restriction to V/W.
3. The sum of commuting semisimple (resp nilpotent) operators is semisimple (resp nilpotent).

We finally present the main results of this section:

Theorem 2.51. (Cartan's solvability criterion)
A Lie algebra $\mathfrak{g}$ is solvable iff $K([\mathfrak{g}, \mathfrak{g}], \mathfrak{g}) = 0$.

Proof. Note that if $\mathfrak{g}$ is a real Lie algebra, it is solvable iff $\mathfrak{g}_{\mathbb{C}}$ is solvable and that $K([\mathfrak{g}, \mathfrak{g}], \mathfrak{g}) = 0$ iff $K([\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}], \mathfrak{g}_{\mathbb{C}}) = 0$ so we may assume $\mathfrak{g}$ is complex.

If $\mathfrak{g}$ is solvable, by Lie's theorem, there is a basis of $\mathfrak{g}$ so ad x are upper-triangular for all $x \in \mathfrak{g}$. In this basis, ad y for $y \in [\mathfrak{g}, \mathfrak{g}]$ is strictly upper-triangular, so $K(x, y) = \mathrm{tr}(\mathrm{ad}\ x \cdot \mathrm{ad}\ y) = 0$.

For the other direction, we'll prove the following lemma:

Lemma 2.52. *Let V be a complex vector space and let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie subalgebra such that for any $x \in [\mathfrak{g}, \mathfrak{g}]$ and $y \in \mathfrak{g}$ we have $\mathrm{tr}(xy) = 0$. Then $\mathfrak{g}$ is solvable.*

Proof. Let $x \in [\mathfrak{g}, \mathfrak{g}]$. Using Jordan decomposition, we write $x = x_s + x_n$. We see that $\mathrm{tr}(x\overline{x}_s) = \sum \lambda_i \overline{\lambda_i} = \sum |\lambda_i|^2$ where λ_i are the eigenvalues of x. Since $x \in [\mathfrak{g}, \mathfrak{g}]$ we can write $x = \sum [y_i, z_i]$ and so:

$$\mathrm{tr}(x\overline{x}_s) = \mathrm{tr}\left(\sum [y_i, z_i]\overline{x}_s\right) = \sum \mathrm{tr}(y_i[z_i, \overline{x}_s]) = -\sum \mathrm{tr}(y_i[\overline{x}_s, z_i])$$

Now note $[\overline{x}_s, z_i] = \mathrm{ad}\ \overline{x}_s.z_i = Q(\mathrm{ad}\ x).z_i \in [\mathfrak{g}, \mathfrak{g}]$ for some polynomial Q so in particular $\mathrm{tr}(x\overline{x}_s) = \sum |\lambda_i|^2 = 0$ so all eigenvalues are 0 and x is nilpotent. Thus $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent and $\mathfrak{g}$ is solvable. □

Now suppose $K(x, y) = 0$ for $x \in \mathfrak{g}, y \in [\mathfrak{g}, \mathfrak{g}]$. Then by the above lemma, $\mathrm{ad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ is solvable. Since $\mathrm{ad}(\mathfrak{g}) = \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ and $\mathfrak{z}(\mathfrak{g})$ is solvable, $\mathfrak{g}$ is solvable. □

Theorem 2.53. (Cartan's semisimplicity criterion)
A Lie algebra $\mathfrak{g}$ is semisimple iff the Killing form is nondegenerate.

Proof. If K is nondegenerate, $\mathfrak{g}$ is reductive. If $x \in \mathfrak{z}(\mathfrak{g})$, ad $x = 0$ so $x \in \mathrm{Ker}(K)$. This implies $\mathfrak{z}(\mathfrak{g}) = 0$ so $\mathfrak{g}$ is semisimple.

If $\mathfrak{g}$ is semisimple, consider $I = \mathrm{Ker}(K)$. The restriction of K to an ideal coincides with the Killing form of the ideal, so the Killing form of I is 0 which implies (by Cartan's solvability criterion) that I is solvable. Since $\mathfrak{g}$ is semisimple, $I = 0$ which implies K is nondegenerate. □

Before we restrict ourselves to discussing only complex semisimple Lie algebras, we will quickly state a few results relating the killing form on real lie algebras and the corresponding compact Lie groups. The proofs of some of these statements will be omitted.

Theorem 2.54. *Let G be a compact real Lie group. Then $\mathfrak{g}$ is reductive and its Killing form is negative semidefinite, with $\mathfrak{z}(\mathfrak{g}) = \mathrm{Ker}(K)$. In particular, the Killing form of $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is negative definite. Moreover, if $\mathfrak{g}$ is a semisimple real Lie algebra with negative definite Killing form, is it the Lie algebra of a compact real Lie group.*

Proof. We will prove later that any representation of a compact group is unitary. Given this fact, $B_V(x, y) = \mathrm{tr}(xy) = -\mathrm{tr}(x\overline{y}^t)$ where x and y are skew-Hermitian. In particular, they have only imaginary eigenvalues and so $tr(x\overline{y}^t)$ is always negative or 0. Thus B_V is negative semidefinite and since $\mathrm{tr}(x^2) = -\sum |x_{ij}^2 \leq 0$ is 0 iff $x = 0$ which implies $\mathrm{Ker}(B_V) = \mathrm{Ker}(\rho)$. Now taking $V = \mathfrak{g}_{\mathbb{C}}$ and ρ to be the complexified adjoint representation proves the Killing form is negative semidefinite with $\mathrm{Ker}(K) = \mathfrak{z}(\mathfrak{g})$.

If $\mathfrak{g}$ is a real Lie algebra with negative definite Killing form, let G be a connected Lie group with $\mathrm{Lie}(G) = \mathfrak{g}$. $B(x, y) := -K(x, y)$ is positive definite and Ad G invariant. This implies Ad $G \subset SO(\mathfrak{g})$. Since $\mathrm{Ad}(G)$ is the connected component of the identity in Aut $\mathfrak{g}$ (this will also be proven later) and Aut $\mathfrak{g} \subset GL(\mathfrak{g})$ is a closed Lie subgroup, $\mathrm{Ad}(G)$ is a closed Lie subgroup of the compact group $SO(\mathfrak{g})$ so it is itself compact. Since $\mathrm{Ad}(G) = G/Z(G)$, $\mathrm{Lie}(\mathrm{Ad}(G)) = \mathfrak{g}/\mathfrak{z}(\mathfrak{g}) = \mathfrak{g}$. □

Remark 2.55. *One can actually prove that if $\mathfrak{g}$ is a real Lie algebra with negative definite Killing form, any connected Lie group G with* $\mathrm{Lie}(G) = \mathfrak{g}$ *is compact.*

Proposition 2.56. *If $\mathfrak{g}$ is a real Lie algebra with positive definite Killing form, $\mathfrak{g} = 0$.*

Proof. Let G such that $\text{Lie}(G) = \mathfrak{g}$. The Killing form is Ad G invariant and positive definite so as above, $\text{Ad}(G)$ is a closed Lie subgroup of $SO(\mathfrak{g})$ and $\mathfrak{g}/\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{so}(\mathfrak{g})$ but $\mathfrak{z}(\mathfrak{g}) = 0$ since K is positive definite. This implies $\mathfrak{g}$ is reductive so the Killing form is negative semidefinite and therefore $\mathfrak{g} = 0$. □

Remark 2.57. *It turns out that the only complex compact Lie groups are Tori. We will probably prove this somewhere else at some point.*

3 Complex Semisimple Lie Algebras

3.1 Basic properties

We'll now turn our attention to our main object of study, complex semisimple Lie algebras. We begin by proving some basic results.

Proposition 3.1. *A real Lie algebra $\mathfrak{g}$ is semisimple iff $\mathfrak{g}_{\mathbb{C}}$ is semisimple. Note this does not hold if we replace semisimple with simple.*

Proof. This follows from Cartan's criteria. □

Theorem 3.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $I \subset \mathfrak{g}$ and ideal. There exists another ideal I' such that $\mathfrak{g} = I \oplus I'$.*

Proof. Let I' be the orthogonal; complement of I with respect to the Killing form. $I \cap I'$ is an ideal with zero Killing form, so by Cartan's criteria, it is solvable. Since $\mathfrak{g}$ is semisimple, this implies $I \cap I' = 0$ so $\mathfrak{g} = I \oplus I'$ □

Corollary 3.3. *A Lie algebra is semisimple iff it is a direct sum of simple Lie algebras*

Proof. Any semisimple Lie algebra is simple and by Cartan's criteria, the direct sum of semisimple Lie algebras is semisimple. The other direction follows from induction using the previous theorem. □

Corollary 3.4. *If $\mathfrak{g}$ is a semisimple Lie algebra, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.*

Proof. If $\mathfrak{g}$ is simple, $[\mathfrak{g}, \mathfrak{g}]$ is a nonzero ideal (because otherwise $\mathfrak{g}$ would be abelian), so we must have $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Since semisimple Lie algebras are direct sums of simple ones, this completes the proof. □

Theorem 3.5. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_k$ be a semisimple Lie algebra with each $\mathfrak{g}_i$ simple. Any ideal $I \subset \mathfrak{g}$ is equal (not just isomorphic) to $\bigoplus_{i \in S} \mathfrak{g}_i$ for some subset $S \subset \{1, ..., k\}$.*

Proof. We induct on k. Consider the projection $\pi_k : \mathfrak{g} \to \mathfrak{g}_k$. $\pi_k(I)$ is either 0 or $\mathfrak{g}_i$ since $\mathfrak{g}_i$ is simple. If $\pi_k(I) = 0$, the induction hypothesis completes the proof. Otherwise, $\pi_k(I) = \mathfrak{g}_i$. Then $[\mathfrak{g}_k, I] = [\mathfrak{g}_k, \pi_k(I)] = [\mathfrak{g}_k, \mathfrak{g}_k] = \mathfrak{g}_k$ since $\mathfrak{g}_k$ is simple. Since I is an ideal, $\mathfrak{g}_k \subseteq I$ which implies $I = J \oplus \mathfrak{g}_k$ for some ideal $\mathfrak{g}_1 \oplus \mathfrak{g}_{k-1}$. Then J can be written as a direct sum in the desired form by the induction hypothesis, so we're done. □

Corollary 3.6. *Any ideal in a semisimple Lie algebra is semisimple.*

Proposition 3.7. *Let G be a connected Lie group with semisimple Lie algebra $\mathfrak{g}$.* Der $\mathfrak{g} = \mathfrak{g}$ *and* Aut $\mathfrak{g}$/Ad G *is discrete where* Ad $G = G/Z(G) = Im(\text{Ad}\colon G \to GL(\mathfrak{g}))$ *is the adjoint group.*

Proof. Recall that for all $x \in \mathfrak{g}, \text{ad } x$ is a derivation. The map $\varphi : \mathfrak{g} \to \text{Der } \mathfrak{g}$ given by $x \mapsto \text{ad } x$ is a morphism of Lie algebras. Since $\text{Ker}(\varphi) = \mathfrak{z}(\mathfrak{g}) = 0$, φ is injective so it is a subalgebra of Der $\mathfrak{g}$. Now, for any derivation δ and $x \in \mathfrak{g}$, $\text{ad}(\delta(x)) = [\delta, \text{ad } x]$.

Since $\text{ad}(\delta(x)) \subset \mathfrak{g}$, this proves $\mathfrak{g}$ is an ideal.

Now, extend the Killing form of $\mathfrak{g}$ to Der $\mathfrak{g}$ by $K(\delta_1, \delta_2) = tr_{\mathfrak{g}}(\delta_1, \delta_2)$. Let $I = \mathfrak{g}^{\perp}$. This is an ideal since K is Der $\mathfrak{g}$-invariant. Since the restriction of K to $\mathfrak{g}$ is nondegenerate (since it's a semisimple ideal), $I \cap \mathfrak{g} = 0$ so Der $\mathfrak{g} = I \oplus \mathfrak{g}$. Since both $\mathfrak{g}$ and I are ideals, $[I, \mathfrak{g}] = 0$ which implies for all $\delta \in I$, $x \in \mathfrak{g}$, $\text{ad}(\delta(x)) = 0$ so $\delta = 0 \implies I = 0$.

To prove the second statement, note that the Lie algebra of Aut $\mathfrak{g}$ is Der $\mathfrak{g} = \mathfrak{g}$. This implies Aut $\mathfrak{g}$ is a covering space of Ad G, which completes the proof. □

3.2 Toral subalgebras and Jordan decomposition

Definition 3.8. *An element x of a Lie algebra $\mathfrak{g}$ is semisimple (resp. nilpotent) if* $\text{ad } x : \mathfrak{g} \to \mathfrak{g}$ *is semisimple (resp. nilpotent). (Recall that an operator A is semisimple if every A-invariant subspace has an invariant complement).*

Proposition 3.9. *Semisimplicity is equivalent to diagonalizability when $\mathfrak{g} = \mathfrak{gl}_n\mathbb{C}$*

Proof. Certainly, if x is semisimple, ad x gives an eigenbasis of $\mathfrak{g}$: Let v_1 be an eigenvector of of ad x. Then $\mathfrak{g} = v_1\mathbb{C} \oplus U$ for some ad x-invariant subspace U. This gives an eigenbasis by induction. Conversely, if x is diagonalizable, any x-invariant subspace U splits into a direct sum $U = \bigoplus(W \cap V_{\lambda_i}) = \bigoplus W_i$ for eigenspaces V_{λ_i} and some collection of subspaces W_i. Since $\mathfrak{g}$ is finite-dimensional, for each V_{λ_i} there exists $W_i^{\perp}$ such that $V_{\lambda_i} = W_i \oplus W^{\perp} = \bigoplus W_i^{\perp}$ and the sum $W^{\perp} = \bigoplus W_i^{\perp}$ is the desired x-invariant subspace. □

Proposition 3.10. *This is equivalent to the usual definition of semisimplicity when $\mathfrak{g} = \mathfrak{gl}_n\mathbb{C}$.*

Proof. We want to show that as operators, ad x is semisimple iff x is. If $x = x_s$, then $\text{ad } x = \text{ad } x_s = (\text{ad } x)_s$. This can be checked by taking a basis of elementary matrices for $\mathfrak{g}$ and seeing that the action of ad x is diagonal. Conversely, if ad x is semisimple, $\text{ad } x = \text{ad } x_s$. If $x = x_s + x_n$ with with x_n, nonzero, we have $\text{ad } x_s = \text{ad } (x_s + x_n)$ which implies $x_n \in Z(\mathfrak{g})$ so it's diagonal. Therefore $x_n = 0$ since it's also nilpotent. □

Theorem 3.11. (Jordan decomposition)
If $\mathfrak{g}$ is a complex semisimple Lie algebra, any $x \in \mathfrak{g}$ can be written uniquely as $x = x_s + x_n$ where x_n is semisimple, x_n is nilpotent, and $[x_s, x_n] = 0$. Further, if there is $y \in \mathfrak{g}$ such that $[x, y] = 0$, then $[x_s, y] = 0$ as well.

We omit the proof of this theorem for now.

Corollary 3.12. *Every semisimple Lie algebra contains at least one nonzero semisimple element.*

Proof. Suppose not. Then every we have $x = x_n$ for every $x \in \mathfrak{g}$. In particular, x is nilpotent so by Engel's theorem, $\mathfrak{g}$ is nilpotent and therefore solvable. This contradicts the semisimplicity of $\mathfrak{g}$. □

Definition 3.13. *A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called toral if it is commutative and consists only of semisimple elements.*

Toral subalgebras can tell us a lot about the structure of the Lie algebra they are contained in.

Theorem 3.14. *Let $\mathfrak{h} \subset \mathfrak{g}$ be a toral subalgebra of a complex semisimple Lie algebra $\mathfrak{g}$. Let $(\cdot, \cdot)$ be an invariant bilinear form on $\mathfrak{g}$ (such as the Killing form). Then:*
1. $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}} \mathfrak{g}_\alpha$ where g_α is a maximal common eigenspace of all operators ad h, *$h \in \mathfrak{h}$, that have weight α. That is:* ad $h.x = \langle \alpha, h \rangle x$ *for all $x \in \mathfrak{g}_\alpha$. In particular, $\mathfrak{h} \subset \mathfrak{g}_0$ since it is commutative.*
2. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.
3. If $\alpha + \beta \neq 0$, $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal.
4. For any α, $(\cdot, \cdot)$ is a non-degenerate pairing $\mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha} \to \mathbb{C}$.

Proof. 1. All operators ad h are diagonalizable since $\mathfrak{h}$ is total so they commute and thus are simultaneously diagonalizable.
2. This is actually a special case of the following:

Theorem 3.15. *If $\mathfrak{g}_\lambda$, $\mathfrak{g}_\mu$ are generalized eigenspaces, $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$*

The proof in the general case is harder, but it is easier in this case where we aren't working with *generalized* eigenspaces. Let $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$. For all $h \in \mathfrak{h}$, ad $h.[x, y =$ [ad $h.x, y] + [x,$ ad $h.y] = \langle \alpha, h \rangle [x, y] + \langle \beta, h \rangle [y, z] = \langle \alpha + \beta \rangle [y, z]$.
3. $([h, x], y) + (x, [h, y]) = 0$ since $(\cdot, \cdot)$ is invariant. $([h, x], y) + (x, [h, y]) = (\langle h, \alpha \rangle + \langle h, \beta \rangle)(x, y) = 0$ so if $(x, y) \neq 0$, we have $\alpha + \beta = 0$. 4. Follows immediately from 3. □

We end this section by proving some properties of $\mathfrak{g}_0$:

Lemma 3.16. -
1. The restriction of $(\cdot, \cdot)$ to $\mathfrak{g}_0$ is nondegenerate.
2. Let $x \in \mathfrak{g}_0$ and $x = x_s + x_n$ be the Jordan decomposition of x. Then $x_s, x_n \in \mathfrak{g}_0$.
3. $\mathfrak{g}_0$ is a reductive subalgebra of $\mathfrak{g}$.

Proof. 1. Follows from part 4 of the last theorem.
2. If $x \in \mathfrak{g}_0$, $[h, x] = 0$ for all $h \in \mathfrak{h}$. By Jordan decomposition, $[h, x_s] = 0$ as well so $x_s \in \mathfrak{g}_0$. This now implies $x_n = x - x_s \in \mathfrak{g}_0$. 3. The restriction of the Killing form on $\mathfrak{g}$ to $\mathfrak{g}_0$ is nondegenerate by 1. Note that this restriction is $(x, y) = tr_{\mathfrak{g}}(\text{ad } x, \text{ad } y)$ which is clearly a trace form. Then by the criteria for reductive Lie algebras, $\mathfrak{g}_0$ is reductive. □

3.3 Cartan subalgebras

Definition 3.17. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. A total subalgebra $\mathfrak{h}$ is a Cartan subalgebra if $\mathfrak{h} = C(\mathfrak{h}) = \{x : [x, h] = 0 \text{ for all } h \in \mathfrak{h}\}$.*

Example 3.18. *Let $\mathfrak{g} = \mathfrak{sl}_n\mathbb{C}$. The set of diagonal matrices with trace 0 is a total subalgebra. Now pick some $h \in \mathfrak{h}$ with distinct eigenvalues. Let $x \in C(\mathfrak{h})$. Then x and h commute so they share all eigenvectors which implies x is diagonal and thus $c(\mathfrak{h}) \subset \mathfrak{h}$. Since $\mathfrak{h}$ is commutative, this is an equality.*

We will see later that Cartan subalgebras are essential to the study of semisimple Lie algebras. A reasonable question to ask given this fact is whether or not they always exist. The answer, a corollary of the following theorem, is yes.

Theorem 3.19. *Every maximal total subalgebra is Cartan.*

Proof. Let $\mathfrak{h}$ be a maximal total subalgebra of $\mathfrak{g}$. Decompose $\mathfrak{g}$ into a direct sum of common eigenspaces $\mathfrak{g} = \bigoplus \mathfrak{g}_\alpha$ for ad h for all $h \in \mathfrak{h}$ as in theorem 3.14. It suffices to show $\mathfrak{g}_0 = C(\mathfrak{h})$ is toral. Note that for any $x \in \mathfrak{g}_0$, ad $x|_{\mathfrak{g}_0}$ is nilpotent. If not, ad $x|_{\mathfrak{g}_0}$ has nonzero eigenvalues so ad $x_s|_{\mathfrak{g}_0} \neq 0$ which implies $x_s \notin \mathfrak{h}$. However, $[\mathfrak{h}, x_s] = 0$ since $x_s \in \mathfrak{g}_0$ which implies $\mathfrak{h} \oplus \mathbb{C}x_s$ is toral. This contradicts the maximality of $\mathfrak{h}$.

By Engel's theorem, $\mathfrak{g}_0$ is nilpotent. Since it is also reductive, it must be commutative. This is because the quotient $\mathfrak{g}_0/\mathfrak{z}(\mathfrak{g}_0)$ is both semisimple and nilpotent, so it must be 0.

It remains to show $x \in \mathfrak{g}_0$ is semisimple. We will do this by showing that any nilpotent element is zero. If $x \in \mathfrak{g}_0$ is nilpotent then ad x is by definition. Since $\mathfrak{g}_0$ is commutative, for any $y \in \mathfrak{g}_0$, ad $x$ad y = ad $y$ad x is nilpotent. Therefore $K_\mathfrak{g}(x, y) = \mathrm{tr}_\mathfrak{g}(\text{ad } x\text{ad } y) = 0$, but the Killing form is nondegenerate on $\mathfrak{g}_0$, so by Jordan decomposition x is semisimple. □

Corollary 3.20. *Every complex semisimple Lie algebra contains a Cartan subalgebra.*

Definition 3.21. *The rank of a Lie algebra is the dimension of any of its Cartan subalgebras.*

We present the following result without proof to demonstrate that this is well-defined.

Theorem 3.22. *All Cartan subalgebras are conjugate.*

The proof of this theorem relies on a different definition of Cartan subalgebras as centralizers of normal elements (elements x whose multiplicity of 0 as a generalized eigenvalue of ad *x is minimal). These definitions are equivalent in the case of complex semisimple Lie algebras.*

3.4 Root decomposition

We will now use what we have established so far in the last two sections to give a concrete description of the structure of complex semisimple Lie algebras. We start by going over the most important results of the last section.

Theorem 3.23. (Root decomposition)
Let $\mathfrak{h}$ be a Cartan subalgebra of a complex semisimple Lie algebra $\mathfrak{g}$.
1. $\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$. $\mathfrak{g}_\alpha = \{x : [h,x] = \langle \alpha, h \rangle x$ *for all* $h \in \mathfrak{h}\}$ *are called the root subspaces and* $R = \{\alpha \in \mathfrak{h}^* \setminus \{0\} : \mathfrak{g}_\alpha \neq 0\}$ *is called the root system of* $\mathfrak{g}$.
2. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset [\mathfrak{g}_{\alpha+\beta}]$ *(here and below* $\mathfrak{h} = \mathfrak{g}_0$*).*
3. If $\alpha + \beta \neq 0$, $\mathfrak{g}_\alpha$ *and* $\mathfrak{g}_\beta$ *are orthogonal with respect to* K.
4. For any α, K *is a nondegenerate pairing* $\mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha} \to \mathbb{C}$. *In particular, the restriction of* K *to* $\mathfrak{h} = \mathfrak{g}_0$ *is nondegenerate.*

The proof can be found in the previous section. We will eventually use this decomposition to completely describe and classify complex semisimple Lie algebras. For now, here is an important consequence:

Theorem 3.24. *Let* $\mathfrak{g}_i$ *be simple Lie algebras and let* $\mathfrak{g} = \bigoplus \mathfrak{g}_i$.
1. Let $\mathfrak{h}_i$ *be Cartan subalgebras of* $\mathfrak{g}_i$ *and* R_i *the root systems of* $\mathfrak{g}_i$. $\mathfrak{h} = \bigoplus \mathfrak{h}_i$ *is Cartan and the corresponding root system is* $R = \bigsqcup R_i$.
2. All Cartan subalgebras of $\mathfrak{g}$ *are of the form* $\mathfrak{h} = \bigoplus \mathfrak{h}_i$.

Proof. 1. Follows immediately from definitions: the sum of toral subalgebras is toral and if $\mathfrak{h}'$ is a Cartan subalgebra containing $\mathfrak{h}$ the projection onto the simple factors $\mathfrak{g}_i$ must be equal to the projection of $\mathfrak{h}$ onto those factor which implies $\mathfrak{h} = \mathfrak{h}'$.
2. Let $\pi_i : \mathfrak{g} \to \mathfrak{g}_i$ be the projection onto the i^{th} factor and write $\mathfrak{h}_i = \pi_i(\mathfrak{h})$. For $x \in \mathfrak{g}_i$ and $h \in \mathfrak{h}$, $[h, x_i] = [\pi_i(h), x_i]$ so $\mathfrak{h}_i$ is Cartan. Now, certainly $\mathfrak{h} \subset \bigoplus \mathfrak{h}_i$. Since $\bigoplus \mathfrak{h}_i$ is toral and $\mathfrak{h}$ is Cartan we have $\bigoplus \mathfrak{h}_i \subset \mathfrak{h}$. □

Example 3.25. *Let* $\mathfrak{g} = \mathfrak{sl}_n\mathbb{C}$ *and let* $\mathfrak{h}$ *be the subalgebra of diagonal matrices with trace 0. Let* e_i *denote the functional which returns the* i^{th} *diagonal entry of* $h \in \mathfrak{h}$ *(denote this entry* h_i*).* $\sum_{e_i} = 0$ *so we have* $\mathfrak{h}^* = \bigoplus \mathbb{C}e_i / \mathbb{C}(e_1 + ...e_n)$. *By a simple computation, we see that for a matrix unit* E_{ij}, ad $h = [h, E_{ij}] = hE_{ij} - E_{ij}h = (h_i - h_j)E_{ij} = (e_i - e_j)(h)E_{ij}$. $(e_i - e_j)(h)$ *is a scalar so each matrix unit is an eigenvector for* ad h. *The root decomposition is given by:* $R = \{e_i - e_j\} \subset \mathfrak{h}^*$ *and* $\mathfrak{g}_{e_i - e_j} = \mathbb{C}E_{ij}$.
The Killing form on $\mathfrak{h}$ *is given by* $(h, h') = \sum_{i \neq j}(h_i - h_j)(h'_i - h'_j) = 2n \sum h_i h'_i = 2\mathrm{tr}(hh')$. *The corresponding form on* $\mathfrak{h}^*$ *is* $\frac{1}{2n} \sum \lambda_i \mu_i$.

Since the restriction of the Killing form to $\mathfrak{h}$ *is always nondegenerate, it defines an isomorphism* $\mathfrak{h} \to \mathfrak{h}^*$ *and an invariant bilinear form* $(\cdot, \cdot)$ *on* $\mathfrak{h}^*$. *For* $\alpha \in \mathfrak{h}^*$, *write* $H_\alpha \in \mathfrak{h}$ *for the corresponding element of* $\mathfrak{h}$ *under this isomorphism. Then* $(\alpha, \beta) = \langle H_\alpha, \beta \rangle = (H_\alpha, H_\beta)$ *for all* $\alpha, \beta \in \mathfrak{h}^*$.

Lemma 3.26. *Let* $e \in \mathfrak{g}_\alpha$ *and* $f \in \mathfrak{g}_{-\alpha}$. *Then* $[e, f] = (e, f)H_\alpha$.

Proof. Let $h \in \mathfrak{h}$. We have $([e, f], h) = (e, [f, h]) = -(e, [h, f]) = \langle h, \alpha \rangle (e, f) = (e, f)(h, H_\alpha) = ((e, f)H_\alpha, h)$ which implies the equality since $(\cdot, \cdot)$ is nondegenerate. □

Lemma 3.27. -
1. Let $\alpha \in R$. $(\alpha, \alpha) = (H_\alpha, H_\alpha) \neq 0$.
2. Let $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\alpha}$ *such that* $(e, f) = \frac{2}{(\alpha, \alpha)}$ *and let* $h_\alpha = \frac{2H_\alpha}{(\alpha, \alpha)} = (e, f)H_\alpha$.

Then $\langle h_\alpha, \alpha\rangle = 2$ and e, f, h_α satisfy the defining relations of $\mathfrak{sl}_2\mathbb{C}$. We denote the subalgebra generated by these elements as $(\mathfrak{sl}_2\mathbb{C})_\alpha \subset \mathfrak{g}$.
3. h_α is independent of the choice of the bilinear form $(\cdot,\cdot)$.

Proof. 1. Assume $(\alpha,\alpha) = 0$. Then $(H_\alpha, H_\alpha) = 0$ as well. Let $e \in \mathfrak{g}_\alpha, f \in g_{-\alpha}$ such that $(e, f) \neq 0$. Let $h = [e, f]$ and consider the subalgebra $\mathfrak{a}$ generated by e, f, and h. We have $[h, e] = \langle h, \alpha\rangle e = (H_\alpha, H_\alpha) = 0$ and similarly, $[h, f] = 0$ so $\mathfrak{a}$ is solvable. By Lie's theorem, there exists a basis of $\mathfrak{g}$ such that ad e, ad f, and ad h are upper triangular. Since $h \in [\mathfrak{a}, \mathfrak{a}]$ and $\mathfrak{a}$ is solvable, ad h is nilpotent but it is also semisimple since it is an element of a Cartan subalgebra. This implies $h = 0$ which is a contradiction since $h = (e, f)H_\alpha$ is nonzero.
2. This follows from the above lemma and a few straightforward computations
3. If $\mathfrak{g}$ is simple, all invariant bilinear forms are multiples of the Killing form, so the result is true in this case. If $\mathfrak{g}$ is only semisimple, by theorem 3.24 the projection of h_α onto any of the simple factors of $\mathfrak{g}$ is invariant of the choice of bilinear form, $\mathfrak{h}_\alpha$ must be as well.
□

This lemma will turn out to be quite powerful since it allows us to study $\mathfrak{g}$ as an $(\mathfrak{sl}_2\mathbb{C})_\alpha$ module and use well-known results about the representations of $\mathfrak{sl}_2\mathbb{C}$. These can be found later in the notes.

Theorem 3.28. (Properties of root systems)
Let $\mathfrak{g}$ be a complex semisimple Lie algebra with a Cartan subalgebra $\mathfrak{h}$ and root decomposition $\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha$. Let $(\cdot,\cdot)$ be a symmetric nondegenerate invariant bilinear form on $\mathfrak{g}$.

1. R spans $\mathfrak{h}^*$ as a vector space and $\mathfrak{h}_\alpha, \alpha \in R$ spans $\mathfrak{h}$.
2. For each $\alpha \in R$ the root subspace $\mathfrak{g}_\alpha$ is one-dimensional.
3. For any two roots $\alpha, \beta \in R$, $\langle h_\alpha, \beta\rangle = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer.
4. Define the reflection operator $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ by $s_\alpha(\lambda) = \lambda - \langle h_\alpha, \lambda\rangle\alpha = \lambda - \frac{2(\alpha,\lambda)}{(\alpha,\alpha)}\alpha$. Then for any roots $\alpha, \beta \in R$, $s_\alpha(\beta)$ is also a root.
5. For any root α, the only multiples of α which are also roots are $\pm\alpha$
6. For roots $\alpha, \beta \neq \pm\alpha$, $V = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{\beta+k\alpha}$ is an irreducible representation of $(\mathfrak{sl}_2\mathbb{C})_\alpha$
7. If α and β are roots such that $\alpha + \beta$ is also a root, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

The proof of this theorem uses the following lemma and various properties of $\mathfrak{sl}_2\mathbb{C}$ representations whose proofs are postponed until the corresponding section.

Lemma 3.29. *For a root α,*

$$V = \mathbb{C}h_\alpha \bigoplus_{k\in\mathbb{Z}^\times} \mathfrak{g}_{k\alpha}$$

is an irreducible $(\mathfrak{sl}_2\mathbb{C})_\alpha$ representation.

Proof. 1. Suppose R does not generate $\mathfrak{h}^*$. Then there exists some nonzero $h \in \mathfrak{h}$ so that $\langle h, \alpha\rangle = 0$. Then the root decomposition implies ad $h = 0$ so $h \in \mathfrak{z}(\mathfrak{g})$. Since $\mathfrak{g}$ is semisimple this implies $h = 0$, a contradiction. 2. This follows from the fact that all irreducible $\mathfrak{sl}_2\mathbb{C}$ representations have one-dimensional weight spaces. 3. Elements of $\mathfrak{g}_\beta$ have weight $\langle h_\alpha, \beta\rangle$ but weights of any finite-dimensional $\mathfrak{sl}_2\mathbb{C}$ representation are integer.

4. Let $\langle h_\alpha, \beta\rangle = n \geq 0$. Elements of $\mathfrak{g}_\beta$ have weight n under the action of $(\mathfrak{sl}_2\mathbb{C})_\alpha$. One can check that f_α^n is an isomorphism of the space of vectors of weight n to the space of vectors of weight $-n$. This means if $v \in \mathfrak{g}_\beta$ is nonzero, then ${}_\alpha f^n(v) \in \mathfrak{g}_{\beta-n\alpha}$ is nonzero as well. This implies $\beta - n\alpha = s_\alpha(\beta) \in R$. For $n < 0$, the proof is the same but using e_α^{-n} instead of f_α^n.
5. Assume α and $\beta = c\alpha$ are roots. By (3), $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = 2c$ is integer so c is a half-integer. The same argument shows $1/c$ is half-integer which implies $c \in \{\pm 1, \pm 2, \pm 1/2\}$. Interchanging the roots and replacing α with $-\alpha$ gives $c = 1$ or $c = 2$.

Now consider the subspace as in lemma 3.28. By (2), $V[2] = \mathfrak{g}_\alpha = \mathbb{C}e_\alpha$ so the map $e_\alpha : \mathfrak{g}_\alpha \to g_{2\alpha}$ is 0. However, the kernel of e is the highest weight subspace so V has highest weight 2. This implies $V = \mathfrak{g}_{-\alpha} \oplus \mathbb{C}h_\alpha \oplus \mathfrak{g}_\alpha$ so only α and $-\alpha$ are roots.
6. This follows immediately from the fact that $\dim(\mathfrak{g}_{\beta+k\alpha}) = 1$
7. We have seen that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$. Since $\dim(\mathfrak{g}_{\alpha+\beta}) = 1$, it suffices to show that for nonzero $e_\alpha \in \mathfrak{g}_\alpha$ and $e_\beta \in \mathfrak{g}_\beta$, $[e_\alpha, e_\beta] \neq 0$. This can be shown using (6) and the fact that if $v \in V[k]$ is nonzero and $V[k+2] \neq 0$, $e.v \neq 0$. □

Theorem 3.30. -
1. Let $\mathfrak{h}_\mathbb{R} \subset \mathfrak{h}$ be the real vector space generated by h_α. $\mathfrak{h} = \mathfrak{h}_\mathbb{R} \oplus i\mathfrak{h}_\mathbb{R}$ and the restriction of the Killing form to $\mathfrak{h}_\mathbb{R}$ is positive definite.
2. Let $\mathfrak{h}_\mathbb{R}^*$ be the real vector space generated by $\alpha \in R$. $\mathfrak{h}^* = \mathfrak{h}_\mathbb{R} \oplus i\mathfrak{h}_\mathbb{R}$ and $\mathfrak{h}_\mathbb{R}^* = (\mathfrak{h}_R)^* = \{\lambda \in \mathfrak{h}^* : \langle \lambda, h\rangle \in \mathbb{R}$ for all $\mathfrak{h} \in \mathfrak{h}_\mathbb{R}\}$.

Proof. (2) follows immediately from (1), so it suffices to prove only the first statement. We first show the restriction of K to $\mathfrak{h}_\mathbb{R}$ is real and positive definite:

$$(h_\alpha, h_\beta) = \mathrm{tr}(\mathrm{ad}\ h_\alpha\ \mathrm{ad}\ h_\beta) = \sum_{\gamma\in R} \langle h_\alpha, \gamma\rangle\langle h_\beta, \gamma\rangle$$

Since both $\langle h_\alpha, \gamma\rangle$ and $\langle h_\beta, \gamma\rangle$ are integers, as is (h_α, h_β) which in particular implies it is real. Now, let $h = \sum c_\alpha h_\alpha \in \mathfrak{h}_\mathbb{R}$ which we can do since R spans $\mathfrak{h}^*$. Then $\langle h, \gamma\rangle = \sum c_\alpha \langle h_\alpha, \gamma\rangle \in \mathbb{R}$ for all $\gamma \in R$ so

$$(h, h) = \mathrm{tr}(\mathrm{ad}\ h)^2 = \sum \langle h, \gamma\rangle^2 \geq 0$$

This implies K is positive definite on $\mathfrak{h}_\mathbb{R}$ so it is negative definite on $i\mathfrak{h}_\mathbb{R}$ which implies their intersection is 0. Let $r = \dim_\mathbb{C}\mathfrak{h}$. Since $\mathfrak{h}_\mathbb{R} \cap i\mathfrak{h}_\mathbb{R} = \{0\}$, $\dim_\mathbb{R}\mathfrak{h}_\mathbb{R} \leq \frac{1}{2}\dim_\mathbb{R}\mathfrak{h}$. However, since h_α span $\mathfrak{h}$ over $\mathbb{C}$, $\dim_R \mathfrak{h}_R \geq r$ so $\dim_\mathbb{R}\mathfrak{h}_R = r$ which implies $\mathfrak{h} = \mathfrak{h}_\mathbb{R} \oplus i\mathfrak{h}_\mathbb{R}$. □

Corollary 3.31. *If $\mathfrak{k}$ is a compact real form of $\mathfrak{g}, \mathfrak{k} \cap \mathfrak{h} = i\mathfrak{h}_\mathbb{R}$.*

Proof. This follows from the above and the fact that semisimple real Lie algebras of compact real Lie groups have negative definite Killing forms (see theorem 2.54). □

Example 3.32. *For $\mathfrak{g} = \mathfrak{sl}_n\mathbb{C}$, $\mathfrak{h}_\mathbb{R}$ is the set of traceless diagonal matrices with real entries and $\mathfrak{su}_n \cap \mathfrak{h} = i\mathfrak{h}_\mathbb{R}$ which is the set of traceless diagonal skew-Hermitian matrices.*

4 Root systems

4.1 Definitions and the Weyl group

In the following sections, we will closely study root systems in order to finally classify semisimple Lie algebras.

Definition 4.1. *A root system is a finite set $R \subset E^\times$ where E is a Euclidean (real with an inner product) vector space such that:*
1. R generates E.
2. For any $\alpha, \beta \in R$,

$$n_{\alpha\beta} = \frac{2(\alpha, \beta)}{(\beta, \beta)}$$

is integer.
3. Let $s_\alpha : E \to E$ be defined by

$$s_\alpha(\lambda) = \lambda - \frac{2(\alpha, \lambda)}{(\alpha, \alpha)}\alpha$$

then for any $\alpha, \lambda \in R$, $s_\alpha(\lambda) \in R$.

Definition 4.2. *A root system R is called reduced if it satisfies the following: If $\alpha, c\alpha$ are roots, $c = \pm 1$.*

Remark 4.3. *We have shown in the last section that we may have $c \in \{\pm 1, \pm 2, \pm 1/2\}$, but we will only consider reduced root systems from now on.*

Remark 4.4. *Conditions 2 and 3 in the definition of a root system have straightforward geometric meanings. 2 says that the projection of β onto α is a half-integer multiple of α and 3 says that the reflection of a root λ around the hyperplane $L_\alpha = \{\lambda \in E : (\alpha, \lambda) = 0\}$ orthogonal to α remains a root.*

We can quickly rephrase most of theorem 3.28 using this definition of a root system:

Theorem 4.5. *Let $\mathfrak{g}$ be a semisimple complex Lie algebra. Given a root decomposition, the set of roots forms a reduced root system.*

Example 4.6. (Root system of $\mathfrak{sl}_n\mathbb{C}$)
Let $\{e_i\}$ be the standard basis of $\mathbb{R}^n$ equipped with the standard inner product $(e_i, e_j) = \delta_{ij}$. Let $E = \{(\lambda_1, ..., \lambda_n) : \sum \lambda_i = 0\}$ and $R = \{e_i - e_j : 1 \leq i, j \leq n, i \neq j\} \cong \mathbb{R}^n/\mathbb{R}(1, ..., 1)$. It is easy to check that R is a reduced root system of rank $n-1$. We call this the root system of type A_{n-1}.

Definition 4.7. *An isomorphism of root systems $\varphi : R_1 \to R_2$ subsets of E_1 and E_2 respectively is a vector space isomorphism $\varphi : E_1 \to E_2$ such that $\varphi(R_1) = R_2$ and $n_{\varphi(\alpha)\varphi(\beta)} = n_{\alpha\beta}$ for all $\alpha, \beta \in R_1$.*

Remark 4.8. *The last condition is automatically satisfied if φ respects the inner product, but a root system isomorphism need not do so. Consider for example the isomorphism $R \to cR$ given by $\alpha \mapsto c\alpha$.*

We will primarily be interested in the automorphisms of a root system generated by reflections.

Definition 4.9. *The Weyl group W of a root system $R \subset E$ is the subgroup of $GL(E)$ generated by reflections s_α, $\alpha \in R$.*

Lemma 4.10. -
1. W is a finite subgroup of $O(E)$ and R is invariant under W.
2. For any $w \in W$ and $\alpha \in R$, $s_{w(\alpha)} = ws_\alpha w^{-1}$.

Proof. $s_\alpha(R) = R$ for all $\alpha \in R$ so $w(R) = R$. Since R is finite, $Aut(R)$ is as well so $W \subset Aut(R)$ must also be. To see (2), note that $ws_\alpha w^{-1}$ is the identity on $wL_\alpha = L_w(\alpha)$ and is a reflection corresponding to the root $w(\alpha)$; it is easy to check that $ws_\alpha w^{-1}$ sends $w(\alpha)$ to $-w(\alpha)$. ☐

Example 4.11. ***(Weyl group of A_{n-1})***
W is generated by transpositions $s_{e_i - e_j}$ which transpose the i^{th} and j^{th} entry of a root $(\lambda_1, ...\lambda_n) \in R$. In the case of $\mathfrak{sl}_2\mathbb{C}$, the root system is A_1 so $W = \mathbb{Z}_2$ and s acts by $\lambda \mapsto -\lambda$. One should note that for $n > 2$, the automorphism $\alpha \mapsto -\alpha$ is not an element of the Weyl group.

4.2 Rank two root systems

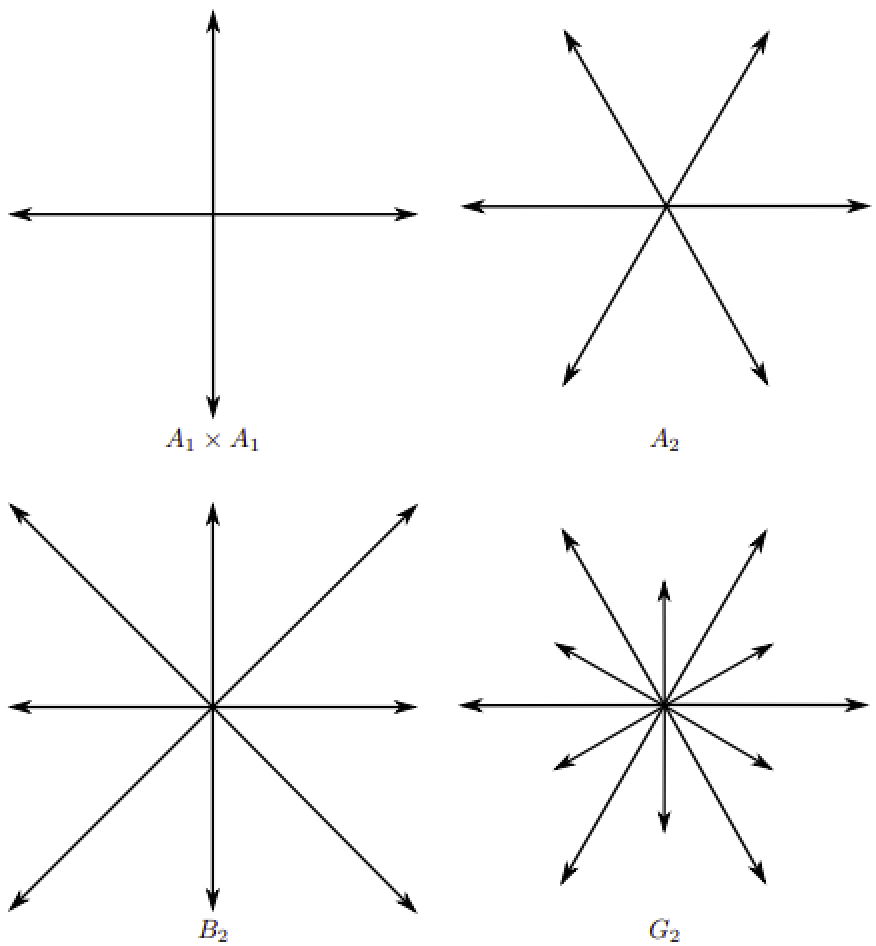
We would like to classify all root systems since this will enable us to classify semisimple Lie algebras. We begin by considering the rank 2 case. The conditions defining a root system impose strong conditions on their relative positions.

Theorem 4.12. *Let R be a reduced root system and let $\alpha, \beta \in R$ be roots that are not multiples of each other such that $|\alpha| \geq |\beta|$. Let φ be the angle between them. We must have one of the following:*
1. $\varphi = \pi/2$ (α and β are orthogonal), $n_\alpha\beta = n_\beta\alpha = 0$
2a. $\varphi = 2\pi/3, |\alpha| = |\beta|, n_\alpha\beta = n_\beta\alpha = -1$
2b. $\varphi = \pi/3, |\alpha| = |\beta|, n_\alpha\beta = n_\beta\alpha = 1$
3a. $\varphi = 3\pi/4, |\alpha| = \sqrt{2}|\beta|, n_\alpha\beta = -2, n_\beta\alpha = -1$
3b. $\varphi = \pi/4, |\alpha| = \sqrt{2}|\beta|, n_\alpha\beta = 2, n_\beta\alpha = 1$
4a. 3a. $\varphi = 5\pi/6, |\alpha| = \sqrt{3}|\beta|, n_\alpha\beta = -3, n_\beta\alpha = -1$
4b. $\varphi = \pi/6, |\alpha| = \sqrt{3}|\beta|, n_\alpha\beta = 3, n_\beta\alpha = 1$.

Proof. Since $(\alpha, \beta) = |\alpha||\beta| \cos\varphi$, we have $n_{\alpha\beta} = 2\frac{|\alpha|}{|\beta|}\cos\varphi$ so $n_\alpha\beta n_\beta\alpha = 4\cos^2\varphi$. $n_\alpha\beta n_\beta\alpha \in \mathbb{Z}$, so $n_\alpha\beta n_\beta\alpha \in \{0, 1, 2, 3\}$. Inspection of each case using $n_\alpha\beta/n_\beta\alpha = |\alpha|^2/|\beta|^2$ when $\cos\varphi \neq 0$ completes the proof. ☐

Each possible root system does indeed exist in $\mathbb{R}^2$. Before proceeding, we note that the product of two root systems $A \times B$ is often written as $A \sqcup B$. I believe this is because this is actually a coproduct and am inclined to prefer this notation because of this.

Theorem 4.13. (Classification of rank 2 root systems)
Each set of vectors in $\mathbb{R}^2$ pictured below is a root system. Any rank two reduced root system is isomorphic to one of them.



Proof. Checking that these are root systems is trivial. Let R be a rank 2 reduced root system. Pick $\alpha, \beta \in R$ such that φ is maximal and $|\alpha| \geq |\beta|$. Note $\varphi \geq \pi/2$ since otherwise, we could take α and $s_\alpha(\beta)$ to get a larger angle. Then we must be in case 1, 2a, 3a, or 4a of theorem 4.12.

Consider 2a. Applying s_α and s_β to α and β gives the root system A_2 so $A_2 \subseteq R$. If R contains a root $\gamma \notin A_2$, γ is between two roots of A_2 since R is reduced but this implies the angle between γ and some other root δ is strictly greater than $2\pi/3$ which cannot occur since we chose the angle between α and β to be maximal. repeating this analysis for cases 1, 3a, and 4a gives $R = A_1 \times A_1, B_2$, and G_2 respectively. □

Lemma 4.14. *Let R be a reduced root system. Let $\alpha, \beta \in R$ not multiples of each other such that $(\alpha, \beta) < 0$. $\alpha + \beta \in R$. Then $\alpha + \beta \in R$.*

Proof. It suffices to check the rank two case. The proof is by direct check of each of the four cases. □

4.3 Positive and simple roots

We will find it useful to have a notion of a minimal "generating set" of a root system.

Definition 4.15. *Fix some $t \in E$ such that for all $\alpha \in R$, $(\alpha, t) \neq 0$ (these are called regular elements). A polarization of a root system R is a decomposition $R = R_+ \sqcup R_-$ defined by $R_+ = \{\alpha : (\alpha, t) > 0\}$ and $R_- = \{\alpha : (\alpha, t) < 0\}$. One should note this is dependent on the choice of t.*

Elements of R_+ are called positive roots and elements of R_- are called negative roots.

Definition 4.16. *A positive root is called simple if it cannot be written as a sum of two positive roots. The set of simple roots is denoted as $\Pi \subset R_+$.*

Lemma 4.17. *Every positive root can be written as a sum of simple roots.*

Proof. If a positive root α is not simple, it can be written as $\alpha' + \alpha''$ with $(\alpha', t), (\alpha'', t) < (\alpha, t)$. Since root systems are finite, (α, t) can take on a finite number of values, so iteration of this process eventually terminates and gives a sum of simple roots. □

Lemma 4.18. *If $\alpha, \beta \in \Pi$ are simple with $\alpha \neq \beta$, $(\alpha, \beta) \leq 0$.*

Proof. Suppose not. Then $(-\alpha, \beta) < 0$ and $\beta' = -\alpha + \beta$ is a root by lemma 4.16. If β' is positive, $\beta = \beta' + \alpha$ is not simple, so β' is a negative root. Then $-\beta$ is positive and $\alpha = \beta - \beta'$ is not simple. Both cases are a contradiction. □

Theorem 4.19. *Let R be a polarized root system. The simple roots form a basis of the vector space E.*

Proof. Each positive root can be written as a linear combination of simple roots and each negative root is a product of a positive root with -1 since it can be given by a reflection: $-\alpha = s_\alpha(\alpha)$. Thus Π spans E. Linear independence follows from the following linear algebra result, the proof of which is omitted. □

Lemma 4.20. *If $B = \{v_i\}$ is a finite set of vectors in a Euclidean vector space such that for $i \neq j$, $(v_i, v_j) \leq 0$ and $(v_i, t) > 0$ for a fixed $t \in E$, B is linearly independent.*

Corollary 4.21. *Each $\alpha \in R$ can be written uniquely as a linear combination of simple roots with integer coefficients:*

$$\alpha = \sum_{i=1}^{n} c_i \alpha_i, \ c_i \in \mathbb{Z}$$

If α is positive, $n_i \geq 0$ for all i. If it is negative, $n_i \leq 0$ for all i.

Lemma 4.22. *A simple reflection s_i sends α_i to α_i and permutes the other positive roots. That is, $s_i(\alpha) \in R_+$ iff $\alpha \in R_+ \setminus \{\alpha_i\}$.*

Example 4.23. (Polarization of A_{n-1})
Choose a polarization of A_{n-1} by $R_+ = \{e_i - e_j : i < j\}$. One can check this defines a polarization. The simple roots are $\alpha_i = e_i - e_{i-1}$ and the height of a root is $ht(e_i - e_j) = j - i$.

We now introduce a useful tool that can be used to prove statements about positive roots via induction.

Definition 4.24. *The height $ht(\alpha)$ of a positive root $\alpha \in R_+$ is*

$$\text{ht}(\sum n_i \alpha_i) = \sum n_i \in \mathbb{Z}_{>0}$$

we should note that simple roots have height 1.

4.4 Root and weight lattices

We begin this section by introducing the notion of a coroot.

Definition 4.25. *The coroot $\alpha^\vee \in E^*$ of a root $\alpha \in R$ is defined by*

$$\langle \alpha^\vee, \lambda \rangle = \frac{2(\alpha, \lambda)}{(\alpha, \alpha)}$$

Remark 4.26. *In the case of a root system of a semisimple Lie algebra, this agrees with our definition of h_α. One should also note:*

$$n_{\alpha\beta} = \langle \alpha, \beta^\vee \rangle \text{ and } s_\alpha(\beta) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$$

Recall that a lattice L in a real vector space E is an abelian group generated by a basis of E. Any lattice $L \subset E$ can be identified with $\mathbb{Z}^n \subset \mathbb{R}^n$.

Definition 4.27. *The root lattice Q of a root system R is the abelian group generated by the elements of R. The coroot lattice $Q^\vee \subset E^*$ is the abelian group generated by all $\alpha^\vee$ with $\alpha \in R$.*

Definition 4.28. *The weight lattice $P \subset E$ of a root system R is*

$$P = \{\lambda \in E : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in R\} = \{\lambda \in E : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha^\vee \in Q^\vee\}$$

So P, not Q, is the dual lattice of $Q^\vee$ (here we are given another chance to complain about the ancient notation in this field). Elements of P are called (integral) weights.

Since simple $Q^\vee$ is generated by $\alpha_i^\vee$ with α_i simple, one can also define P as $\{\lambda \in E : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Pi\}$

We would like to define a $\mathbb{Z}$-basis of P.

Definition 4.29. *A fundamental weight $w_i \in E$ is defined by the property that $\langle w_i, \alpha_j^\vee \rangle = \delta_{ij}$.*

We can see right away that fundamental weights form an $\mathbb{R}$-basis of E and a $\mathbb{Z}$-basis of P. We can also see that since $n_{\alpha\beta} = \langle \alpha, \beta^\vee \rangle$ that $Q \subset P$, though this is rarely an equality.

Example 4.30. (Lattices in A_1)
Recall that A_1 has a single positive root α so $Q = \mathbb{Z}\alpha$ and $Q^\vee = \mathbb{Z}\alpha^\vee$. We can define $(\alpha, \alpha) = 2$ and get the identification $E \cong E^*$. Under this identification $\alpha \mapsto \alpha^\vee$ and $Q \mapsto Q^\vee$. The fundamental weight is $\alpha/2$ since $\langle \alpha, \alpha^\vee \rangle = 2$ so $P = \mathbb{Z}(\alpha/2)$.

4.5 Weyl chambers and simple reflections

In this section, we will answer two important questions about root systems:
1) Do different polarizations give equivalent sets of simple roots? (Spoiler: Yes)
2) Is it possible to recover the root system R from its simple roots? (Spoiler: Yes)

The Weyl group will provide us with the answers to these questions. Recall a polarization is defined by some $t \in E$ not in any hyperplane orthogonal to a root. The polarization depends not on t, but on the sign of (t, α) which remains unchanged as long as t does not cross any of the hyperplanes. This motivates the following

Definition 4.31. *A Weyl chamber C is a connected component of the complement to the hyperplanes orthogonal to the roots. That is to say C is a connected component of $E \backslash \bigcup_{\alpha \in R} L_\alpha$ where $L_\alpha = \{\lambda \in E : (\alpha, \lambda = 0\}$.*

We can specify a Weyl chamber by specifying, for each hyperplane, which side of it the Weyl chamber is on. Thus a Weyl chamber is described by a system of inequalities of the form $\pm(\alpha, \lambda) > 0$ with one inequality for each hyperplane. Any such system describes either a Weyl chamber or an empty set. By defining Weyl chambers as subsets of Euclidean space cut out by a finite number of inequalities, we get some results about their geometry.

Lemma 4.32. -
1. The closure of a Weyl chamber (denoted $\overline{C}$) is an unbounded convex cone.
2. The boundary $\partial\overline{C}$ is a finite union of codimension one faces F_i, each of with is a closed convex unbounded cone in one of the hyperplanes L_α and as such can be given by a system of inequalities. We call the hyperplanes containing some F_i the walls of C.

Proof. These properties apply to any subset of Euclidean space cut out by a finite number of inequalities. □

Proposition 4.33. *There is a bijection between the set of polarization of R and the set of Weyl chambers.*

Proof. Any Weyl chamber C defines a polarization by $R_+ = \{\alpha : (\alpha, t > 0)\}$ for any $t \in C$. Conversely, Given a polarization $R = R_+ \sqcup R_-$ the *positive Weyl chamber* is $C_+ = \{\lambda \in E : (\lambda, \alpha_i) > 0$ for all $\alpha_i \in \Pi\}$ □

We now state two lemmas which will be of immediate use

Lemma 4.34. -
1) Any two Weyl chambers can be connected by a sequence of adjacent Weyl chambers.
2) Adjacent Weyl chambers C and C' separated by a hyperplane L_α are mapped to each other by $s_\alpha(C) = C'$.

Together, these imply the following

Corollary 4.35. *The Weyl group acts transitively on the set of Weyl chambers*

Which in turn implies the following

Corollary 4.36. *Every Weyl chamber has $rank(R)$ walls.*

Proof. For the positive Weyl chamber, this is immediate from the definition. Since the Weyl group acts transitively, all Weyl chambers have the same number of walls. □

This brings us to one of the two main results of this section

Theorem 4.37. *If Π and Π' are two sets of simple roots obtained from two different polarization, there is an element $w \in W$ such that $\Pi = w(\Pi')$.*

Proof. Since polarizations are in bijection with Weyl chambers, the Weyl group acts transitively on polarizations. □

So we see that modulo the action of the Weyl group, all sets of simple roots are equivalent in a reasonable sense.

Theorem 4.38. *Let R be a reduced root system with a fixed polarization and a set of simple roots $\Pi = \{\alpha_i\}$. Consider the reflections s_{α_i} which we will denote s_i for simplicity. These are called simple reflections.*
1) The set of simple reflections generates W.
2) $W(\Pi) = R$. That is, every $\alpha \in R$ can be written as $\alpha = w(\alpha_i)$ for some $w \in W$, $\alpha_i \in \Pi$.

Proof. Note that any Weyl chamber can be reached from the positive chamber by a series of reflections, and therefore also a series of simple reflections. In particular, any wall L_α can be written as $w(L_{\alpha_i})$ for some simple root α_i and some $w \in W$. This implies $\alpha = \pm w(\alpha_i)$. □

Corollary 4.39. *The root system can be recovered from the set of simple roots*

Proof. We can recover W as the group generated by simple reflections and then compute $W(\Pi) = R$. □

4.6 Dynkin diagrams, Cartan matrices, and classification of root systems

Any two root systems $R_1 \subset E_1$ and $R_2 \subset E_2$ can be joined to make a new root system $R_1 \sqcup R_2 \subset E_1 \oplus E_2$ (also written as $R_1 \times R_2$ with an inner product defined on $E_1 \oplus E_2$ such that $E_1 \perp E_2$.

Definition 4.40. *A root system R is reducible if it can be written nontrivially as $R = R_1 \sqcup R_2$ with $R_1 \perp R_2$ and R_1, R_2. A root system that is not reducible is called irreducible. One should note that the property of being reduced is independent of the property of being irreducible.*

It can be shown that all reducible root systems can be written uniquely as a disjoint union of mutually orthogonal irreducible root systems. Given this, it suffices to classify irreducible root systems.

Proposition 4.41. *Let R be a reduced root system equipped with a polarization and a set of simple roots Π.*
1. If $R = R_1 \sqcup R_2$ then $\Pi = \Pi_1 \sqcup \Pi_2$ where $\Pi_i = R_i \sqcup \Pi$ are the simple roots of R_i.
2. If $\Pi = \Pi_1 \sqcup \Pi_2$ with $\Pi_1 \perp \Pi_2$ then $R = R_1 \sqcup R_2$ where R_i is generated by Π_i.

Proof. 1 is immediate from the definitions. Let $\alpha \in R_1$ and $\beta \in R_2$, then $s_\alpha(\beta) = \beta$ and s_α commutes with s_β. Let W_i be the Weyl group generated by the simple reflections of R_i. Then W_i acts trivially on R_j when $i \neq j$ and $W = W_1 \times W_2$. Then $R = W(\Pi_1 \sqcup \Pi_2) = W(\Pi_1) \sqcup W(\Pi_2)$. □

We'll need a way to keep track of the relative positions of simple roots. The inner product won't suffice since it isn't invariant under the Weyl group.

Definition 4.42. *Fix an irreducible root system R with a set of simple roots Π and an order on the simple roots $\Pi = \{\alpha_1, ...\alpha_r\}$. The Cartan matrix $A = (a_{ij})$ of Π is the $r \times r$ matrix given by*

$$a_{ij} = n_{\alpha_j \alpha_i} = \langle \alpha_i^\vee, \alpha_j \rangle = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$$

The definition immediately implies the following:

Lemma 4.43. -
1. $a_{ii} = 2$ for all i.
2. For any $i \neq j$, a_{ij} is a nonpositive integer.
3. For any $i \neq j$, $a_{ij}a_{ji} = 4\cos^2(\varphi)$ where φ is the angle between α_i and α_j. If α_i and α_j are orthogonal (that is, $\varphi = \pi/2$), then $\frac{|\alpha_i|^2}{|\alpha_j|^2} = \frac{a_{ji}}{a_{ij}}$.

We use a visual tool called a Dynkin diagram to represent the information in a Cartan matrix.

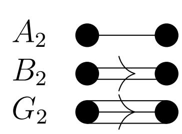
Definition 4.44. *The Dynkin diagram D of a set of simple roots Π is the semidirected multigraph defined by the following rules:*

1. Each simple root α_i corresponds to a vertex v_i of D.
2. Two vertices v_i, v_j with $i \neq j$ are connected by n edges where n depends on the angle φ between α_i and α_j as follows:

- $\varphi = \pi/2$: $n = 0$
- $\varphi = 2\pi/3$: $n = 1$ *(the A_2 system)*
- $\varphi = 3\pi/4$: $n = 2$ *(the B_2 system)*
- $\varphi = 5\pi/6$: $n = 3$ *(the G_2 system)*

3. The edges between vertices corresponding to two distinct, non-orthogonal simple roots α_i, α_j with $|\alpha_i| \neq |\alpha_j|$ point to the shorter of the two roots. If $|\alpha_i| = |\alpha_j|$, the edges are undirected.

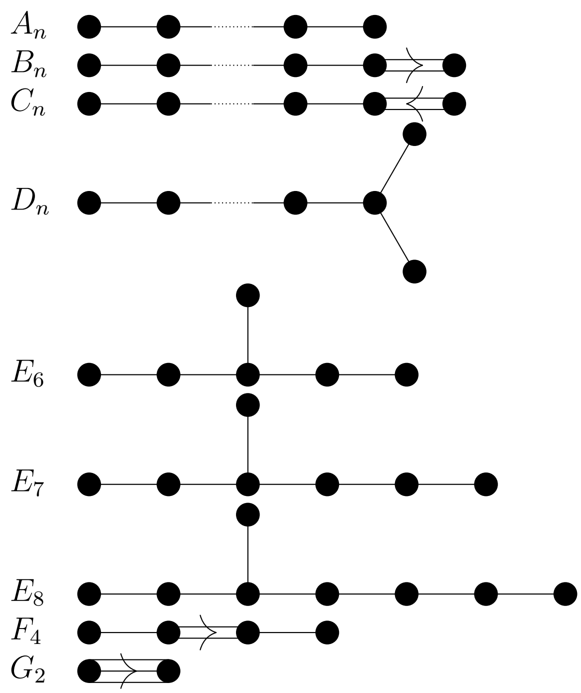
Example 4.45. *The Dynkin diagrams of the irreducible dimension two root systems:*



Theorem 4.46. *Let Π be the set of simple roots of a reduced root system R.*
1. The Dynkin diagram of R is connected iff R is irreducible.
2. The Dynkin diagram of R determines its Cartan matrix.
3. R is uniquely determined up to isomorphism by its Dynkin diagram.

So the problem of classifying root systems has been reduced to the problem of classifying possible Dynkin diagrams.

Theorem 4.47. *Let R be a reduced irreducible root system. Its Dynkin diagram is isomorphic to one of the diagrams below and each diagram is the Dynkin diagram of some reduced irreducible root system (each diagram below has n vertices). We assume that $n \geq 1$ for A_n, $n \geq 2$ for B_n and C_n, and $n \geq 4$ for D_n.*



Remark 4.48. *The assumptions on n are because $A_1 = B_1 = C_1$, $A_1 \sqcup A_1 = D_2$, and $A_3 = D_3$. Also note that $B_2 \cong C_2$.*

Corollary 4.49. *If R is a reduced irreducible root system, (α, α) takes on at most two distinct values.*

$$m = \frac{\max(\alpha, \alpha)}{\min(\alpha, \alpha)}$$

is the maximum edge multiplicity of the Dynkin diagram. Therefore:
$m = 1$ for types A, D, and E
$m = 2$ for types B, C, and F
$m = 3$ for type G.

Diagrams of types A, D, and E are called ***simply-laced****.*

4.7 Serre relations and classification of semisimple Lie algebras

Let's quickly review our progress in the classification of semisimple Lie algebras:

Every semisimple Lie algebra defined a reduced root system. This system is irreducible if and only if the Lie algebra is simple. Irreducible root systems are classified by their Dynkin diagrams, which we classified in the last section. If we can recover a Lie algebra from its root system, then simple Lie algebras are classified by the Dynkin diagrams in theorem 4.45.

Theorem 4.50. *Let $\mathfrak{g}$ be a semisimple complex Lie algebra with root system $R \subset \mathfrak{h}^*$ and a non-degenrate invariant symmetric bilinear form $(\cdot, \cdot)$. Let $\Pi = \{\alpha_1, ...\alpha_r\}$ be a set of simple roots given by some fixed polarization of R. Then:*
1. The subspaces

$$\mathfrak{n}_\pm = \bigoplus_{\alpha \in R_\pm} \mathfrak{g}_\alpha$$

are subalgebras of $\mathfrak{g}$ and

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

as a vector space.
2. Let $e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i \in \mathfrak{g}_{-\alpha_i}$ such that $(e_i, f_i) = 2/(\alpha_i, \alpha_i)$ and let $h_i = h_{\alpha_i} = 2H_\alpha/(\alpha, \alpha) \in \mathfrak{h}$. Then $\{e_i\}_{i=1}^r$ generates $\mathfrak{n}_+$, $\{f_i\}_{i=1}^r$ generates $\mathfrak{n}_-$ and $\{h_i\}_{i=1}^r$ is a basis of $\mathfrak{h}$. This implies $\{e_i, f_i, h_i\}_{i=1}^r$ generates $\mathfrak{g}$.
3. e_i, f_i, and h_i satisfy the following relations, called ***Serre relations****:*

$$\textbf{\textit{1)}}\ [h_i, h_j] = 0$$

$$\textbf{\textit{2)}}\ [h_i, e_j] = a_{ij} e_j$$

$$\textbf{\textit{3)}}\ [h_i, f_j] = -a_{ij} f_j$$

$$\textbf{\textit{4)}}\ [e_i, f_j] = \delta_{ij} h_i$$

$$\textbf{\textit{5)}}\ (\text{ad } e_i)^{1-a_{ij}} e_j = 0 \text{ for } i \neq j$$

$$\textbf{\textit{6)}}\ (\text{ad } f_i)^{1-a_{ij}} f_j = 0 \text{ for } i \neq j$$

where $a_{ij} = n_{\alpha_j \alpha_i} = \langle \alpha_i^\vee, \alpha_j \rangle$ are the entries of the Cartan matrix. We remind ourselves that $n_{\alpha_j \alpha_i}$ is twice the component of α_j along α_i.

Proof. 1.$\mathfrak{n}_+$ and $\mathfrak{n}_-$ are subalgebras since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ and the sum of positive/negative roots is positive/negative. The triangular decomposition follows from root decomposition.
2. Recall that the simple roots form a basis of $E = \mathfrak{h}^*$ so the h_i are a basis of $\mathfrak{h}$.
We'll need the following lemma to complete the proof of (2):

Lemma 4.51. *Let R be a reduced root system with a set of simple roots $\{\alpha_i\}$. If α is a positive root which is not simple, then $\alpha = \beta + \alpha_i$ for some positive root β. That is, positive roots differ exactly by a simple root.*

Proof. If all inner produces (α, α_i) are nonpositive, then $\{\alpha, \alpha_1, ...\alpha_r\}$ by lemma 4.20. This isn't possible since $\{\alpha_i\}$ is a basis, so there is some i so that $(\alpha, \alpha_i) > 0$. This implies $(\alpha, -\alpha_i) < 0$. Since the sum of two roots that are not scalars of each other whose inner product is negative is a root, $\beta = \alpha - \alpha_i$ is a root so $\alpha = \beta + \alpha_i$. One can check case-by-case that β is positive. □

We can now complete the proof of (2). We have $\mathfrak{g}_\alpha = [\mathfrak{g}_\beta, e_i]$. The result follows from induction on $\text{ht}(\alpha)$. The proof for f_i is similar.
3. The first three relations are immediate from the definitions of Cartan subalgebras and root subspaces. The fourth, when $i = j$ is because elements e_i, f_i and h_i satisfy the relations

of $\mathfrak{Sl}_2\mathbb{C}$. When $i \neq j$, $[e_i, f_j] = 0$ because $[e_i, f_j] \in \mathfrak{g}_{\alpha_i - \alpha_j}$ but $\alpha_i - \alpha_j$ isn't a root (since the coefficients on the simple roots must be the same sign).

To prove the last two relations, consider the subspace $\bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\alpha_j + k\alpha_i} \subset \mathfrak{g}$ as an $\mathfrak{sl}_2\mathbb{C}$ module. Since ad $e_i.f_j = 0$, f_j is a highest weight vector with weight $-a_{ij}$. Using results from the representation theory of $\mathfrak{sl}_2\mathbb{C}$ (to be proved in the next section), we know that a highest weight vector v of weight λ with $e.v = 0$ must also have $f^{\lambda+1}.v = 0$. The proof of (5) is similar. □

The Serre relations completely describe complex semisimple Lie algebras, though the proof of this fact is too difficult to include in these notes.

Theorem 4.52. *Let R be a reduced root system. Let $\mathfrak{g}(R)$ be the complex Lie algebra generated by $\{e_i, f_i, h_i\}_{i=1}^r$ subject to the Serre relations. Then $\mathfrak{g}(R)$ is a finite-dimensional semisimple Lie algebra with root system R.*

Corollary 4.53. -
1. If $\mathfrak{g}$ is a semisimple Lie algebra with root system R, $\mathfrak{g} \cong \mathfrak{g}(R)$.
2. There is a natural bijection between isomorphism classes of reduced root systems and isomorphism classes of finite-dimensional complex semisimple Lie algebras. $\mathfrak{g}$ is simple iff R is irreducible.

With this corollary, we can finally describe finite-dimensional complex semisimple Lie algebras

Theorem 4.54. *Simple, finite-dimensional complex Lie algebras are classified by Dynkin diagrams $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4,$ and G_2.*

4.8 Length of a Weyl group element

We define in this section the notion of the length $l(w)$ of a Weyl group element $w \in W$ which we will need later. Let R be a reduced root system. Define $l(w)$ by

$l(w) =$ the number of root hyperplanes separating C_+ and $w(C_+) = |\{\alpha \in R_+ : w(\alpha) \in R_-\}$

We note that $l(w)$ depends on the choice of polarization and that simple roots have length 1.

Lemma 4.55. *Let ρ be 1/2 the sum of all positive roots. then $\langle \rho, \alpha_i^\vee \rangle = 2(\rho, \alpha_i)/(\alpha_i, \alpha_i) = 1$.*

Proof. We have $s_i(\rho) = \rho - \alpha_i$ and $s_i(\lambda) := \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i$ □

Theorem 4.56. *Let $w = s_{i_1} ... s_{i_n}$ be a reduced product of simple reflections. Then $l(w) = n$.*

Proof. We can connect C_+ and $w(C_+)$ by a path going through each root hyperplane L_{α_i} so $l(w) \leq n$. The proof can be completed by showing that if we cross a root hyperplane more than once, the expression we began with is not reduced. □

Corollary 4.57. *The action of W on Weyl chambers is simply transitive.*

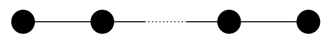
Proposition 4.58. *There is a unique $w_0 \in W$ such that $w_0(C_+) = C_-$ (this exists by the last corollary). Then $l(w_0) = |R_+|$ and is the unique longest element of W.*

4.9 Examples:

In this section, we list the properties of A_n and G_2.

4.9.1 $A_n = \mathfrak{sl}_{n+1}\mathbb{C}$, $n \geq 1$

Dynkin diagram:

A_n 

Lie algebra:

- $\mathfrak{g} = \mathfrak{sl}_{n+1}\mathbb{C} = \{\text{traceless matrices}\}$
- $dim(\mathfrak{g}) = (n+1)^2 - 1$
- $\mathfrak{z}(\mathfrak{g}) = \{0\}$
- *Simple: YES*
- *Real form:* $\mathfrak{sl}_n\mathbb{C} \cong \mathfrak{su}_n \otimes \mathbb{C}$

Cartan subalgebra:
(Let $e_i \in \mathfrak{h}^*$ be the functional which sends a matrix to its i^{th} diagonal entry)

- $\mathfrak{h} = \mathfrak{g} \cap \{\text{diagonal matrices}\}$
- $\mathfrak{h}^* = \bigoplus \mathbb{C}e_i/\mathbb{C}(e_1 + ... + e_{n+1})$
- $E = \mathfrak{h}^*_{\mathbb{R}} = \bigoplus \mathbb{R}e_i/\mathbb{R}(e_1 + ... + e_{n+1})$ *with inner product* $(\lambda, \mu) = \sum \lambda_i \mu_i$ *where* λ *and* μ *are representatives chosen to be traceless*

Root system:

- $R = \{e_i - e_j : i \neq j\}$, $|R| = n(n+1)$, *simply-laced*
- $\mathfrak{g}_\alpha = \mathbb{C}E_{ij}$
- $h_\alpha = \alpha^\vee = E_{ii} - E_{jj}$
- $R_+ = \{e_i - e_j : i < j\}$, $|R_+| = \frac{n(n+1)}{2}$
- $\Pi = \{e_i - e_{i+1}\}_{i=1}^n$, $|\Pi| = n$
- *Highest weight/maximal root:* $\theta = e_1 - e_{n+1} = (1, 0, .., 0, -1)$

Cartan matrix:

$$\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}$$

Weight and root lattices:

- $P = \{(\lambda_1, ...\lambda_{n+1}) : \lambda_i - \lambda_j \in \mathbb{Z}\}/\mathbb{R}(1, ..., 1) = \{(\lambda_1, ...\lambda_n, 0)\lambda_i \in \mathbb{Z}\}$
- $Q = \{(\lambda_1, ...\lambda_{n+1}) : \lambda_i - \lambda_j \in \mathbb{Z}, \sum \lambda_i = 0\}$
- $P/Q \cong \mathbb{Z}_{n+1}$

Fundamental and dominant weights

- $\hat{\Pi} = \{\sum_{i=2}^{k} e_i\}_{k=2}^{n+1} = \{\omega_1, ..., \omega_n\}$
- $P_+ = \{(\lambda_1, ..., \lambda_{n+1}) : \lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}\}/\mathbb{R}(1, ..., 1) =$
  $\{(\lambda_1, ..., \lambda_n, 0) : \lambda_i \in \mathbb{Z}, \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \geq 0\}$
- $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha = (n, n-1, ..., 1, 0) = (n/2, (n-2)/2, ..., (-n)/2)$

Weyl group:

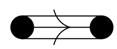
- $W = S_{n+1}$, $|W| = (n+1)!$
- $s_i = (i \ i+1)$
- $C_+ = \{(\lambda_1, ..., \lambda_{n+1}) : \lambda_1 > \lambda_2 > ... > \lambda_{n+1}\}/\mathbb{R}(1, ..., 1) =$
  $\{(\lambda_1, ..., \lambda_n, 0) : \lambda_1 > \lambda_2 > ... > \lambda_n > 0\}$

Lie groups with this Lie algebra:

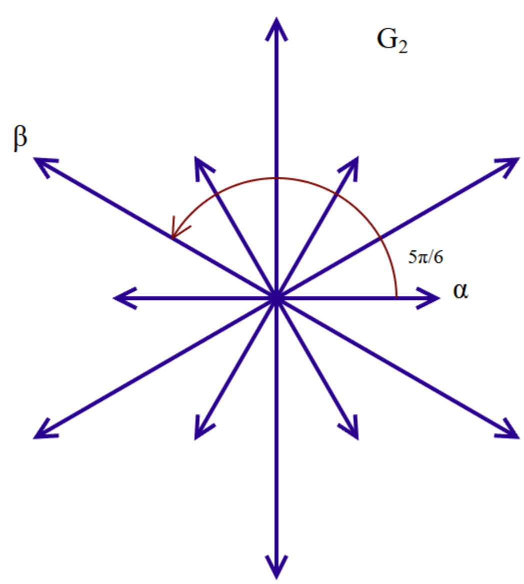
- $G = SL_{n+1}\mathbb{C} = \{\text{determinant 1 matrices}\}$, $Z(G) = \{\lambda I_{n+1} : \lambda^{n+1} = 1\} \cong \mathbb{Z}$
- $G = PSL_{n+1} = PGL_{n+1}\mathbb{C} = SL_{n+1}\mathbb{C}/\mathbb{Z}$, $Z(G) = \{0\}$

4.9.2 $G_2 = \mathfrak{g}_2$

Dynkin diagram:

G_2 

Let $\alpha = e_1 - e_2$ and $\beta = 2e_2 - e_1 - e_3$



Root system:

- $R = \{\pm\alpha, \pm(c_1\alpha+\beta), \pm(3\beta+2\alpha) : c_1 \in \{0,1,2,3\}\}$, $|R| = 12$
- $R_+ = \{\alpha, c_1\alpha+\beta, 3\beta+2\alpha\}$, $|R_+| = 6$
- $\Pi = \{(e_1 - e_2), (2e_2 - e_1 - e_3)\}$ $|\Pi| = 2$
- $\varphi = \pi/6$, $|\alpha| = 1$, $|\beta| = \sqrt{3}$

Weight and root lattices

- $P = Q = \{(c_1+c_2)e_1 + c_2e_2 - (3c_1+2c_2)e_3\}$

Cartan matrix

$$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

Fundamental and dominant weights:

- $\hat{\Pi} = \{2\alpha+\beta, 3\alpha+2\beta\} = \{e_1 - e_3, e_1 + e_2 - 2e_3\} = \{\omega_1, \omega_2\}$
- $P_+ = \{c_1\omega_1 + c_2\omega_2 : c_1, c_2 \in \mathbb{Z}_{\geq 0}\} = \{(c_1+c_2)e_1 + c_2e_2 - (3c_1+2c_2)e_3 : c_1, c_2 \geq 0\}$
- $\rho = 3\alpha + \frac{3}{2}\beta$

Weyl group:

- $W = D_6$, $|W| = 12$
- $\{s_i\} = \{f, fr^2\}$

4.10 Accidental Isomorphisms

In this section, we will describe the "accidental" low-dimensional Lie algebra isomorphisms corresponding to the previously mentioned isomorphisms of Dynkin diagrams. We can use this to find Lie groups that are locally isomorphic.

The isomorphisms of Dynkin diagrams $A_1 = B_1 = C_1$ *correspond to the fact that* $\mathfrak{sl}_2\mathbb{C} \cong \mathfrak{so}_3\mathbb{C} \cong \mathfrak{sp}_2\mathbb{C}$ *are isomorphic. Recall* $\mathfrak{so}_3\mathbb{C}$ *has a basis*

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \; J_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \; J_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

subject to the relations $[J_x, J_y] = J_z,\ [J_y, J_z] = J_x,\ [J_z, J_x] = J_y$.

$\mathfrak{sl}_2\mathbb{C}$ *has a basis*

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

subject to the relations $[e, f] = h,\ [h, e] = 2e,\ [h, f] = -2f$. *The isomorphism* $\mathfrak{so}_3\mathbb{C} \to \mathfrak{sl}_2\mathbb{C}$ *can be given by*

$$J_x \mapsto -\frac{i}{2}(e+f),\ J_y \mapsto \frac{1}{2}(f-e),\ J_Z \mapsto -\frac{ih}{2}$$

We also note that when $n = 2$, the an element of $GL_2\mathbb{C}$ is in $SP_2\mathbb{C}$ iff its determinant is 1, so $\mathfrak{sp}_2\mathbb{C}$ and $\mathfrak{sl}_2\mathbb{C}$ are equal, not just isomorphic.

We also want to note that there is an isomorphic $\mathfrak{su}_2 \to \mathfrak{so}_3\mathbb{R}$, which explains why $\mathfrak{su}_{2\mathbb{C}} \cong \mathfrak{sl}_2\mathbb{C}$. Knowing this fact, we could take the basis of $\mathfrak{sl}_2\mathbb{C}$ as our basis of $\mathfrak{su}_2$, but the convention is to use the Pauli matrices σ_i as follows:

$$i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},\ i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

(It is easy to construct a mapping of this basis to the one we chose for $\mathfrak{sl}_2\mathbb{C}$.) Then the isomorphism $\mathfrak{su}_2 \to \mathfrak{so}_3\mathbb{R}$ can be given by

$$i\sigma_1 \mapsto -2J_X,\ i\sigma_2 \mapsto -2J_y,\ i\sigma_3 \mapsto -2J_z$$

and in fact, this lifts to a double cover of Lie groups $SU_2 \to SO_3\mathbb{R}$.

We now list, but won't prove the three other exceptional Lie algebra isomorphisms:

- *$A_1 \sqcup A_1 = D_2$ corresponds to the fact that $\mathfrak{sl}_2\mathbb{C} \otimes \mathfrak{sl}_2\mathbb{C} \cong \mathfrak{so}_4\mathbb{C}$*
- *$B_2 = C_2$ corresponds to the fact that $\mathfrak{sp}_4\mathbb{C} \cong \mathfrak{so}_5\mathbb{C}$*
- *$A_3 = D_3$ corresponds to the fact that $\mathfrak{sl}_4\mathbb{C} \cong \mathfrak{so}_6\mathbb{C}$*

These Lie algebra isomorphisms imply local isomorphism between their Lie groups. We might ask which of these Lie groups are isomorphic and which are covers of each other? We present the answers here for complex Lie groups, without proof. For those interested, Terence Tao has an article available on his website in which he proves these.

- *$SL_2\mathbb{C} \cong SP_2\mathbb{C}$*
- *$SL_2\mathbb{C}$ is a double cover of $SO_3\mathbb{C}$, both are 3-dimensional*
- *$SL_2\mathbb{C} \otimes SL_2\mathbb{C}$ is a double cover of $SO_4\mathbb{C}$, both are 6-dimensional*
- *$SP_4\mathbb{C}$ is a double cover of $SO_5\mathbb{C}$, both are 10-dimensional*
- *$SL_4\mathbb{C}$ is a double cover of $SO_6\mathbb{C}$, both are 15-dimensional*

5 Representation Theory Primer

5.1 Definitions and basic properties

We begin this section with a quick review of some important definitions and properties of representations. Most proofs will be omitted.

Definition 5.1. *A representation of a Lie group (resp. Lie algebra) G (resp. $\mathfrak{g}$) is a vector space V together with a morphism $\rho : G \to GL(V)$ (resp. $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. If G (resp. $\mathfrak{g}$) is real and V is complex, we require that ρ be smooth and view $GL(V)$ as a real manifold.*

We will often write $g.v$ to mean $\rho(g)v$.

Definition 5.2. *A morphism $f : V \to W$ of two representations V, W of the same object G is a linear map $f : V \to W$ that commutes with the action of G: $f \circ \rho(g) = \rho(g) \circ f$. The space of all such morphisms, called G-morphisms, is denoted $\mathrm{Hom}_G(V, W)$.*

Remark 5.3. *An algebra representation is equivalent to a module over the algebra.*

Theorem 5.4. -
1) Every representation $\rho : G \to GL(V)$ defines a representation $\rho_* : \mathfrak{g} \to \mathfrak{gl}(V)$ and every G-morphism is also a $\mathfrak{g}$-morphism.
2. If G is simply connected then $\rho \mapsto \rho_*$ is an equivalence of categories. In particular, every $\mathfrak{g}$-representation lifts uniquely to a G-representation and $\mathrm{Hom}_G(V, W) = \mathrm{Hom}_{\mathfrak{g}}(V, W)$.
3. If G is connected but not necessarily simply connected, write $G = \tilde{G}/Z$ with $\tilde{G}$ simply connected and Z a discrete central subgroup. Then a G representation is the same as a $\tilde{G}/Z$ representation with $\rho(Z) = \mathrm{id}$.

Lemma 5.5. *Any complex representation of a Lie algebra $\mathfrak{g}$ has a unique structure as a $\mathfrak{g}_{\mathbb{C}}$-representation. Moreover, $\mathrm{Hom}_{\mathfrak{g}}(v, W) = \mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}}}(V, W)$. This implies the categories of complex representations of $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$ are equivalent.*

Proof. Given $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, define $\rho : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}(V)$ by $\rho'(x + iy) = \rho(x) + i\rho(y)$. One checks this is $\mathbb{C}$-linear and respects the Lie bracket. □

Remark 5.6. *These results imply the categories of finite-dimensional representations of $SL_2\mathbb{C}, SU_2, \mathfrak{sl}_2\mathbb{C}, and \mathfrak{su}_2$ are equivalent since $\mathfrak{sl}_2\mathbb{C} = (\mathfrak{su}_2)_{\mathbb{C}}$.*

Definition 5.7. *A subrepresentation $W \subset V$ is a subspace stable under the action of $\rho(g)$ for all $\mathfrak{g} \in G$. The quotient space V/W also has the structure of a representation.*

Theorem 5.8. *If V and W are two representations of G or $\mathfrak{g}$, then $V \oplus W, V \otimes W$, and V^* are also representations of G.*

Proof. The action of G or $\mathfrak{g}$ on $V \oplus W$ is given by $\rho(g)(v + w) = \rho(g)v + \rho(g)w$.

The action of G on $V \otimes W$ is simply $\rho(g)(v \otimes w) = \rho(g)v \otimes \rho(g)w$. The action of $\mathfrak{g}$ is $\rho(x)(v \otimes w) = \rho(x)v \otimes w + v \otimes \rho(x)w$. This comes from the Leibniz rule.

To define the action of G on V^*, we force the pairing $V \otimes V^* \to \mathbb{C}$ to be a morphism of representations. Thus, $\langle \rho(g)v, \rho(g)v^* \rangle = \langle v, v^* \rangle$ so $\rho_{V^*}(g) = \rho(g^{-1})^t$ where $A^t : V^* \to V^*$ is the adjoint operator to $A : V \to V$. For $\mathfrak{g}$, we similarly get $\rho_{V^*}(x) = -\rho_{V^*}(x)^t$. □

We now go over some important examples

Example 5.9. (Coadjoint representation)
Any Lie algebra $\mathfrak{g}$ has a representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{g}^*$ called the coadjoint representation given by the following action on $\mathfrak{g}^*$:

$$\langle ad^*x.f, y \rangle = \langle f, ad\ x.y \rangle$$

Example 5.10. *($End(V)$* **and** $\mathrm{Hom}(V,W)$*)*
Let V be a G-(resp $\mathfrak{g}$-)representation. $End(V) \cong V \otimes V^*$ is a representation with action given by $g.A = \rho_V(g)A\rho_V(g^{-1})$ (resp. $x.A = \rho_V(x)A - A\rho_V(x)$. Similarly, for two representations V and W, $\mathrm{Hom}(v,W)$ is a representation with action $g.A = \rho_W(g)A\rho_V(g^{-1})$ (resp. $x.A = \rho_W(x)A - A\rho_V(x)$

Theorem 5.11. $(\mathrm{Hom}(v,W))^G = \mathrm{Hom}_G(V,w)$ *the space of intertwining operators. In particular, $V^G = \mathrm{Hom}(\mathbb{C},V)^G = Hom_G(\mathbb{C},V)$.*

Example 5.12. (Space of bilinear forms)
The space of bilinear forms on a representation V is also a representation given by

$$g.B(v,w) = B(g^{-1}.vg^{-1}.w)$$

and

$$x.B(v,w) = -(B(x.v,w) + B(v,x.w))$$

Definition 5.13. *Let V be a G-representation. $v \in V$ is called invariant if $g.v = v$ for all $g \in G$. We denote the subspace of invariant vectors by V^G.*

Similarly, if V is a $\mathfrak{g}$-representation, $v \in V$ is called invariant if $x.v = 0$ for all $x \in \mathfrak{g}$. The corresponding subspace is denoted $V^{\mathfrak{g}}$.

Now consider the space of bilinear forms on V as a representation. If B is a bilinear form on V as in the above definition, B is invariant under the action of G iff $B(g.v,g.w) = B(v,w)$ and invariant under $\mathfrak{g}$ iff $B(x.v,w) + B(v,x.w) = 0$.

Theorem 5.14. *A bilinear form B is invariant iff the map $V \to V^*$ defined by $v \mapsto B(v,\cdot)$ is a morphism of representations.*

Definition 5.15. *A representation is called simple or irreducible if it has no proper subrepresentations. If it does, it is called reducible. A representation is called semisimple or completely reducible if it is isomorphic to a direct sum of simple representations. That is, $V = \bigoplus n_i V_i$ where V_i are simple and n_i is called the multiplicity of V_i in V.*

Example 5.16. *Not every representation is semisimple. For instance, consider $G = \mathbb{R}$, $\mathfrak{g} = \mathbb{R}$. Fix a matrix $A \in End(V)$ and consider the $\mathfrak{g}$-representation defined by $\rho(t) = tA$. The corresponding representation of G is given by $\rho(t) = exp(tA)$. Writing this representation as a direct sum of irreducibles is equivalent to diagonalizing A, which is not always possible.*

Lemma 5.17. *Let $\rho : G \to GL(V)$ be a representation (one can replace G with $\mathfrak{g}$ and the theorem holds). Let $A : V \to V$ be a diagonalizable intertwining operator where $V_\lambda \subset V$ is an eigenspace of A with eigenvalue λ. Each V_λ is a subrepresentation and $V = \bigoplus V_\lambda$*

Corollary 5.18. *Let $Z \in Z(G)$ such that $\rho(Z)$ is diagonalizable. Then V is a direct sum of the eigenspaces of $\rho(Z)$. The same is true for central elements of $\mathfrak{g}$.*

5.2 Schur's lemma and intertwining operators

Theorem 5.19. (Schur's Lemma)
1. Let V be an irreducible complex representation of G. The space of intertwining operators $\mathrm{Hom}_G(V,V) = \mathbb{C}\mathrm{id}$. *That is, any endomorphism of an irreducible representation is constant.*
2. If V and W are nonisomorphic irreducible complex representations, $\mathrm{Hom}_G(V,W) = 0$

Proof. Let $\Phi : V \to W$ be an intertwining operator. $\mathrm{Ker}\Phi$ and $Im\Phi$ are subrepresentations of V and W respectively. Since V is irreducible, $\mathrm{Ker}\Phi = 0$ or $\mathrm{Ker}\Phi = V$ so either Φ is injective or 0. We can do the same for $\mathrm{Im}\Phi \subset W$, so either Φ is an isomorphism or 0. This completes the proof of (2).

To prove (1), notice that by (2), any nonzero intertwining operator $\Phi : V \to V$ is an isomorphism. Let λ be an eigenvalue of Φ. $\Phi - \lambda\mathrm{id}$ is not invertible but is also an intertwining operator. This implies it is 0, so $\Phi = \lambda\mathrm{id}$. □

Example 5.20. (The center of a Lie group)
Since $\mathbb{C}^n$ is irreducible as a $GL_n\mathbb{C}$ representation, all operators that commute with $GL_n\mathbb{C}$ are scalars. Thus $Z(GL_n\mathbb{C}) = \{\lambda\mathrm{id}, \lambda \in \mathbb{C}^\times\}$ and similarly $\mathfrak{z}(\mathfrak{gl}_n\mathbb{C}) = \{\lambda\mathrm{id}, \lambda \in \mathbb{C}\}$. We can use a similar argument to find that:

$$Z(SL_n\mathbb{C}) = Z(SU_n) = \{\lambda\mathrm{id} : \lambda^n = 1\} \quad \mathfrak{z}(\mathfrak{sl}_n\mathbb{C}) = \mathfrak{z}(\mathfrak{su}_n) = 0$$

$$Z(U_n) = \{\lambda\mathrm{id} : |\lambda| = 1\} \quad \mathfrak{z}(\mathfrak{sl}_n\mathbb{C}) = \mathfrak{z}(\mathfrak{u}_n) = \{\lambda\mathrm{id}, \lambda \in i\mathbb{R}\}$$

$$Z(SO_n\mathbb{C}) = Z(SO_n\mathbb{R}) = \left\{ \begin{array}{ll} \pm 1 & \textit{for } n \text{ even} \\ 1 & \textit{for } n \text{ odd} \end{array} \right\} \quad \mathfrak{z}(\mathfrak{so}_n\mathbb{C}) = \mathfrak{z}(\mathfrak{so}_n\mathbb{R}) = 0$$

Corollary 5.21. (Classification of intertwining operators)
Let V be a ***completely reducible*** *representation of a Lie group G or Lie algebra $\mathfrak{g}$.*
1. If $V = \bigoplus V_i$ with each V_i irreducible, pairwise non-isomorphic, then any intertwining operator is of the form $\bigoplus \lambda_i \mathrm{id}_{V_i}$.
2. If $V = \bigoplus n_i V_i$ irreducible, pairwise non-isomorphic, then any intertwining operator is of the form $\bigoplus (A_i \otimes \mathrm{id}_{V_i}$ with $A_i \in End(\mathbb{C}^{n_i})$.

Theorem 5.22. *Any irreducible complex representation of an abelian group or commutative Lie algebra is one-dimensional.*

Proof. Every $\rho(g)$ commutes with the action of G so $\rho(g) = \lambda\mathrm{id}$ for some scalar λ since $\rho(g) \in GL_n\mathbb{C}$. □

Example 5.23. (Irreducible representations of $\mathbb{R}$)
Complex irreducible representations of $\mathfrak{g} = \mathbb{R}$ are $a \mapsto \lambda a$ for some $\lambda \in \mathbb{C}$. The representations of $G = \mathbb{R}$ would then be $exp(\lambda a)$ but exp is the identity on $\mathbb{R}$.

Example 5.24. (Irreducible representations of S^1)
Note $S^1 = \mathbb{R}/\mathbb{Z}$ so an S^1 representation is exactly an $\mathbb{R}$ representation such that every integer acts trivially. Thus they are one-dimensional complex vector spaces $V_k, k \in \mathbb{Z}$ where $\rho(a) = e^{2\pi i k a}$. If instead, we view S^1 as the unit complex numbers, then in V_k $z \in S^1$ acts by z^k.

5.3 Unitary representations

We want to determine what kinds of representations will be completely reducible. It turns out that a large family of them are.

Definition 5.25. *A complex representation V of a real Lie group G is called unitary if there exists a G-invariant inner product (positive definite Hermitian form) on V: $(g.v, g.w) = (v, w)$. A representation V of a real Lie algebra $\mathfrak{g}$ is called unitary if there exists a $\mathfrak{g}$-invariant inner product: $(x.v, w) + (v, x.w) = 0$.*

Example 5.26. *Let V be the space of complex-valued functions on a finite set S. Let G be a finite group acting by permutations on S. Then G acts on V by $g.f(s) = f(g^{-1}s)$. $(f_1, f_2) = \sum_{s\in S} f_1(s)\overline{f_2(s)}$ is an invariant inner product so this representation is unitary.*

Theorem 5.27. *Unitary representations are completely reducible.*

Proof. We induct on the dimension of V. The statement is trivial in dimension 1. If V is not irreducible, it has a subrepresentation W and $V = W \oplus W^\perp$. We claim $W^\perp$ is also a subrepresentation: Let $w \in W^\perp$. Then $(g.w, v) = (w, g^{-1}.v) = 0$ for any $v \in W$. Thus $gw \in W^\perp$. A similar argument holds in the case of Lie algebras. □

Theorem 5.28. *Finite group representations are unitary.*

Proof. Let $B(\cdot, \cdot)$ be an inner product on V. The inner product defined by

$$\frac{1}{|G|}\sum_{g\in G} B(g.v, g.w)$$

is positive definite and G-invariant. □

Corollary 5.29. *Finite group representations are completely reducible*

We would like to extend this argument to Lie groups, but it isn't immediately clear how. The following theorem, whose proof is omitted and relies on some measure theory beyond the scope of these notes, allows us to do this.

Theorem 5.30. *Let G be a compact real Lie group. Then G admits a canonical Borel measure dg that is left- and right-invariant, invariant under the map $g \mapsto g^{-1}$ and that satisfies $\int_G \mathrm{d}g = 1$. This is called the Haar measure on G*

This allows us to extend the argument we made to prove 5.28 to any compact Lie group. Before doing so, we mention that explicitly writing the Haar measure of a group is usually very difficult since a group rarely comes equipped with a reasonable choice of a coordinate system.

Theorem 5.31. *Any finite-dimensional representation of a compact Lie group is unitary and therefore completely reducible.*

Proof. Let $B(\cdot, \cdot)$ be an inner product on V. The inner product defined by

$$\tilde{B}(v, w) = \int_G B(g.v, g.w)\mathrm{d}g$$

is positive definite and G-invariant.It is positive definite since $\tilde{B}(v, v)$ is the integral of a positive function. G-invariance follows from right-invariance of the Haar measure. □

5.4 Characters and Peter-Weyl theorem

Having determined that any representation V of a compact Lie group is completely reducible in the form $V = \bigoplus n_i V_i$, we might be interested in figuring out how to compute this decomposition. In this subsection, G is a compact real Lie group with a Haar measure dg.

Definition 5.32. *Fix a basis of a representation V. $\rho(g)$ is matrix-valued. We define the matrix coefficients $\rho_{ij} : G \to \mathbb{F}$ by $\rho_{ij}(g) = (\rho(g))_{ij}$.*

Remark 5.33. *Matrix coefficients are continuous by the continuity of ρ.*

Theorem 5.34. -
1. Let V and W be non-isomorphic irreducible representations of G. Fix bases of V and W. Note we have an inner product on $C^\infty(G, \mathbb{C})$ given by

$$(f_1, f_2) = \int_G f_1(g)\overline{f_2(g)}\mathrm{d}g$$

Then For any i, j, k, l, the matrix coefficients ρ^V_{ij}, ρ^W_{kl} are orthogonal.
2. Let V be an irreducible representation of G and fix an orthonormal basis with respect to a G-invariant inner product (such an inner product exists by theorem 5.31). The matrix coefficients ρ^V_{ij} are pairwise orthogonal with respect to this inner product and each has norm $1/\dim V$. Equivalently

$$(\rho^V_{ij}, \rho^V_{kl}) = \frac{1}{\dim(V)}\delta_{ik}\delta_{jl}$$

The proof relies on the following lemma

Lemma 5.35. -
1. Let V and W be non-isomorphic irreducible representations of G and f some linear map $V \to W$. Then $\int_G gfg^{-1}\mathrm{d}g = 0$.
2. If f is a linear map $V \to V$ (V is still irreducible), then $\int_G gfg^{-1}\mathrm{d}g = \frac{\mathrm{tr}(f)}{\dim(V)}\mathrm{id}$.

Proof. Let $F = \int_G gfg^{-1}\mathrm{d}g$. Then for any $h \in G$, $hFh^{-1} = F$ so by Schur's lemma, F $= 0$ for $V \neq W$ and $\lambda\mathrm{id}$ for $V \cong W$. $\mathrm{tr}(F) = \mathrm{tr}(gfg^{-1}) = \mathrm{tr}(f)$, we must have $\lambda = (\mathrm{tr}(f)/\dim(V))\mathrm{id}$ □

We are now ready to prove theorem 5.34.

Proof. Fix orthonormal bases $\{v_i\}, \{w_i\}$ of V and W respectively. Applying the above lemma to the map $E_{ki} : V \to W$ given by $E_{ki}(v_i) = w_k$, $E_{ki}(v_j) = 0$ for $i \neq j$ gives

$$\int_G \rho^W(g)E_{ki}\rho^V(g^{-1})\mathrm{d}g = 0$$

Since ρ is unitary we have $\rho(g^{-1}) = \overline{\rho(g)^t}$. Using this and rewriting the above in matrix form we have

$$\int_G \rho^W_{kl}(g)\overline{\rho^V_{ji}(g)}\mathrm{d}g = 0$$

If our bases are not orthonormal, the previous expression differs only by a change of basis, so it still resolves to 0. This proves the first part of the theorem.

Now, to prove the second statement apply the lemma to a matrix unit E_{ki} to see

$$\sum_{l,j} E_{lj} \int_G \rho^V_{lk}(g)\overline{\rho^V_{ji}}(g) = \frac{\mathrm{tr}(E_{ki})}{\dim(V)}\mathrm{id}$$

which completes the proof. □

Now that we have a way of constructing an orthonormal set of functions on a G, we would like it to be coordinate-free. We can do this with one particular choice of matrix coefficients:

Definition 5.36. *The character of a representation V is the function $\chi_V : G \to \mathbb{C}$ defined by*

$$\chi_V(g) = \mathrm{tr}_V(\rho(g)) = \sum \rho^V_{ii}(g)$$

Theorem 5.37. (Properties of characters)
1. If $V = \mathbb{C}$ is the trivial representation, $\chi_V = 1$.
2. $\chi_{V\oplus W} = \chi_V + \chi_W$.
3. $\chi_{V\otimes W} = \chi_V \cdot \chi_W$.
4. $\chi_V(ghg^{-1}) = \chi_V(h)$ i.e. characters are invariant under conjugation by elements of G.
5. $\chi_{V^*} = \overline{\chi_V}$.
6. χ_V is independent of the choice of basis of V.

Since characters are essentially a special case of matrix coefficients, we immediately have the following:

Theorem 5.38. -
1. Let V and W be non-isomorphic complex irreducible representations of a compact Lie group G. χ_V and χ_W are orthogonal with respect to the inner product defined by

$$(f_1, f_2) = \int_G f_1(g)\overline{f_2(g)}\mathrm{d}g$$

2. For any irreducible representation V, $(\chi_v, \chi_V) = 1$.

In other words, the set $\{\chi_V : V \in \hat{G}\}$ is an orthonormal family of functions $G \to \mathbb{C}$. Here $\hat{G}$ denotes the set of isomorphism classes of irreducible representations of G. We have the following corollary.

Corollary 5.39. *Let V be a complex representation of a compact real Lie group.*
1. V is irreducible iff $(\chi_V, \chi_V) = 1$.
2. V can be written uniquely as $V \cong n_i V_i$ with V_i pairwise non-isomorphic irreducible representations and $n_1 = (\chi_v, \chi_v)$

Remark 5.40. *While this gives us a method to count multiplicities in a representation, in practice, this isn't often a feasible way to do so. We will develop a better way to count multiplicities when our Lie algebra is semisimple in section 6.6.*

Now we are going to reformulate theorem 5.34 without a choice of basis. Define a function $\rho_{v^*,v} : G \to \mathbb{C}$ *by* $g \mapsto \langle v^*, \rho(g)v\rangle$. *When* $v = v_j$ *and* $v^* = v_i^*$, *the is a matrix coefficient so we can view* $\rho_{v^*,v}$ *as a generalization of matrix coefficients. This associates to any representation* V *a map* $V^* \otimes V \to C^\infty(G, \mathbb{C})$ *given by* $v^* \otimes v \mapsto \langle v^*, \rho(g)v\rangle$.

We note that $V^* \otimes V$ *is a* $G-$*bimodule with the module structures given by the action of* G *on the factors* V^* *and* V. *Also note that if* V *is unitary, it defines an inner product on* V^* *(which can be done by taking the dual basis of an orthogonal basis and declaring it to be orthogonal). We can define an inner product on* $V^* \otimes V$ *by*

$$(v_1^* \otimes w_1, v_2^* \otimes w_2) = \frac{1}{\dim(V)}(v_1^*, v_2^*)(w_1, w_2)$$

Lemma 5.41. *Define the map*

$$m : \bigoplus_{V_i \in \hat{G}} V_i^* \otimes V \to C^\infty(G, \mathbb{C})$$

by $m(v^* \otimes v)(g) = \langle v^*, g.v\rangle$. *Then:*
1. m *is a G-bimodule isomorphism:* $m((g.v^* \otimes v) = L_g(m(v^* \otimes v))$ *and* $m((v^* \otimes g.v) = R_g(m(v^* \otimes v))$.
2. m *preserves the inner product.*

Proof. (1) can be done via explicit computation and (2) follows from theorem 5.34. □

Corollary 5.42. m *is injective*

Proof. Orthogonal transformations are injective. □

This map is also surjective if we replace the direct sum by a Hilbert direct sum:

Theorem 5.43. (Peter-Weyl theorem)
m *gives an isomorphism*

$$\hat{\bigoplus}_{V_i \in \hat{G}} V_i^* \otimes V_i \to L^2(G, \mathrm{d}g)$$

where $\hat{\bigoplus}$ *is the Hilbert space direct sum (the completion of* $\bigoplus$ *with respect to the metric induced by the inner product).*

Equivalently, the set of linear combinations of matrix coefficients is dense in $L^2(G, \mathrm{d}g)$.

The proof requires too much analysis to be included in these notes.

Corollary 5.44. (Peter-Weyl theorem II)
The set of characters $\{\chi_V, V \in \hat{G}\}$ *is an orthonormal Hilbert basis of* $L^2(G, \mathrm{d}g)^G$, *the conjugation-invariant functions* L^2 *functions on* G.

5.5 Representations of $\mathfrak{sl}_2\mathbb{C}$

We'll now discuss the representation theory of $\mathfrak{sl}_2\mathbb{C}$, the results of which were taken for granted in sections 3 and 4. The representation theory of $\mathfrak{sl}_2\mathbb{C}$ forms the groundwork for that of all semisimple complex Lie algebras, as we have already seen glimpses of in the previous sections.

Proposition 5.45. $\mathfrak{sl}_2\mathbb{C}$ *representations are completely reducible*

Proof. $\mathfrak{sl}_2\mathbb{C}$ representations are the same as $\mathfrak{su}_2$ representations which are the same as SU_2 representations which is compact. □

Recall that $\mathfrak{sl}_2\mathbb{C}$ has a basis $\{e, f, h\}$ with relations

$$[e, f] = h \quad [h, e] = 2e \quad [h, f] = -2f$$

The main idea in the representation theory of $\mathfrak{sl}_2\mathbb{C}$ is to diagonalize h.

Definition 5.46. *Let V be a representation of $\mathfrak{sl}_2\mathbb{C}$. A vector $v \in V$ is a vector of weight $\lambda \in \mathbb{C}$ if it is an eigenvector for h with eigenvalue λ. We denote the space of vectors of weight λ by $V[\lambda] \subset V$.*

Lemma 5.47. *$eV[\lambda] \subset V[\lambda + 2]$ and $fV[\lambda] \subset V[\lambda - 2]$. In other words, e is a raising operator and f is a lowering operator.*

Proof. $hev = [h, e]v + ehv = 2ev + \lambda ev = (\lambda + 2)ev$. The proof for f is similar. □

Theorem 5.48. *Every finite-dimensional $\mathfrak{sl}_2\mathbb{C}$ representation be written in the form*

$$V = \bigoplus_{\lambda} V[\lambda]$$

This is called the weight decomposition of V.

Proof. Since $\mathfrak{sl}_2\mathbb{C}$ are completely reducible, it suffices to prove this for irreducible V. Let V' be the subspace spanned by eigenvectors of h. Then $V' = \sum_\lambda V[\lambda]$. Eigenvectors with distinct eigenvalues are linearly independent so this is a direct sum. By the above lemma, V' is stable under the action of $\mathfrak{sl}_2\mathbb{C}$ so it is a subrepresentation. Since V is irreducible and $V' \neq 0$ (since an eigenvector always exists), $V = V'$. □

Definition 5.49. *Let λ be a weight of V such that $Re\lambda \geq Re\lambda'$ for all other weights λ'. We call λ a highest weight and eigenvectors with eigenvalue λ are called highest weight vectors.*

Lemma 5.50. *Let $v \in V[\lambda]$ be a highest weight vector. Then:*
1. $ev = 0$.
2. Let

$$v^k = \frac{f^k}{k!}v, \; k \geq 0$$

Then

$$hv^k = (\lambda - 2k)v^k$$
$$fv^k = (k+1)v^{k+1}$$
$$ev^k = (\lambda - k + 1)v^{k-1}, \; k \geq 0$$

Proof. We have $ev \in V[\lambda+2]$ but $V[\lambda+2]=0$ since λ is a highest weight. This proves (1).

The formula for the action of f follows immediately from the definition of v^k. The formula for the action of h follows from the fact that $fV[\lambda] \subset V[\lambda-2]$. The formula for the action of e can be proven by induction: For $k=1$, $ev^1 = efv = [e,f]v + fev = hv = \lambda v$. Now

$$ev^{k+1} = \frac{1}{k+1} efv^k = \frac{1}{k+1}(hv^k + fev^k)$$

Applying the induction hypothesis, this becomes

$$\frac{1}{k+1}\left((\lambda-2k)v^k + (\lambda-k+1)fv^{k-1}\right) = \frac{1}{k+1}((\lambda-2k+(\lambda-k+1)k)v^k = (\lambda-k)v^k$$

□

It will be useful to view V as a finite-dimensional quotient of an infinite dimensional vector space with basis $\{v_k\}$:

Lemma 5.51. *Let $\lambda \in \mathbb{C}$. Define M_λ as the vector space with basis $\{v^i\}_{i=0}^{\infty}$. This is called a Verma module, and we will study them in more detail later.*
1) The formulas from the previous lemma, with the added condition that $ev^0 = 0$ give M_λ the structure of an (infinite-dimensional) $\mathfrak{sl}_2\mathbb{C}$ representation.
2) Every irreducible finite-dimensional representation is a quotient of M_λ. In particular, if V is a representation with highest weight λ, $V = M_\lambda/W$ for some subrepresentation $W \subset M_\lambda$.

Proof. (1) can be shown immediately via explicit calculation. (2) can be seen by first noting that M_λ is irreducible iff $\lambda \in \mathbb{Z}_{>0}$. Suppose that the irreducible finite-dimensional representations of $\mathfrak{sl}_2\mathbb{C}$ have positive integer highest weight. Knowing this, we can take W to be the subrepresentation generated by $\{v_i\}_{i=\lambda+1}^{\infty}$ and weight decomposition completes the proof. □

Lemma 5.52. *The irreducible finite-dimensional representations of $\mathfrak{sl}_2\mathbb{C}$ have non-negative integer highest weight.*

Proof. Let V be such a representation with highest weight λ. Let V^0 be a highest weight vector and defined $v^k = \frac{f^k}{k!}v^0$ as before. Since V is finite-dimensional, there is some $n \geq 0$ such that $v^n \neq 0$ but $V^{n+1} = 0$. Then

$$0 = ev^{n+1} = (\lambda-(n+1)+1)v^n = (\lambda-n)v^n$$

which implies $\lambda = n$. □

Theorem 5.53. (Classification of $\mathfrak{sl}_2\mathbb{C}$ representations)
Fix $n \geq 0$ and define V_n to be the finite-dimensional vector space with basis $\{v_i\}_{i=0}^{n}$. Define the action of $\mathfrak{sl}_2\mathbb{C}$ by:

$$hv^k = (n-2k)v^k$$

$$fv^k = (k+1)v^{k+1}, \text{ for } k<n \text{ and } fv^n = 0$$

$$ev^k = (n+1-k)v^{k-1}, \text{ for } k>n \text{ and } fv^0 = 0$$

1. V_n is an irreducible representation of $\mathfrak{sl}_2\mathbb{C}$ with highest weight n.
2. V_m and V_n are not isomorphic for $n \neq m$.
3. Every finite-dimensional irreducible $\mathfrak{sl}_2\mathbb{C}$ representation is isomorphic to some V_n.

Proof. Let M_λ be the Verma module as defined above. If $\lambda = n$ is a non-negative integer, the subspace $J_\lambda \subset M_\lambda$ spanned by $v^{n+1}, V_{n+2}, ...$ is a subrepresentation so $V_\lambda = M_\lambda / J_\lambda$ is a finite-dimensional representation of $\mathfrak{sl}_2\mathbb{C}$. It is irreducible since any v^k for $0 \leq k \leq n$ generates V_λ. Since V_n is $n+1$ dimensional, $V_n \ncong V_m$ for $n \neq m$.

Now let V be an irreducible representation of highest weight λ and let $v \in V[\lambda]$. By lemma 5.52, λ is integer. By lemma 5.51, $V \cong M_\lambda / J_\lambda$ and we are done. □

Theorem 5.54. (Structure of $\mathfrak{sl}_2\mathbb{C}$ representations)
Let V be a finite-dimensional complex $\mathfrak{sl}_2\mathbb{C}$ representation.
1. V admits a decomposition, called a weight decomposition, with integer weights:

$$V = \bigoplus_{n \in \mathbb{Z}} V[n]$$

2. $dim(V[n]) = dim(V[-n])$ and for $n \geq 0$, the maps $e^n : V[n] \to V[-n]$ and $f^n : V[-n] \to V[n]$ are isomorphisms.

Proof. It suffices to show this when V is an irreducible representation V_n. Then using 5.50.2, this can be shown explicitly. □

We state quickly, without proof, how to decompose tensor products of irreducible representations into a direct sum.

Theorem 5.55. (Clebsch–Gordan decomposition for $\mathfrak{sl}_2\mathbb{C}$)
Let V_n be the $n+1$ dimensional complex irreducible representation of $\mathfrak{sl}_2\mathbb{C}$. Then:

$$V_n \otimes V_m \cong V_{n+m} \oplus V_{n+m-2} \oplus ... \oplus V_{m-n}$$

Using the what we know about the representations of $\mathfrak{sl}_2\mathbb{C}$, we can also study the representation theory of $\mathfrak{so}_3\mathbb{R}$:

Theorem 5.56. (Representation theory of $\mathfrak{so}_3\mathbb{R}$)
1. Every finite-dimensional $\mathfrak{so}_3\mathbb{R}$ representation admits a weight decomposition:

$$V = \bigoplus_{n \in \mathbb{Z}} V[n]$$

where $V[n] = \{v \in V : J_z v = \frac{in}{2} v\}$. Here,

$$J_z = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

2. A representation V of $\mathfrak{so}_3\mathbb{R}$ lifts to a representation of $SO_3\mathbb{R}$ iff all weights are even. That is, if $V[k] = 0$ for odd k.

Proof. We only briefly sketch the proof. Recall there is an isomorphism $(\mathfrak{so}_3\mathbb{R})_\mathbb{C} = \mathfrak{so}_3\mathbb{C} \to \mathfrak{sl}_2\mathbb{C}$ which sends J_z to $-\frac{ih}{2}$. Using the weight decomposition of $\mathfrak{sl}_2\mathbb{C}$, (1) follows. Now, recall that $\mathfrak{sl}_2\mathbb{C}$ is the complexification of $\mathfrak{su}_2$ and that SU_2 is a double cover of $SO_3\mathbb{R}$ with kernel $\{1, -1\}$. Note $-1 = e^{i\pi h}$. Thus, representations of $\mathfrak{sl}_2$ where $e^{i\pi h}$ acts trivially lift to representations of $SO_3\mathbb{R}$. This completes the proof. □

Remark 5.57. *In the physics literature, $j = \lambda/2$ is called the spin of the representation where λ is the highest weight.*

6 Representations of Complex Semisimple Lie Algebras

6.1 Complete reducibility and Casimir element

We will now show that the representations of a semisimple Lie algebra are completely reducible.

Remark 6.1. *This can be proven using the theory of compact groups as follows. Any semisimple complex Lie algebra $\mathfrak{g}$ is the complexification of a real Lie algebra $\mathfrak{k} = Lie(K)$ where K is compact and simply connected. Representations of such groups were shown to be completely reducible in section 5.3, which implies representations of $\mathfrak{g}$ are reducible since $\mathfrak{k}_{\mathbb{C}}$ representations are equivalent to $\mathfrak{k}$-representations. This is called Weyl's unitary trick.*

Proposition 6.2. (Casimir element)
Let $\mathfrak{g}$ be a Lie algebra, B a non-degenerate symmetric bilinear form on $\mathfrak{g}$, X_i a basis of $\mathfrak{g}$, x^i the dual basis with respect to B. The element

$$C_B = \sum x_i x^i \in U\mathfrak{g}$$

is central and does not depend on the choice of basis. C_B is called the Casimir element determined by.B When $\mathfrak{g}$ is semisimple, it is assumed that B is the Killing form.

Proof. The element $I = \sum x_i \otimes x^i \in \mathfrak{g} \otimes \mathfrak{g}^* \cong End(\mathfrak{g}, \mathfrak{g})$ is the identity element of $End(\mathfrak{g}, \mathfrak{g})$ which also implies I is ad $\mathfrak{g}$-invariant. The map $\mathfrak{g} \otimes \mathfrak{g}^* \to \mathfrak{g} \otimes \mathfrak{g} \to U\mathfrak{g}$ is a morphism of representations, so $C_B = \sum x_i x^i$ is ad $\mathfrak{g}$-inaviant and thus central. □

Example 6.3. (Casimir element of $\mathfrak{sl}_2\mathbb{C}$)
We first compute e^*, f^*, and h^* using the fact that for $x, y \in \{e, f, h\}$, $tr(xy^*) = \delta_{xy}$. This gives $C = ef + fe + \frac{1}{2}h^2$. Since C is central, $\rho(C) : V \to V$ for a representation V commutes with the action of $\mathfrak{sl}_2\mathbb{C}$ and is thus an intertwining operator. By Schur's lemma, this means it acts as a constant on irreducible representations and that reducible representations decompose into subrepresentations which are eigenspaces of C.

Proposition 6.4. *If $\mathfrak{g}$ is simple, the Casimir element is unique up to a constant.*

Proof. This follows from the fact that any symmetric invariant bilinear form on a simple Lie algebra is a scalar multiple of the Killing form. □

Remark 6.5. *C always acts nontrivially in a nontrivial representation. This is not very easy to prove, so we won't.*

Theorem 6.6. (Weyl's theorem)
Any complex finite-dimensional representation of a semisimple Lie algebra $\mathfrak{g}$ is completely reducible.

The proof is omitted since it uses some basic notions of Lie algebra cohomology which are not covered in these notes. Unfortunately, this means we won't see the Casimir element in action, but you can take my word for it that it is important.

As an immediate consequence, we get the following result, stated without proof in section 2.5. Levi decomposition can be proved similalry.

Corollary 6.7. (Reductive decomposition)
Any reductive Lie algebra can be written as a direct sum of semisimple and commutative ideals:

$$\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_{ss}$$

Proof. Since $\mathfrak{z}(\mathfrak{g})$ acts trivially in the adjoint representation of $\mathfrak{g}$, the representation decends to a representation of $g' = \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$. Since $\mathfrak{g}$ is reductive, $\mathfrak{g}'$ is semisimple so $\mathfrak{g}$ considered as a representation of $\mathfrak{g}'$ is completely reducible. Since $\mathfrak{z}(\mathfrak{g})$ is stable under the adjoint action of $\mathfrak{g}'(and\mathfrak{g})$, it is a subrepresentation which we will denote $\mathfrak{z}$ and we can thus write $\mathfrak{g} = \mathfrak{z} \oplus I$ for some $I \subset \mathfrak{g}$ which is stable under the adjoint action of $\mathfrak{g}'$. Since this implies I is stable under the disjoint action of $\mathfrak{g}$, I is an ideal so $\mathfrak{g} = \mathfrak{z} \oplus I$ not just as a representation but as a Lie algebra. Since $I = \mathfrak{g}/\mathfrak{g}'$, it is semisimple. □

6.2 Weight decomposition and character theory

We will now finally begin our journey towards the classification of complex finite-dimensional representations of semi-simple Lie algebras. As was the case with $\mathfrak{sl}_2\mathbb{C}$, the trick will be to decompose representations into eigenspaces for a Cartan subalgebra. In the case of $\mathfrak{sl}_2\mathbb{C}$, we had $\mathfrak{h} = \langle h \rangle$. We'll need to introduce a few concepts before we can give a classification theorem.

Definition 6.8. *Let V be a representation of a complex semisimple Lie algebra $\mathfrak{g}$. A vector $v \in V$ is called a vector of weight $\lambda \in \mathfrak{h}^*$ if for all $h \in \mathfrak{h}$, $hv = \langle \lambda, h \rangle v$. The space of all vectors of weight λ is called a weight space and is denoted $V[\lambda]$. When the weight space is nonempty, λ is called a weight of V. We denote the set of all weights of V as $P(V) = \{\lambda \in \mathfrak{h}^8 : V[\lambda] \neq 0\}$.*

Remark 6.9. *Vectors of different weights are linearly independent, so $P(V)$ is finite when V is finite-dimensional.*

Recall that the weight lattice $P \subset E$ is the lattice $\{\lambda \in E : \langle \lambda, \alpha_i^\vee \in \mathbb{Z}$ for all simple roots $\alpha_i\}$. Also recall that the coroot $\alpha^\vee \in E^*$ is defined by $\langle \alpha^\vee, \lambda \rangle = \frac{2(\alpha,\lambda)}{(\alpha,\alpha)}$. That is, $\langle \alpha^\vee, \lambda \rangle$ is twice the component of λ along α.

Theorem 6.10. *Every finite-dimensional representation of $\mathfrak{g}$ admits a weight decomposition*

$$V = \bigoplus_{\lambda \in P(V)} V[\lambda]$$

and all weights of V are integral, that is, $P(V) \subset P$.

Proof. Let α be a root and recall that there is an $\mathfrak{sl}_2\mathbb{C}$ subalgebra in $\mathfrak{g}$ generated by elements e_α, f_α, and h_α as follows:

$e_\alpha \in \mathfrak{g}_\alpha$ and $f_\alpha \in \mathfrak{g}_-\alpha$ are chosen such that $(e, f) = 2/(\alpha, \alpha)$ and

$$h_\alpha = \frac{2H_\alpha}{(\alpha, \alpha)}$$

where $H_\alpha \in \mathfrak{h}$ is defined by the property that $(\alpha, \beta) = \langle H_\alpha, \beta\rangle$ for all $\beta \in \mathfrak{h}^*$.

We have seen $\mathfrak{h}_\alpha$ is diagonalizable in V considered as a module over $(\mathfrak{sl}_2\mathbb{C})_\alpha$. Since $\{h_\alpha : \alpha \in R\}$ spans $\mathfrak{h}$ and the sum of commuting diagonalizable operators is diagonalizable, any $h \in \mathfrak{h}$ is diagonalizable. Since $\mathfrak{h}$ is commutative, they can be simultaneously diagonalized. Since $\mathfrak{sl}_2\mathbb{C}$ has integer weights, this implies $P(V) \subset P$ and concludes the proof. □

Remark 6.11. *As expected, the weight decomposition agrees with the root decomposition.*

Lemma 6.12. *For $x \in \mathfrak{g}_\alpha$, $x.V[\lambda] \subset V[\lambda + \alpha]$*

Proof. Using the Serre relations, one can show this explicitly as was done with $\mathfrak{sl}_2\mathbb{C}$. □

We will be often interested in the dimensions of the weight subspaces $V[\lambda]$. Let $\mathbb{C}[P]$ be the algebra generated by the symbols $e^\lambda, \lambda \in P$ subject to the relations $e^\lambda e^\mu = e^{\lambda+\mu}$ and $e^0 = 1$. $\mathbb{C}[P]$ is isomorphic to the algebra of complex-valued polynomial functions on the torus $T = \mathfrak{h}/2\pi i Q^\vee$ by defining $e^\lambda(t) = e^{\langle t,\lambda\rangle}$ for $t \in T$. This is isomorphic to the algebra of Laurent polynomials in rank($\mathfrak{g}$) variables.

Definition 6.13. *The character* $\mathrm{ch}(V) \in \mathbb{C}[P]$ *of a finite-dimensional representation V of $\mathfrak{g}$ is*

$$\mathrm{ch}(V) = \sum(\dim(V[\lambda])e^\lambda$$

Remark 6.14. *The use of the term character here is closely related to its use in group representations. Every $t \in \mathfrak{h}$ corresponds to* $\exp(t) \in G$*. Then*

$$\mathrm{ch}(V)(t) = \mathrm{tr}_V(\exp(t))$$

which justifies the choice of notation e^λ.

Example 6.15. *Let $\mathfrak{g} = \mathfrak{sl}_2\mathbb{C}$. The weight lattice $P = \mathbb{Z}\frac{\alpha}{2}$ so $\mathbb{C}[P]$ is generated by $e^{n\alpha/2}, n \in \mathbb{Z}$. Writing $x = e^{\alpha/2}$ and the results of 5.5, we have*

$$\mathrm{ch}(V_n) = x^n + x^{n-2} + ... + x^{-n} = \frac{x^{n+1} - x^{-n-1}}{x - x^{-1}}$$

Lemma 6.16. -
1. $\mathrm{ch}(\mathbb{C}) = 1$.
2. $\mathrm{ch}(V_1 \oplus V_2) = \mathrm{ch}(V_1) + \mathrm{ch}(V_2)$.
3. $\mathrm{ch}(V_1 \otimes V_2) = \mathrm{ch}(V_1)\mathrm{ch}(V_2)$.
4. $\mathrm{ch}(V^*) = \overline{\mathrm{ch}(V)}$ *where* $\overline{e^\lambda} = e^{-\lambda}$.

We might recall that the characters of $\mathfrak{sl}_2\mathbb{C}$ are invariant under its Weyl group $\mathbb{Z}_2$ which acts by inversion. This turns out to always be the case:

Theorem 6.17. *Let V be a finite -dimensional representation of $\mathfrak{g}$. The set of weights and the dimensions of the weight subspaces are invariant under the action of the Weyl group. That is to say, for all $w \in W$, $\dim(V[\lambda]) = \dim(V[w(\lambda)])$. Equivalently, $w(\mathrm{ch}(V)) = \mathrm{ch}(V)$ where the action of W on $\mathbb{C}[P]$ is $w(e^\lambda) = e^{w(\lambda)}$.*

Proof. It suffices to prove this for simple reflections s_i since these generate W. Let $\langle \lambda, \alpha_i^\vee\rangle = n \geq 0$. The operators $f_i^n : V[\lambda] \to V[\lambda - n\alpha_i]$ and $e_i^n : V[\lambda - n\alpha_i] \to V[\lambda]$ are isomorphism. Therefore, $\dim(V[\lambda]) = \dim(V[\lambda - n\alpha_i])$. By definition, $s_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee\rangle\alpha_i = \lambda - n\alpha_i$, so we are done. □

6.3 Highest weight representations and Verma modules

Recall that when we studied the representation theory of $\mathfrak{sl}_2\mathbb{C}$*, we constructed infinite-dimensional representations* M_λ *called, Verma modules, that are in some sense "universal:" Every finite-dimensional irreducible representation of* $\mathfrak{sl}_2\mathbb{C}$ *is a quotient of* M_λ*. In this section, we will do the same for a general complex semisimple Lie algebra.*

Recall the triangular decomposition of a Lie algebra $\mathfrak{g}$ *given by a choice of polarization:*

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \ n_\pm = \bigoplus_{\alpha \in R_\pm} \mathfrak{g}_\alpha$$

The subalgebras $n_\pm$ *are called the positive and negative nilpotent subalgebras.*

Definition 6.18. *A representation* V *of* $\mathfrak{g}$ *is called a highest weight representation of highest weight* λ *if it is generated by a vector* $v \in V[\lambda]$ *such that* $e_\alpha v = 0$ *for all* $\alpha \in R_+$*.* v *is called a highest weight vector.*

Theorem 6.19. *Every finite-dimensional irreducible representation of a semisimple Lie algebra is a highest weight representation.*

Proof. Let $\lambda \in P(V)$ be such that for all $\alpha \in R$, $\lambda + \alpha \notin P(V)$. Such λ always exists: Take $h \in \mathfrak{h}$ such that $\langle h, \alpha_i \rangle > 0$ for all $\alpha \in R_+$. Then pick λ such that $\langle h, \lambda \rangle$ is maximal.

Let $v \in V[\lambda] benonzero$. We have $e_\alpha v = 0$ for any $\alpha \in R_+$ since $\lambda + \alpha$ is not a weight of V. The representation generated by v is a nontrivial highest weight subrepresentation, but since V is irreducible V must be this representation □

Remark 6.20. *We briefly note that highest weight representations of the same highest weight need not be isomorphic if they are not irreducible. We will explore this further in the next section.*

Lemma 6.21. *Any highest weight vector* v *of weight* λ *in a highest weight representation satisfies the following:*
1. $hv_\lambda = \langle h, \lambda \rangle v_\lambda$ *for all* $h \in \mathfrak{h}$*.*
2. $ev_\lambda = 0$ *for all* $e \in \mathfrak{n}_+$*.*

Definition 6.22. *Define the universal highest weight representation* M_λ*, called a Verma module, as the representation generated by a single vector* v_λ *subject to the relations of the previous lemma. That is*

$$M_\lambda = U\mathfrak{g}/I_\lambda$$

where I_λ *is the left ideal in* $U\mathfrak{g}$ *generated by* $e \in \mathfrak{n}_+$ *and* $(h - \langle h, \lambda \rangle$ *with* $h \in \mathfrak{h}$*.*

There is an equivalent definition of M_λ *given by a Borel subalgebra. We will take a quick detour to describe this since it will prove useful later.*

Definition 6.23. *The Borel subalgebra* $\mathfrak{b} \subset \mathfrak{g}$ *with respect to a given triangular decomposition is the subalgebra* $\mathfrak{b} = \mathfrak{n}_+ \oplus \mathfrak{h}$*.*

Lemma 6.24. -
1. n_+ *and* n_- *are nilpotent.*
2. $\mathfrak{b}$ *is solvable with derived algebra* $\mathfrak{n}_+$*.*

Theorem 6.25. (Borel-Morozov)
$\mathfrak{b}$ is a maximal solvable subalgebra and any solvable subalgebra of $\mathfrak{g}$ can be mapped to a subalgebra of $\mathfrak{b}$ by an inner automorphism. In particular, any two maximal solvable subalgebras are conjugate. One can actually take this as the definition of a Borel subalgebra.

Remark 6.26. *We can also define M_λ as the product $M_\lambda = U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda$ where $\mathbb{C}_\lambda$. is the one-dimensional representation defined by lemma 6.29. We mention for completeness that this is the same as saying $M_\lambda = \mathrm{Ind}_{U\mathfrak{b}}^{U\mathfrak{g}}\mathbb{C}_\lambda$, although we won't cover induced representations in these notes.*

Theorem 6.27. *If V is a highest weight representation of weight λ can be realized as M_λ/W for some subrepresentaiton $W \subset M_\lambda$*

Theorem 6.28. -
1. Every $v \in M_\lambda$ can be written as uv_λ for some $u \in U\mathfrak{n}_-$. That is to say, we have an isomorphism $U\mathfrak{n}_- \to M_\lambda$ given by $u \mapsto uv_\lambda$.
2. M_λ admits a weight space decomposition

$$M_\lambda = \bigoplus_{\mu \in P(M_\lambda)} M_\lambda[\mu]$$

where the set of weights of $M\lambda$ is

$$P(M_\lambda) = \lambda - Q_+ \text{ where } Q_+ = \left\{\sum n_i\alpha_i, n_i \in \mathbb{Z}_{\geq 0}\right\}$$

3. $dim(M_\lambda[\lambda] = 1$.

Proof. Recall that by the PBW theorem, if $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, $U\mathfrak{g} = U\mathfrak{g}_1 \otimes U\mathfrak{g}_2$. In our case, $U\mathfrak{g} \cong U\mathfrak{n}_- \otimes U\mathfrak{b}$ as $U\mathfrak{n}_-$ modules so

$$M_\lambda = U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda = U\mathfrak{n}_- \otimes U\mathfrak{b} \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda = U\mathfrak{n}_- \otimes \mathbb{C}_\lambda$$

This implies all 3 parts of the theorem. □

Having seen that all highest weight representations are quotients of M_λ, we will generalize the above to any highest weight representation.

We can define a partial order on $\mathfrak{h}^*$ by $\lambda \leq \mu$ iff $\lambda - \mu \in Q_+$ and $\lambda < \mu$ iff $\lambda \leq \mu$ and $\lambda \neq \mu$.

Theorem 6.29. *Let V be a highest weight representation with highest weight vector λ.*
1. Every $v \in V$ can be written as $v = uv_\lambda$ with $u \in U\mathfrak{n}_-$. That is to say, the map $U\mathfrak{n}_- \to V$ given by $u \mapsto uv_\lambda$ is surjective.
2. V admits a weight decomposition:

$$V = \oplus_{\mu \leq \lambda} V[\mu]$$

where each weight subspace is finite-dimensional.
3. $dim(V[\lambda]) = 1$.

Proof. (1) follows from the previous theorem on Verma modules. To prove (2), we will use this lemma, the proof of which follows from elementary linear algebra:

Lemma 6.30. *Let $\mathfrak{h}$ be a commutative finite-dimensional Lie algebra and M an $\mathfrak{h}$-module with a weight decomposition*

$$M = \bigoplus M[\lambda], \quad M[\lambda] := \{v : hv = \langle h, \lambda\rangle v \textit{ for all } h \in \mathfrak{h}$$

Then any quotient and submodule of M admits a weight decomposition.

(2) follows immediately. To see (3), note that $\dim(V[\lambda]) \leq \dim(M_\lambda[\lambda]) = 1$ and $V[\lambda]$ is nonempty. □

The following is a useful corollary:

Corollary 6.31. *Any highest weight representation has a unique highest weight and a unique highest weight vector up to a scalar.*

Proof. By 6.29.2, we must have $\lambda \leq \mu$ and $\mu \leq \lambda$. □

6.4 Classification of finite-dimensional irreducible representations

Finally, we will classify all finite-dimensional irreducible representations of semisimple Lie algebras. We have already seen that all of these are highest weight representations, so it will suffice to classify irreducible finite-dimensional highest weight representations. We start by proving the following important result:

Theorem 6.32. *For any $\lambda \in \mathfrak{h}^*$, there is a unique irreducible highest weight representation with highest weight λ with highest weight λ denoted L_λ.*

Since all highest weight representations with highest weight λ are of the form $V = M_\lambda/W$ for some subrepresentation W, V is irredcuble iff W is maximal so it suffices to show there exist a unique maximal proper submodule. Every submodule of M_λ admits a weight decomposition so we have $W[\lambda] = 0$ since otherwise, $W[\lambda] = M_\lambda[\lambda]$ which would imply $W = M_\lambda$. Let J_λ be the sum of all submodules W such that $W[\lambda] = 0$. This is clearly proper and contains every other submodule of M_λ, so $L_\lambda = M_\lambda/J_\lambda$ is the unique irreducible representation of highest weight λ.

Remark 6.33. *As was the case with $\mathfrak{sl}_2$, for generic λ, M_λ is irreducible.*

Corollary 6.34. *Every irreducible finite-dimensional representation V is isomorphic to some L_λ*

We need to determine which of these L_λ are finite-dimensional. To do this, we define the following:

Definition 6.35. *A weight $\lambda \in \mathfrak{h}^*$ is dominant integral if for all $\alpha \in R_+$ (or equivalently, $\alpha \in \Pi$),*

$$\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{>0}$$

We denote the set of dominant integral weights by P_+

Lemma 6.36. *Recall the weight lattice $P \subset \mathfrak{h}^*$,*

$$P = \{\lambda \in E : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in R\} = \{\lambda \in E : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha^\vee \in Q^\vee\}$$

where $Q^\vee$ is the coroot lattice.
1. We have $P_+ = P \cap \overline{C}_+$ where

$$C_+ = \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha_i^\vee > 0 \text{ for all } \alpha \in \Pi\}$$

where C_+ is the positive Weyl chamber.
2. For any $\lambda \in P$, its Weyl group orbit $W\lambda$ contains exactly one element of P_+.

Proof. (1) follows from the definitions and (2) follows from the fact that any W-orbit in $\mathfrak{h}^*_\mathbb{R}$ contains exactly one element of $\overline{C_+}$. To see that, note that W acts transitively on Weyl chambers and fixes the root hyperplanes. □

This lemma will help us prove the following result, which will be the key to the classification we are seeking:

Lemma 6.37. *An irreducible highest weight representation L_λ is finite-dimensional iff $\lambda \in P_+$.*

Proof. We only give a sketch of the proof: If L_λ is finite-dimensional, recall we have a subalgebra $(\mathfrak{sl}_2\mathbb{C})_i$ defined by $e_i \in \mathfrak{g}_{\alpha_i}$, $f_i \in \mathfrak{g}_{-\alpha_i}$ such that $(e, f) = 2/(\alpha, \alpha)$ and

$$h_\alpha = \frac{2H_\alpha}{(\alpha, \alpha)}$$

where H_α is the element of $\mathfrak{h}$ such that for all $\beta \in \mathfrak{h}^*$,

$$(\alpha, \beta) = \langle H_\alpha, \beta \rangle = (H_\alpha, H_\beta)$$

The highest weight vector $v_\lambda \in \mathfrak{sl}_2\mathbb{C}$ satisfies $e_i v_\lambda = 0$ and $h_i v_\lambda = \langle h_i, \lambda \rangle v_\lambda = \langle \alpha_i^\vee, \lambda \rangle v_\lambda$ and generates a finite-dimensional highest weight $(\mathfrak{sl}_2\mathbb{C})_i$ submodule. By the representation theory of $\mathfrak{sl}_2\mathbb{C}$, $\langle h_i, \lambda \rangle \in \mathbb{Z}_+$ for any simple root α_i and therefore, we have $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_+$ for any positive root.

Now suppose λ is finite-dimensional. The proof breaks into three steps:
1. Define $v_{s_i(\lambda)} = f_i^{n_i+1} v_\lambda \in M_\lambda[s_i(\lambda)]$ where $n_i = \langle \alpha_i^\vee, \lambda \rangle \in \mathbb{Z}_{>0}$.
2. Let M_i be the submodule generated by $v_{si(\lambda)}$. This is a highest weight submodule of weight $\lambda_{s_i(\lambda)}$ and therefore cannot have $\lambda > s_i(\lambda)$ as a weight.
3. Define

$$\tilde{L_\lambda} = M_\lambda / \sum M_i$$

and note it is a nonzero highest weight representation. When λ is dominant integral, this is finite-dimensional (although this is nontrivial to show). Since L_λ is a quotient of M_λ by a maximal proper submodule, it must be contained in $\tilde{L_\lambda}$ which is finite dimensional. This completes the proof. □

We finally get the following result as a corollary:

Theorem 6.38. *For every $\lambda \in P_+$, L_λ is an irreducible finite-dimensional representation. These are pairwise non-isomorphic and every irreducible finite-dimensional representation is isomorphic to some L_λ.*

Example 6.39. (Weights of the adjoint representation)
Let $\mathfrak{g}$ be a simple Lie algebra and consider its adjoint representation. This is irreducible with weights $\alpha \in R$ and 0 with multiplicity dim($\mathfrak{h}$). This must have a highest weight θ i.e. a weight $\theta \in R_+$ such that $\theta + \alpha \notin R \cup \{0\}$ for all $\alpha \in R_+$. This is sometimes called a maximal root. In the case of $\mathfrak{sl}_n\mathbb{C}$, $\theta = e_1 - e_n$. This root is also the unique root with maximal height.

6.5 Weyl character formula and dimension

Recall that the character of a finite-dimensional representation V is

$$\mathrm{ch}(V) = \sum \dim(V)[\lambda]e^\lambda \in \mathbb{C}[P]$$

Since the character of a representation tells us a lot about its weight decomposition, it will be nice to have an explicit formula for computing it. We state here some results about this, the proofs of which are omitted for brevity.

Lemma 6.40. (Character of M_λ)

$$\mathrm{ch}(M_\lambda) = \frac{e^\lambda}{\prod_{\alpha \in R_+}(1 - e^{-\alpha})}$$

where $\frac{1}{1-e^{-\alpha}}$ is a formal series $1 + e^{-\alpha} + e^{-2\alpha} + \ldots$

Theorem 6.41. (Weyl character formula)
Let L_λ be the irreducible finite-dimensional (highest weight) representation with highest weight $\lambda \in P_+$. We have:

$$\mathrm{ch}(L_\lambda) = \frac{\sum_{w \in W}(-1)^{l(w)}e^{w(\lambda)}}{\prod_{\alpha \in R_+}(1 - e^{-\alpha}} = \frac{\sum_{w \in W}(-1)^{l(w)}e^{w(\lambda+\rho)}}{\prod_{\alpha \in R_+}(e^{\alpha/2} - e^{-\alpha/2}}$$

We note that one intuitive proof of the Weyl character formula uses the BGG resolution for its key steps. Sadly, that is not covered in these notes.

There are two useful corollaries:

Corollary 6.42. (Weyl denominator identity)

$$\prod_{\alpha \in R_+}(e^{\alpha/2} - e^{-\alpha/2} = \sum_{w \in W}(-1)^{l(w)}e^{w(\rho)}$$

This polynomial is called the Weyl denominator. In particular the Weyl denominator is ske-symmetric: $w(\delta) = (-1)^{l(w)}\delta$.

Proof. Apply the Weyl denominator and is denoted δ.l character formula when $\lambda = 0$. We see that $L_\lambda = \mathbb{C}$ and $\mathrm{ch}(L_\lambda) = 1$. The equality follows □

Corollary 6.43. *For $\lambda \in P_+$*

$$\mathrm{ch}(L_\lambda) = A_{\lambda+\rho}/A_\rho$$

where

$$A_\mu = \sum_{w \in W} (-1)^{l(w)} e^{w(\mu)}$$

We would like to use the following:

$$\dim(V) = \sum_{\lambda \in R} \dim(V[\lambda]) = \mathrm{ch}(V)(0)$$

but both the denominator and numerator of the Weyl character formula vanish at 0. Thus, we introduce the following:

Definition 6.44. *Let V be a finite-dimensional representation of a Lie algebra $\mathfrak{g}$. Define the q-dimension, $dim_q(V) \in \mathbb{C}[q^{\pm 1}]$ by*

$$\dim_q V = \mathrm{tr}_V(q^{2\rho}) = \sum_\lambda \dim(V[\lambda]) q^{2(\rho,\lambda)}$$

where $(\cdot,\cdot)$ is a W-invariant symmetric bilinear form on $\mathfrak{h}^*$ such that $(\lambda,\mu) \in \mathbb{Z}$ for any $\lambda, \mu \in P$. Note that at $q = 1$, this is just the usual notion of dimension.

Proposition 6.45.

$$\dim_q V = \pi_\rho(\mathrm{ch}(V))$$

where $\pi_\rho : \mathbb{C}[P] \to C[q^{\pm 1}]$ is given by $\pi_\rho(e^\lambda) = q^{2(\lambda,\rho)}$.

Theorem 6.46. *For $\lambda \in P_+$ dominant integral:*

$$\dim_q(L_\lambda) = \prod_{\alpha \in R_+} \frac{q^{(\lambda+\rho,\alpha} - q^{-(\lambda+\rho,\alpha}}{q^{(\rho,\alpha)} - q^{-(\rho,\alpha)}}$$

Corollary 6.47.

$$\dim(L_\lambda) = \prod_{\alpha \in R_+} \frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)} = \prod_{\alpha \in R_+} \frac{\langle \lambda+\rho, \alpha^\vee \rangle}{\langle \rho,^\vee \rangle}$$

Proof. We only note that this follows from the fact that

$$\lim_{q \to 1} \frac{q^n - q^{-m}}{q^m - q^{-m}} = \frac{n}{m}$$

□

6.6 Multiplicities

We have now seen that any finite-dimensional representation of a semisimple Lie algebra is completely reducible and thus can be written as

$$V = \bigoplus_{\lambda \in P_+} n_\lambda L_\lambda$$

for some $n_\lambda \in \mathbb{Z}_{\geq 0}$. *We will now attempt to compute the multiplicities* n_λ.

Theorem 6.48. *The characters* $\mathrm{ch}(L_\lambda)$ *with* $\lambda \in P_+$ *are a basis of the algebra of* W*-invariant polynomials* $\mathbb{C}[P]^W$.

Proof. Note we have a basis $\{m_\lambda\}_{\lambda \in P_+}$ of $\mathbb{C}[P]^W$ where

$$m_\lambda = \sum_{\mu \in W\lambda} e^\mu$$

where $W\lambda$ is the Weyl group orbit of λ. Any such orbit contains a unique dominant integral root, so this is indeed a basis. By weight decomposition, we have

$$\mathrm{ch}(L_\lambda) = \sum_{\mu \leq \lambda} c_\mu e^\mu = m_\lambda + \sum_{\mu \in P+, \mu < \lambda} c_\mu m_m u$$

Then the matrix representing $\mathrm{ch}(L_\lambda)$ in our chosen basis is upper-triangular with 1s on the diagonal and thus invertible. □

Corollary 6.49. -
1. We can count multiplicities by writing $\mathrm{ch}(V)$ *in the basis* $\mathrm{ch}(L_\lambda)$:

$$\mathrm{ch}(V) = \sum_{\lambda \in P_+} n_\lambda \mathrm{ch}(L_\lambda)$$

2. We have a way to recursively compute these coefficients: If $\lambda \in P(V)$ *is maximal, then* $n_\lambda = \dim(V[\lambda])$. *Now, consider* $\mathrm{ch}(V) - n_\lambda \mathrm{ch}(L_\lambda)$ *and repeat this until you reach the 0 polynomial.*

6.7 Example: Representations of $\mathfrak{sl}_n\mathbb{C}$ and Young diagrams

To do:

6.8 Example: Representations of $\mathfrak{g}_2$

To do: