Small Fragments of the Martin's Maximum

Nam Trang

University of North Texas
Berkeley Inner Model Conference 2025

June 26, 2025

Contents

1. **Introduction**
2. Consistency Strength
3. Forcing Fragments of Martin's Maximum over Models of AD^+
4. Questions

Introduction

Introduction

- Part of this talk is on-going joint work with M. Zeman.

Introduction

- Part of this talk is on-going joint work with M. Zeman.
- The main topic of this talk centers around small fragments of the Martin's Maximum and their variations that are forcable over models of determinacy.

Introduction

- Part of this talk is on-going joint work with M. Zeman.
- The main topic of this talk centers around small fragments of the Martin's Maximum and their variations that are forcable over models of determinacy.

This work is partially supported by the National Science Foundation via career grant DMS-1945592 and the Simons Fellowship.

Martin's Maximum

Definition

Martin's Maximum (MM) is the statement that whenever $\mathbb{P}$ is a stationary-set preserving forcing poset, and $(D_\alpha : \alpha < \omega_1)$ is a sequence of dense subsets of $\mathbb{P}$ then there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$.

Martin's Maximum

Definition

Martin's Maximum (MM) is the statement that whenever $\mathbb{P}$ is a stationary-set preserving forcing poset, and $(D_\alpha : \alpha < \omega_1)$ is a sequence of dense subsets of $\mathbb{P}$ then there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$.

Stationary-set preserving forcing posets include a large class of posets like proper and semi-proper forcings. Therefore, MM implies PFA, SPFA.

Martin's Maximum

Definition

Martin's Maximum (MM) is the statement that whenever $\mathbb{P}$ is a stationary-set preserving forcing poset, and $(D_\alpha : \alpha < \omega_1)$ is a sequence of dense subsets of $\mathbb{P}$ then there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$.

Stationary-set preserving forcing posets include a large class of posets like proper and semi-proper forcings. Therefore, MM implies PFA, SPFA.

Theorem (Foreman-Magidor-Shelah)

*Con(*ZFC$+$ *there is a supercompact cardinal) implies Con(*MM*).*

Question: What is the consistency strength of MM?

Martin's Maximum (cont.)

Some small fragments of MM are much weaker, consistency-wise, than full MM.
Write MM(κ) for MM restricted to posets of size $\leq \kappa$.

Martin's Maximum (cont.)

Some small fragments of MM are much weaker, consistency-wise, than full MM. Write $\mathsf{MM}(\kappa)$ for MM restricted to posets of size $\leq \kappa$.

Theorem (Dobrinen-Krueger-Marun-Mota-Zapletal, 2023)

The theories ZFC *and* $\mathsf{MM}(\omega_1)$ *are equiconsistent.*

Martin's Maximum (cont.)

Some small fragments of MM are much weaker, consistency-wise, than full MM. Write $\mathrm{MM}(\kappa)$ for MM restricted to posets of size $\leq \kappa$.

Theorem (Dobrinen-Krueger-Marun-Mota-Zapletal, 2023)

The theories ZFC *and* $\mathrm{MM}(\omega_1)$ *are equiconsistent.*

Find me all relevant papers concerning the consistency of Martin's Maximum in set theory

Talked to scispace.com

Here are five key papers addressing the **consistency of Martin's Maximum (MM)** in set theory, particularly its consistency strength, definability implications, and forcing techniques:

Relevant Papers on Martin's Maximum Consistency

No.	Title & Authors	Summary	Link
1	On the Consistency Strength of $\mathrm{MM}(\omega_1)$ Natasha Dobrinen et al. (2023)	Shows that MM restricted to partial orders of cardinality ω_1 has consistency strength derivable from ZFC alone.	Read here
2	On the consistency strength of $\mathrm{MM}(\omega_1)$ Dobrinen et al. (2024)	Formal journal version of the above result in *Proc. AMS*. Explores $\mathrm{MM}(\omega_1)$ as consistent under ZFC without large cardinals.	Read here

Martin's Maximum (cont.)

Theorem (Woodin)

Con($\mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular*) *implies Con*($\mathsf{MM}(\mathfrak{c})$).

Martin's Maximum (cont.)

Theorem (Woodin)

Con($\mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular) implies Con*$(\mathsf{MM}(\mathfrak{c}))$.

On the other hand, inner model theoretic techniques show that $\mathsf{MM}(\mathfrak{c})$ is strictly stronger than ZFC:

- (Woodin) $\mathsf{MM}(\mathfrak{c})$ implies Projective Determinacy.
- (Steel-Zoble) $\mathsf{MM}(\mathfrak{c})$ implies $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$.

Martin's Maximum (cont.)

Theorem (Woodin)

Con($\mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular) implies Con*($\mathsf{MM}(\mathfrak{c})$).

On the other hand, inner model theoretic techniques show that $\mathsf{MM}(\mathfrak{c})$ is strictly stronger than ZFC:

- (Woodin) $\mathsf{MM}(\mathfrak{c})$ implies Projective Determinacy.
- (Steel-Zoble) $\mathsf{MM}(\mathfrak{c})$ implies $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$.

Question: What is the consistency strength of $\mathsf{MM}(\mathfrak{c})$? What about larger fragments of $\mathsf{MM}(\mathfrak{c})$ and their consequences?

Martin's Maximum (cont.)

Theorem (Woodin)

Con($\mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular) implies Con*($\mathsf{MM}(\mathfrak{c})$).

On the other hand, inner model theoretic techniques show that MM(c) is strictly stronger than ZFC:

- (Woodin) MM(c) implies Projective Determinacy.
- (Steel-Zoble) MM(c) implies $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$.

Question: What is the consistency strength of MM(c)? What about larger fragments of MM(c) and their consequences?

In this talk, we will mostly consider the following "small" fragments of MM:

1. MM(c)+there is a semi-saturated ideal on ω_2.
2. $\mathsf{MM}(\mathfrak{c}) + \neg\Box_{\omega_2}$.
3. $\mathsf{MM}(\mathfrak{c}) + \neg\Box(\omega_3)$.

Martin's Maximum (cont.)

Theorem (Woodin)

Con($\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular) implies Con$(\mathsf{MM}(\mathsf{c}))$.

On the other hand, inner model theoretic techniques show that MM(c) is strictly stronger than ZFC:

- (Woodin) MM(c) implies Projective Determinacy.
- (Steel-Zoble) MM(c) implies $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$.

Question: What is the consistency strength of MM(c)? What about larger fragments of MM(c) and their consequences?

In this talk, we will mostly consider the following "small" fragments of MM:

1. MM(c)+there is a semi-saturated ideal on ω_2.
2. $\mathsf{MM}(\mathsf{c}) + \neg\square_{\omega_2}$.
3. $\mathsf{MM}(\mathsf{c}) + \neg\square(\omega_3)$.

Consequences of Martin's Maximum

For an infinite cardinal λ, the principle $\square_\lambda$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ such that for each $\alpha < \lambda^+$,

- C_α is club in α;
- for each limit point β of C_α, $C_\beta = C_\alpha \cap \beta$;
- the order type of C_α is at most λ.

Consequences of Martin's Maximum

For an infinite cardinal λ, the principle $\square_\lambda$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ such that for each $\alpha < \lambda^+$,

- C_α is club in α;
- for each limit point β of C_α, $C_\beta = C_\alpha \cap \beta$;
- the order type of C_α is at most λ.

The principle $\square(\lambda)$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \lambda \rangle$ such that

1. for each $\alpha < \lambda$,
   - each C_α is club in α;
   - for each limit point β of C_α, $C_\beta = C_\alpha \cap \beta$; and
2. there is no thread through the sequence, i.e., there is no club $E \subseteq \lambda$ such that $C_\alpha = E \cap \alpha$ for each limit point α of E.

Consequences of Martin's Maximum

For an infinite cardinal λ, the principle $\square_\lambda$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ such that for each $\alpha < \lambda^+$,

- C_α is club in α;
- for each limit point β of C_α, $C_\beta = C_\alpha \cap \beta$;
- the order type of C_α is at most λ.

The principle $\square(\lambda)$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \lambda \rangle$ such that

1. for each $\alpha < \lambda$,
   - each C_α is club in α;
   - for each limit point β of C_α, $C_\beta = C_\alpha \cap \beta$; and
2. there is no thread through the sequence, i.e., there is no club $E \subseteq \lambda$ such that $C_\alpha = E \cap \alpha$ for each limit point α of E.

Observation: $\neg\square(\lambda^+) \Rightarrow \neg\square_\lambda$.

Consequences of Martin's Maximum

For an infinite cardinal λ, the principle $\square_\lambda$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ such that for each $\alpha < \lambda^+$,

- C_α is club in α;
- for each limit point β of C_α, $C_\beta = C_\alpha \cap \beta$;
- the order type of C_α is at most λ.

The principle $\square(\lambda)$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \lambda \rangle$ such that

1. for each $\alpha < \lambda$,
   - each C_α is club in α;
   - for each limit point β of C_α, $C_\beta = C_\alpha \cap \beta$; and
2. there is no thread through the sequence, i.e., there is no club $E \subseteq \lambda$ such that $C_\alpha = E \cap \alpha$ for each limit point α of E.

Observation: $\neg\square(\lambda^+) \Rightarrow \neg\square_\lambda$.

A semi-saturated ideal I on ω_2 can be extended to a semi-saturated ideal I^+_g on $V[g]$ for a V-generic $g \subseteq Coll(\omega, \omega_1)$; in particular, if H is $V[g]$-generic for the boolean algebra $\wp(\omega_1^{V[g]})/I^+_g$, then letting $j_H : V[g] \to N$ be the generic embedding, $j_H(\omega_1^{V[g]}) = \omega_2^{V[g]}$.

Consequences of Martin's Maximum (cont.)

MM implies:

- $2^\omega = 2^{\omega_1} = \omega_2$.
- The non-stationary ideal on ω_1 is saturated.
- Strong and Weak reflection principles.
- $\forall\kappa \geq \omega_2 \neg\Box(\kappa)$.

Consequences of Martin's Maximum (cont.)

MM implies:

- $2^\omega = 2^{\omega_1} = \omega_2$.
- The non-stationary ideal on ω_1 is saturated.
- Strong and Weak reflection principles.
- $\forall\kappa \geq \omega_2 \neg\square(\kappa)$.

MM does not imply the existence of a semi-saturated ideal on ω_2. But Woodin shows

Theorem (Woodin)

Suppose $M \vDash V = L(\wp(\mathbb{R})) + \mathsf{AD}_{\mathbb{R}} + \Theta$ is regular. Let $G \subseteq \mathbb{P}_{\max}$ be M-generic and $H \subseteq Add(\Theta^M, 1)$ be $M[G]$-generic, then $M[G][H] \vDash \mathsf{MM(c)}+$ there is a semi-saturated ideal on ω_2.

Consequences of Martin's Maximum (cont.)

MM implies:

- $2^\omega = 2^{\omega_1} = \omega_2$.
- The non-stationary ideal on ω_1 is saturated.
- Strong and Weak reflection principles.
- $\forall\kappa \geq \omega_2 \neg\Box(\kappa)$.

MM does not imply the existence of a semi-saturated ideal on ω_2. But Woodin shows

Theorem (Woodin)

Suppose $M \vDash V = L(\wp(\mathbb{R})) + \mathsf{AD}_{\mathbb{R}} + \Theta$ is regular. Let $G \subseteq \mathbb{P}_{\max}$ be M-generic and $H \subseteq Add(\Theta^M, 1)$ be $M[G]$-generic, then $M[G][H] \vDash \mathsf{MM(c)}+$ there is a semi-saturated ideal on ω_2.

Consistency Strength

MM(c)'s consistency

Question: Why is the consistency of MM(c) hard to determine?

MM(c)'s consistency

Question: Why is the consistency of MM(c) hard to determine?

Conjecture (Woodin)

Assume MM(c). *Suppose* $M \vDash \mathsf{AD}^+$ *such that* $\mathbb{R} \cup ON \subset M$ *and* $\Theta^M = \omega_3$. *Then* $M \vDash \mathsf{AD}_{\mathbb{R}}$.

MM(c)'s consistency

Question: Why is the consistency of MM(c) hard to determine?

Conjecture (Woodin)

Assume MM(c)*. Suppose* $M \vDash \mathsf{AD}^+$ *such that* $\mathbb{R} \cup ON \subset M$ *and* $\Theta^M = \omega_3$*. Then* $M \vDash \mathsf{AD}_{\mathbb{R}}$.

In the core model induction context, we have this version of Woodin's Conjecture: Suppose MM(c) holds, and that there are no models of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular". Let

$$\Gamma = \{A \subset \mathbb{R} : L(A, \mathbb{R}) \vDash \mathsf{AD}^+\}.$$

Then $\Theta^{L(\Gamma)} < \omega_3$ if $L(\Gamma) \vDash \mathsf{AD}^+$.

MM(c)'s consistency

Question: Why is the consistency of MM(c) hard to determine?

Conjecture (Woodin)

Assume MM(c)*. Suppose* $M \vDash \mathsf{AD}^+$ *such that* $\mathbb{R} \cup ON \subset M$ *and* $\Theta^M = \omega_3$*. Then* $M \vDash \mathsf{AD}_{\mathbb{R}}$.

In the core model induction context, we have this version of Woodin's Conjecture: Suppose MM(c) holds, and that there are no models of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular". Let

$$\Gamma = \{A \subset \mathbb{R} : L(A, \mathbb{R}) \vDash \mathsf{AD}^+\}.$$

Then $\Theta^{L(\Gamma)} < \omega_3$ if $L(\Gamma) \vDash \mathsf{AD}^+$.
Note that $\Gamma \neq \emptyset$ by Steel-Zoble and that MM(c) implies $\mathbb{R}^\sharp$ exists and Steel-Zoble's proof gives $\mathbb{R}^\sharp \in \Gamma$, so $\Theta^{L(\mathbb{R})} < \omega_3$.

MM(c)'s consistency

Question: Why is the consistency of MM(c) hard to determine?

Conjecture (Woodin)

Assume MM(c). *Suppose* $M \models \mathsf{AD}^+$ *such that* $\mathbb{R} \cup ON \subset M$ *and* $\Theta^M = \omega_3$. *Then* $M \models \mathsf{AD}_{\mathbb{R}}$.

In the core model induction context, we have this version of Woodin's Conjecture: Suppose MM(c) holds, and that there are no models of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular". Let

$$\Gamma = \{A \subset \mathbb{R} : L(A, \mathbb{R}) \models \mathsf{AD}^+\}.$$

Then $\Theta^{L(\Gamma)} < \omega_3$ if $L(\Gamma) \models \mathsf{AD}^+$.
Note that $\Gamma \neq \emptyset$ by Steel-Zoble and that MM(c) implies $\mathbb{R}^\sharp$ exists and Steel-Zoble's proof gives $\mathbb{R}^\sharp \in \Gamma$, so $\Theta^{L(\mathbb{R})} < \omega_3$.
If $\Gamma \models \Theta = \theta_0$, then every $A \in \Gamma$ is in an $\mathbb{R}$-mouse , the union of such mice is denoted by Lp($\mathbb{R}$), so $L(\Gamma) \cap \wp(\mathbb{R}) = Lp(\mathbb{R})$.

MM(c)'s consistency

Question: Why is the consistency of MM(c) hard to determine?

Conjecture (Woodin)

Assume MM(c). *Suppose* $M \vDash \mathsf{AD}^+$ *such that* $\mathbb{R} \cup ON \subset M$ *and* $\Theta^M = \omega_3$. *Then* $M \vDash \mathsf{AD}_{\mathbb{R}}$.

In the core model induction context, we have this version of Woodin's Conjecture: Suppose MM(c) holds, and that there are no models of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular". Let

$$\Gamma = \{A \subset \mathbb{R} : L(A, \mathbb{R}) \vDash \mathsf{AD}^+\}.$$

Then $\Theta^{L(\Gamma)} < \omega_3$ if $L(\Gamma) \vDash \mathsf{AD}^+$.
Note that $\Gamma \neq \emptyset$ by Steel-Zoble and that MM(c) implies $\mathbb{R}^\sharp$ exists and Steel-Zoble's proof gives $\mathbb{R}^\sharp \in \Gamma$, so $\Theta^{L(\mathbb{R})} < \omega_3$.
If $\Gamma \vDash \Theta = \theta_0$, then every $A \in \Gamma$ is in an $\mathbb{R}$-mouse , the union of such mice is denoted by Lp($\mathbb{R}$), so $L(\Gamma) \cap \wp(\mathbb{R}) = Lp(\mathbb{R})$.

MM(c)'s consistency (cont.)

We consider the following extensions of MM(c):

1. MM(c)+there is a semi-saturated ideal on ω_2.
2. MM(c) $+ \neg\Box_{\omega_2}$.
3. MM(c) $+ \neg\Box(\omega_3)$.

MM(c)'s consistency (cont.)

We consider the following extensions of MM(c):

1. MM(c)+there is a semi-saturated ideal on ω_2.
2. MM(c) $+ \neg\Box_{\omega_2}$.
3. MM(c) $+ \neg\Box(\omega_3)$.

From MM(c) $+ \neg\Box(\omega_3)$, one can show that $\Theta^{Lp(\mathbb{R})} < \omega_3$. For if not, let $\vec{C}$ be the canonical coherent sequence build over $Lp(\mathbb{R})$. Then $\Box(\omega_3)$ implies there is a thread D through the sequence and D defines a new sound mouse projecting to $\mathbb{R}$. Contradiction.

MM(c)'s consistency (cont.)

We consider the following extensions of MM(c):

1. MM(c)+there is a semi-saturated ideal on ω_2.
2. MM(c) $+ \neg\Box_{\omega_2}$.
3. MM(c) $+ \neg\Box(\omega_3)$.

From MM(c) $+ \neg\Box(\omega_3)$, one can show that $\Theta^{Lp(\mathbb{R})} < \omega_3$. For if not, let $\vec{C}$ be the canonical coherent sequence build over $Lp(\mathbb{R})$. Then $\Box(\omega_3)$ implies there is a thread D through the sequence and D defines a new sound mouse projecting to $\mathbb{R}$. Contradiction.
$\vec{C}$ can in fact be turned into a $\Box_{\omega_2}$-sequence (T.-Zeman), so MM(c) $+ \neg\Box_{\omega_2}$ suffices as well. It turns out theory (1) also suffices.

Theorem (T., 2025)

Let (T) be one of the 3 theories above. Assume $(T) + (\dagger)$. Then there is a model $M \vDash \mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$.

Here $(\dagger)$ is the statement: Whenever A is a set of ordinals that is OD from a countable set of ordinals, for any $X \in \wp_{\omega_1}(A)$ there is a transitive model M of ZFC containing $\{A, X\}$ such that $M \vDash$ "ω_1^V is measurable."

Forcing Fragments of Martin's Maximum over Models of AD^+

Some results

Theorem (Woodin)

Suppose $M \vDash V = L(\wp(\mathbb{R})) + \mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular. Let* $G \subseteq \mathbb{P}_{\max}$ *be* M*-generic and* $H \subseteq Add(\Theta^M, 1)$ *be* $M[G]$*-generic, then* $M[G][H] \vDash \mathsf{MM(c)}+$ *there is a semi-saturated ideal on* ω_2*.*

Some results

Theorem (Woodin)

Suppose $M \vDash V = L(\wp(\mathbb{R})) + \mathsf{AD}_{\mathbb{R}} + \Theta$ is regular. Let $G \subseteq \mathbb{P}_{\max}$ be M-generic and $H \subseteq Add(\Theta^M, 1)$ be $M[G]$-generic, then $M[G][H] \vDash \mathsf{MM(c)}+$ there is a semi-saturated ideal on ω_2.

It is easy to see that $(\dagger)$ holds in $M[G][H]$ and it's very plausible that $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular is consistent with $\mathsf{MM(c)} + (\dagger)+$ there is a semi-saturated ideal on ω_2. In which case, we'd obtain an equiconsistency result.

Some results

Theorem (Woodin)

Suppose $M \vDash V = L(\wp(\mathbb{R})) + \mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular. Let* $G \subseteq \mathbb{P}_{\max}$ *be* M*-generic and* $H \subseteq Add(\Theta^M, 1)$ *be* $M[G]$*-generic, then* $M[G][H] \vDash \mathsf{MM}(\mathsf{c}) +$ *there is a semi-saturated ideal on* ω_2*.*

It is easy to see that $(\dagger)$ holds in $M[G][H]$ and it's very plausible that $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular is consistent with $\mathsf{MM}(\mathsf{c}) + (\dagger) +$ there is a semi-saturated ideal on ω_2. In which case, we'd obtain an equiconsistency result.

Theorem (Caicedo-Larson-Sargsyan-Schindler-Steel-Zeman)

If Con($\mathsf{AD}_{\mathbb{R}} + \Theta$ *is Mahlo), then Con(*$\mathsf{MM}(\mathsf{c}) + \neg\square(\omega_3)$*).*

Some results

Theorem (Woodin)

Suppose $M \vDash V = L(\wp(\mathbb{R})) + \mathsf{AD}_{\mathbb{R}} + \Theta$ is regular. Let $G \subseteq \mathbb{P}_{\max}$ be M-generic and $H \subseteq Add(\Theta^M, 1)$ be $M[G]$-generic, then $M[G][H] \vDash \mathsf{MM(c)}+$ there is a semi-saturated ideal on ω_2.

It is easy to see that $(\dagger)$ holds in $M[G][H]$ and it's very plausible that $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular is consistent with $\mathsf{MM(c)} + (\dagger) +$ there is a semi-saturated ideal on ω_2. In which case, we'd obtain an equiconsistency result.

Theorem (Caicedo-Larson-Sargsyan-Schindler-Steel-Zeman)

If Con($\mathsf{AD}_{\mathbb{R}} + \Theta$ is Mahlo), then Con($\mathsf{MM(c)} + \neg\square(\omega_3)$).

Theorem (Blue-Larson-Sargsyan, Sargsyan)

For any $k < \omega$, the theory $\mathsf{MM(c)} + \forall \mathsf{n} \leq \mathsf{k} \neg\square(\omega_{2+\mathsf{n}})$ is consistent with $\mathsf{ZFC}+$ there is a Woodin limit of Woodin cardinals.

Optimal results

Theorem (Dichotomy Theorem 1, T.-Zeman, 2025)

1. *If* $\mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular holds and the set* $S = \{\alpha = \theta_\alpha : HOD_{\wp_\alpha(\mathbb{R})} \vDash \Theta \text{ is regular}\}$ *is non-stationary, then there is a coherent sequence* $\vec{C} = \{C_\alpha : \alpha < \Theta\}$ *without a thread.*

Optimal results

Theorem (Dichotomy Theorem 1, T.-Zeman, 2025)

1. *If* $\mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular holds and the set* $S = \{\alpha = \theta_\alpha : HOD_{\wp_\alpha(\mathbb{R})} \vDash \Theta \textit{ is regular}\}$ *is non-stationary, then there is a coherent sequence* $\vec{C} = \{C_\alpha : \alpha < \Theta\}$ *without a thread.*
2. *If* $\mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular holds and the set* $\Theta - \{\alpha = \theta_\alpha : HOD_{\wp_\alpha(\mathbb{R}} \vDash \Theta \textit{ is singular}\}$ *is stationary then* $V^{\mathbb{P}_{\max} \star Add(\omega_3,1)} \vDash \mathsf{MM(c)} + \neg\square_{\omega_2}$.

Optimal results

Theorem (Dichotomy Theorem 1, T.-Zeman, 2025)

1. *If* $\mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular holds and the set* $S = \{\alpha = \theta_\alpha : HOD_{\wp_\alpha(\mathbb{R})} \vDash \Theta \text{ is regular}\}$ *is non-stationary, then there is a coherent sequence* $\vec{C} = \{C_\alpha : \alpha < \Theta\}$ *without a thread.*
2. *If* $\mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular holds and the set* $\Theta - \{\alpha = \theta_\alpha : HOD_{\wp_\alpha(\mathbb{R}} \vDash \Theta \text{ is singular}\}$ *is stationary then* $V^{\mathbb{P}_{\max} \star Add(\omega_3,1)} \vDash \mathsf{MM(c)} + \neg\Box_{\omega_2}$.

Theorem (Dichotomy Theorem 2, T.-Zeman, 2025)

1. *If* $\mathsf{AD}_{\mathbb{R}} + \Theta$ *is not weakly compact (i.e. there is a* Θ*-tree* T *on* Θ *without a cofinal branch) then there is a coherent sequence* $\vec{C} = \{C_\alpha : \alpha < \Theta\}$ *that has no thread.*

Optimal results

Theorem (Dichotomy Theorem 1, T.-Zeman, 2025)

1. *If* $\mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular holds and the set* $S = \{\alpha = \theta_\alpha : HOD_{\wp_\alpha(\mathbb{R})} \vDash \Theta \textit{ is regular}\}$ *is non-stationary, then there is a coherent sequence* $\vec{C} = \{C_\alpha : \alpha < \Theta\}$ *without a thread.*
2. *If* $\mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular holds and the set* $\Theta - \{\alpha = \theta_\alpha : HOD_{\wp_\alpha(\mathbb{R}} \vDash \Theta \textit{ is singular}\}$ *is stationary then* $V^{\mathbb{P}_{\max} \star Add(\omega_3,1)} \vDash \mathsf{MM(c)} + \neg\Box_{\omega_2}$.

Theorem (Dichotomy Theorem 2, T.-Zeman, 2025)

1. *If* $\mathsf{AD}_{\mathbb{R}} + \Theta$ *is not weakly compact (i.e. there is a* Θ*-tree* T *on* Θ *without a cofinal branch) then there is a coherent sequence* $\vec{C} = \{C_\alpha : \alpha < \Theta\}$ *that has no thread.*
2. *If* $\mathsf{AD}_{\mathbb{R}} + \Theta$ *is weakly compact then* $V^{\mathbb{P}_{\max} \star Add(\omega_3,1)} \vDash \mathsf{MM(c)} + \neg\Box(\omega_3)$.

Optimal results

Theorem (Dichotomy Theorem 1, T.-Zeman, 2025)

1. *If* $\mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular holds and the set* $S = \{\alpha = \theta_\alpha : HOD_{\wp_\alpha(\mathbb{R})} \models \Theta \textit{ is regular}\}$ *is non-stationary, then there is a coherent sequence* $\vec{C} = \{C_\alpha : \alpha < \Theta\}$ *without a thread.*
2. *If* $\mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular holds and the set* $\Theta - \{\alpha = \theta_\alpha : HOD_{\wp_\alpha(\mathbb{R}} \models \Theta \textit{ is singular}\}$ *is stationary then* $V^{\mathbb{P}_{\max} \star Add(\omega_3,1)} \models \mathsf{MM(c)} + \neg\Box_{\omega_2}$.

Theorem (Dichotomy Theorem 2, T.-Zeman, 2025)

1. *If* $\mathsf{AD}_{\mathbb{R}} + \Theta$ *is not weakly compact (i.e. there is a* Θ*-tree* T *on* Θ *without a cofinal branch) then there is a coherent sequence* $\vec{C} = \{C_\alpha : \alpha < \Theta\}$ *that has no thread.*
2. *If* $\mathsf{AD}_{\mathbb{R}} + \Theta$ *is weakly compact then* $V^{\mathbb{P}_{\max} \star Add(\omega_3,1)} \models \mathsf{MM(c)} + \neg\Box(\omega_3)$.

Towards the equiconsistency

Theorem (Jensen-Schimmerling-Schindler-Steel, 2007)

Assume $2^\omega = \omega_2 + 2^{\omega_2} = \omega_3 + \neg\Box(\omega_3) + \neg\Box_{\omega_3}$. *Then there are non-domestic mice.*

Towards the equiconsistency

Theorem (Jensen-Schimmerling-Schindler-Steel, 2007)

Assume $2^\omega = \omega_2 + 2^{\omega_2} = \omega_3 + \neg\square(\omega_3) + \neg\square_{\omega_3}$. *Then there are non-domestic mice.*

Theorem (T., 2016)

Assume $2^\omega = \omega_2 + 2^{\omega_2} = \omega_3 + \neg\square(\omega_3) + \neg\square(\omega_4)$. *Then there are models of* $\mathrm{AD}_{\mathbb{R}} + \Theta$ *is weakly compact (and more).*

Towards the equiconsistency

Theorem (Jensen-Schimmerling-Schindler-Steel, 2007)

Assume $2^\omega = \omega_2 + 2^{\omega_2} = \omega_3 + \neg\square(\omega_3) + \neg\square_{\omega_3}$. *Then there are non-domestic mice.*

Theorem (T., 2016)

Assume $2^\omega = \omega_2 + 2^{\omega_2} = \omega_3 + \neg\square(\omega_3) + \neg\square(\omega_4)$. *Then there are models of* $\mathsf{AD}_{\mathbb{R}} + \Theta$ *is weakly compact (and more).*

Theorem (T.-Zeman, 2025)

Assume $2^\omega = \omega_2 + 2^{\omega_2} = \omega_3 + \neg\square(\omega_3) + \neg\square_{\omega_3}$. *Then there are models of* $\mathsf{AD}_{\mathbb{R}} + \Theta$ *is weakly compact (and more).*

Towards the equiconsistency

Theorem (Jensen-Schimmerling-Schindler-Steel, 2007)

Assume $2^\omega = \omega_2 + 2^{\omega_2} = \omega_3 + \neg\square(\omega_3) + \neg\square_{\omega_3}$. *Then there are non-domestic mice.*

Theorem (T., 2016)

Assume $2^\omega = \omega_2 + 2^{\omega_2} = \omega_3 + \neg\square(\omega_3) + \neg\square(\omega_4)$. *Then there are models of* $\mathrm{AD}_{\mathbb{R}} + \Theta$ *is weakly compact (and more).*

Theorem (T.-Zeman, 2025)

Assume $2^\omega = \omega_2 + 2^{\omega_2} = \omega_3 + \neg\square(\omega_3) + \neg\square_{\omega_3}$. *Then there are models of* $\mathrm{AD}_{\mathbb{R}} + \Theta$ *is weakly compact (and more).*

Towards the equiconsistency (cont.)

- Assume $\mathsf{MM}(c) + \neg\Box_{\omega_2}$. If $\Theta^\Gamma = \omega_3$ and there are no models of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular holds and the set $\Theta - \{\alpha = \theta_\alpha : HOD_{\wp_\alpha(\mathbb{R}} \vDash \Theta \text{ is singular}\}$ is stationary", we can build a coherent sequence $\vec{C}' = \{C'_\alpha : \alpha < \omega_3\}$ as in Dichotomy Theorem 1, but this sequence can be turned into a $\Box_{\omega_2}$-sequence $\vec{C}$. $\neg\Box_{\omega_2}$ gives a thread through $\vec{C}$. Contradiction.

Towards the equiconsistency (cont.)

- Assume $\mathsf{MM(c)} + \neg\square_{\omega_2}$. If $\Theta^\Gamma = \omega_3$ and there are no models of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular holds and the set $\Theta - \{\alpha = \theta_\alpha : HOD_{\wp_\alpha(\mathbb{R}} \vDash \Theta \text{ is singular}\}$ is stationary", we can build a coherent sequence $\vec{C}' = \{C'_\alpha : \alpha < \omega_3\}$ as in Dichotomy Theorem 1, but this sequence can be turned into a $\square_{\omega_2}$-sequence $\vec{C}$. $\neg\square_{\omega_2}$ gives a thread through $\vec{C}$. Contradiction.
- Assume $\mathsf{MM(c)} + \neg\square(\omega_3)$. If $\Theta^\Gamma = \omega_3$ and there are no models of $\mathsf{AD}_{\mathbb{R}} + \Theta$ is weakly compact, then by Dichotomy Theorem 2, we can build a coherent sequence a coherent sequence $\vec{C} = \{C_\alpha : \alpha < \omega_3\}$ from a ω_3-tree T that has no cofinal branch in $Lp^\Sigma(\Gamma)$ (**this can't in general be turned into a $\square_{\omega_2}$-sequence**). $\neg\square(\omega_3)$ gives a thread through $\vec{C}$ which witnesses T has a cofinal branch in $Lp^\Sigma(\Gamma)$. Contradiction.

Towards the equiconsistency (cont.)

- Assume $\mathrm{MM}(c) + \neg\square_{\omega_2}$. If $\Theta^\Gamma = \omega_3$ and there are no models of "$\mathrm{AD}_{\mathbb{R}} + \Theta$ is regular holds and the set $\Theta - \{\alpha = \theta_\alpha : HOD_{\wp_\alpha(\mathbb{R}} \vDash \Theta \text{ is singular}\}$ is stationary", we can build a coherent sequence $\vec{C}' = \{C'_\alpha : \alpha < \omega_3\}$ as in Dichotomy Theorem 1, but this sequence can be turned into a $\square_{\omega_2}$-sequence $\vec{C}$. $\neg\square_{\omega_2}$ gives a thread through $\vec{C}$. Contradiction.
- Assume $\mathrm{MM}(c) + \neg\square(\omega_3)$. If $\Theta^\Gamma = \omega_3$ and there are no models of $\mathrm{AD}_{\mathbb{R}} + \Theta$ is weakly compact, then by Dichotomy Theorem 2, we can build a coherent sequence a coherent sequence $\vec{C} = \{C_\alpha : \alpha < \omega_3\}$ from a ω_3-tree T that has no cofinal branch in $Lp^\Sigma(\Gamma)$ (**this can't in general be turned into a $\square_{\omega_2}$-sequence**). $\neg\square(\omega_3)$ gives a thread through $\vec{C}$ which witnesses T has a cofinal branch in $Lp^\Sigma(\Gamma)$. Contradiction.
- If $\Theta^\Gamma < \omega_3$, one needs to show either hypothesis (maybe $+(\dagger)$) implies the existence of models of "$\mathrm{AD}_{\mathbb{R}} + \Theta$ is measurable".

Towards the equiconsistency (cont.)

- Assume $\mathsf{MM(c)} + \neg\square_{\omega_2}$. If $\Theta^\Gamma = \omega_3$ and there are no models of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular holds and the set $\Theta - \{\alpha = \theta_\alpha : HOD_{\wp_\alpha(\mathbb{R}} \vDash \Theta \text{ is singular}\}$ is stationary", we can build a coherent sequence $\vec{C}' = \{C'_\alpha : \alpha < \omega_3\}$ as in Dichotomy Theorem 1, but this sequence can be turned into a $\square_{\omega_2}$-sequence $\vec{C}$. $\neg\square_{\omega_2}$ gives a thread through $\vec{C}$. Contradiction.
- Assume $\mathsf{MM(c)} + \neg\square(\omega_3)$. If $\Theta^\Gamma = \omega_3$ and there are no models of $\mathsf{AD}_{\mathbb{R}} + \Theta$ is weakly compact, then by Dichotomy Theorem 2, we can build a coherent sequence a coherent sequence $\vec{C} = \{C_\alpha : \alpha < \omega_3\}$ from a ω_3-tree T that has no cofinal branch in $Lp^\Sigma(\Gamma)$ (**this can't in general be turned into a $\square_{\omega_2}$-sequence**). $\neg\square(\omega_3)$ gives a thread through $\vec{C}$ which witnesses T has a cofinal branch in $Lp^\Sigma(\Gamma)$. Contradiction.
- If $\Theta^\Gamma < \omega_3$, one needs to show either hypothesis (maybe $+(\dagger)$) implies the existence of models of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is measurable".

Questions

Conjectures and Questions

Conjecture

The theories "$\mathsf{MM}(\mathfrak{c})+$*there is a semi-saturated ideal on* ω_2 *(+ (†))" and "*$\mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular" are equiconsistent.*

Conjectures and Questions

Conjecture

The theories "$\mathsf{MM}(\mathsf{c})$+*there is a semi-saturated ideal on* ω_2 *(+* $(\dagger)$*)" and "*$\mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular" are equiconsistent.*

Question

- *Can we make do without* $(\dagger)$*?*
- *Can we weaken the hypothesis "semi-saturation" to "weak-presaturation of* $\mathcal{I}_{NS}$*"?*

Conjectures and Questions

Conjecture

*The theories "*MM(c)+*there is a semi-saturated ideal on* ω_2 *(+ (†))" and "*$\mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular" are equiconsistent.*

Question

- *Can we make do without (†)?*
- *Can we weaken the hypothesis "semi-saturation" to "weak-presaturation of* $\mathcal{I}_{NS}$*"?*

Conjecture

The theories "$\mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular holds and the set* $\Theta - \{\alpha = \theta_\alpha : HOD_{\wp_\alpha(\mathbb{R}} \vDash \Theta \textit{ is singular}\}$ *is stationary " and "*MM(c) $+ \neg\square_{\omega_2}$ *(+ (†))" are equiconsistent.*

Conjectures and Questions

Conjecture

The theories "MM(c)+there is a semi-saturated ideal on ω_2 $(+\ (\dagger))$" and "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular" are equiconsistent.

Question

- *Can we make do without $(\dagger)$?*
- *Can we weaken the hypothesis "semi-saturation" to "weak-presaturation of $\mathcal{J}_{NS}$"?*

Conjecture

The theories "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular holds and the set $\Theta - \{\alpha = \theta_\alpha : HOD_{\wp_\alpha(\mathbb{R}} \vDash \Theta \textit{ is singular}\}$ is stationary " and "MM(c) $+ \neg\square_{\omega_2}$ $(+\ (\dagger))$" are equiconsistent.

Conjecture

The theories "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is weakly compact" and "MM(c) $+ \neg\square(\omega_3)$ $(+\ (\dagger))$" are equiconsistent.

Thank you!