Degree structures in canonical inner models

Xianghui Shi

Beijing Normal University

Inner Model Theory Conference
UC Berkeley, June 25, 2025

Contents

1. Classical Recursion Theory
   - Turing degrees
   - Structure of the Turing degrees
   - Generalizations of Turing degrees
2. Higher Degree Theory

Turing degrees

- Classical Recursion theory studies the structure of Turing degrees, induced by the notion of Turing reducebility.
- For $A, B \subseteq \omega$,
  $A <_T B$ if A, $\omega \backslash A$ are both recursively computable from B.

  $<_T$ induces an equivalence relation – Turing degrees, $\boldsymbol{a} = [a]_T$

  Turing degrees $= \Delta^0_1$-degrees.

Structure of the Turing degrees

Here are some properties of $(\mathscr{D}, <_T)$.

- $(\mathscr{D}, <_T)$ has height $\aleph_1$.
- $\forall \boldsymbol{a} \neq \mathbf{0}$, $\exists \boldsymbol{b}\,(\boldsymbol{a}, \boldsymbol{b}$ are incomparable).
  $\Longrightarrow <_T$ is not a linear order.
  In fact, there are $2^{\aleph_0}$ many pairwise incomparable Turing degrees, therefore the width of $(\mathscr{D}, <_T) = 2^{\aleph_0}$.
- There are minimal degrees.[1]
  $\Longrightarrow <_T$ is not dense.
- Every $\boldsymbol{a}, \boldsymbol{b}$ have the least upper bound, $\boldsymbol{a} \vee \boldsymbol{b}$. But there are pairs of degrees with no greatest lower bound.
  Thus $(\mathscr{D}, <_T)$ is only an upper semi-lattice.
- Every countable partially ordered set can be embedded into $(\mathscr{D}, <_T)$.

[1] A degree $\boldsymbol{a}$ is minimal (w.r.t. $\mathbf{0}$) if $\boldsymbol{a} > \mathbf{0}$ and $\neg\exists \boldsymbol{b}\,(\mathbf{0} <_T \boldsymbol{b} <_T \boldsymbol{a})$.

For $X \subset \omega$, $X' := \{e \mid \{e\}^X(e)\downarrow\}$ is called the Turing jump of X.

Properties involving the jump:

- For any $\boldsymbol{a} \in \mathscr{D}$, there exists $\boldsymbol{b} \in \mathscr{D}$ s.t. $\boldsymbol{a} < \boldsymbol{b}$ and $\boldsymbol{b}' = \boldsymbol{a}'$.
- There is a Post (1944) sequence: $\langle \boldsymbol{a}_i : i < \omega \rangle$ s.t. $\boldsymbol{a}'_{i+1} \leqslant \boldsymbol{a}_i$, $\forall i$.
- (Friedberg, 1957). $\forall \boldsymbol{a} \geqslant \mathbf{0}'$, $\exists \boldsymbol{b}$ s.t. $\boldsymbol{a} = \boldsymbol{b}'$.
- (Posner-Robinson, 1981). $\forall \boldsymbol{a}$, $\boldsymbol{a} \neq \mathbf{0}$ iff $\exists \mathbf{g}$ s.t. $\boldsymbol{a} \vee \boldsymbol{g} = \boldsymbol{g}'$.
- (Shore-Slaman, 1999). The jump operator is 1^{st}-order definable in the structure $(\mathscr{D}, <_T)$.
- (Simpson, 1977). The 1^{st}-order theory of $(\mathscr{D}, <_T)$ is "equivalent" to the theory of 2^{nd}-order arithmetic.

Generalizations of Turing degrees

- Turing reduction is the simplest form of definability reduction. The study of degree notions is naturally extended to higher degrees of definability reduction, such as
    - arithmetic degrees,
    - hyperarithmetic degrees,
    - constructible degrees,
    - degrees induced by inner model operators, etc.
- One can also extend the notion of Turing degree on subsets of ω to subsets of larger ordinals. This is so called α-recursion theory. However, this study was mainly conducted inside L.

Contents

1. Classical Recursion Theory
2. Higher Degree Theory
   - Degree theoretic questions
   - In L, $L[\mu]$
   - In $L[\bar{\mu}]$
   - In $L[\mathcal{E}]$
   - Degree structures under I_0

Higher Degree Theory

- Classical recursion theory has been extended/generalized to
  - higher levels of computability (e.g. hyperarithmetic degrees)
  - higher ordinals/cardinals (e.g. α-degrees)
  - both directions (e.g. generalized hyperarithmetic α-degrees)

  However these do not go beyond ZFC.
- Around 2010, I began to study generalized degree structures at uncountable cardinals, particularly at strong limit singular cardinals of countable cofinality, within canonical models for large cardinals.
- New focus is on the connection between
  - the complexity of the degree structures and
  - the large cardinal strength of the canonical inner models in which they reside.

Generalize Turing degree

- A is Turing reducible to B, i.e. $A \leqslant_T B$ iff $A \in \Delta^0_1(B)$, where $\Delta^0_1(B)$ is the collection of subsets of ω that is Δ^0_1-definable over the structure $(\omega, \in, B)$.
- A is hyperarithmetically reducible to B, i.e. $A \leqslant_h B$ iff $A \in \Delta^1_1(B)$, where $\Delta^1_1(B)$ is the collection of subsets of ω that is closed under recursive operators. By Kleene, $\Delta^1_1(B) = L_{\omega_1^B}[B] \cap \mathscr{P}(\omega)$.

In general, one can define

Let Γ be a reasonable theory in the language of set theory, and $A, B \subset \omega$. A is Γ-reducible to B if $A \in M[B] \cap \mathscr{P}(\omega)$, where $M[B]$ is the smallest model of Γ that contains $\{B\}$.

Γ-degrees

Let $\Gamma = \mathsf{ZF}^{-\varepsilon}$, a fragment of ZF sufficient for the core model construction.

Definition

Suppose λ is a limit of strongly inaccessible cardinals. Fix a well-ordering $w : H(\lambda) \to \lambda$. For $a, b \subset \lambda$:

- $M[a]$ denotes the minimal Γ-model of the form $L_\alpha[w][a]$, $\alpha > \lambda$. Let α_a, Γ-ordinal for a, denote the height of $M[a]$.
- $a \leqslant_\Gamma b$ if $M[a] \subseteq M[b]$. $a \equiv_\Gamma b$ if $a \leqslant_\Gamma b$ and $b \leqslant_\Gamma a$
- The degree of a, $[a]_\Gamma$ or $\boldsymbol{a}$, is the set $\{b \subset \lambda \mid b \equiv_\Gamma a\}$.
- $J_\Gamma(a)$, Γ-jump of G, is the theory of $M[a]$. It can be coded by a subset of λ.

Degree structures in forcing extensions

Not very much of degree structures at uncountable cardinals can be determined by ZFC alone, even with large cardinals.

Example

- Assume $\mathrm{ZFC} + \mathrm{GCH}$ and a measurable cardinal κ of Mitchell order $o(\kappa) = \kappa^{++}$ plus a measurable cardinal $\kappa' > \kappa$.
- With a small forcing, one can arrange that in $V[G]$: $\kappa = \aleph_\omega$, GCH remains true below $\aleph_\omega$, $2^{\aleph_\omega} = \aleph_{\omega+2}$ while the measurability of κ' is preserved (This combines results of Woodin and Gitik).
- In $V[G]$: $\mathscr{D}^{\Gamma}_{\aleph_\omega}$ can't be well ordered, as every degree has only $\aleph_\omega$ many predecessors in the degree partial ordering.

Study degrees in canonical models

On the other hand,

- fine structure models provide very complete settings for answering most questions.
- In these models, objects are constructed in a well organized manner – the rigidity of these models clears out many structural chaos, and makes the impacts of large cardinal axioms on the structure of degrees somewhat more transparent.

Degrees at regular cardinals

Consider degrees at λ inside an L-like models.

- $\mathrm{cf}(\lambda) = \lambda$, i.e. λ is regular. *Not very interesting*.

  Most degree theoretic constructions at ω can be generalized to such λ.

[The main techniques used in recursion theory are priority, forcing, and injury arguments. Most of the combinatorics of ω needed for these techniques to run at regular λ can be obtained by assuming $\lambda^{<\lambda} = \lambda$, which follows from GCH.]

At singular cardinals with uncountable cofinality

- $\mathrm{cf}(\lambda) > \omega$, e.g. $\lambda = \aleph_{\omega_1}$. *Nothing interesting left.*

Theorem (Sy Friedman, 81) ($V = L$)

The $\aleph_{\omega_1}$-degrees are well-ordered, and the successor of $A \subset \aleph_{\omega_1}$ is the $\aleph_{\omega_1}$-jump of A.

Sy Friedman's argument is to analyze *stationary subsets* of $\mathrm{cf}(\lambda)$. His argument works in all $L[E]$-like inner models for most definablity degree notions (that are coarser than α-degree) at any singular cardinals of uncountable cofinality.

Theorem (Sy Friedman) ($V =$ pure extender model)

The generalized degrees at singular cardinals of uncountable cofinality are well-ordered above some degree/on a cone.

Pictures in L

- $\mathrm{cf}(\lambda) = \omega$, e.g. $\lambda = \aleph_\omega$. *Where the fun is.*

Theorem (Woodin) ($V = L$)

If $\mathrm{cf}(\lambda) = \omega$, then Γ-degrees at λ are well-ordered on a cone. In particular, Γ-degrees at $\aleph_\omega$ is well-ordered.

SKETCH. Use Covering Lemma. Every minimal Γ-model is an initial segment of L, therefore the Γ-degrees at λ are well-ordered via the Γ-ordinals.

Covering for L

A key ingredient of the argument is

Covering Lemma for L. (Jensen, 74)

Assume $\neg\exists 0^\sharp$. Then every set $x \subset \mathrm{Ord}$ is covered by a $y \in L$, with $|y| = |x| + \omega_1$.

Pictures in canonical models below one measurable

For inner models below one measurable cardinal, the same picture follows from the Covering Lemmas for K^{DJ}:

Γ-degrees at every singular cardinal are wellordered on a cone.

There is a little wrinkle in canonical models for finitely many measurable cardinals – the covering lemma for $L[\mu]$ have different format.

Degrees in $L[\langle \mu_i : i < n \rangle]$

Covering Lemma for one measurable. (Dodd-Jensen, 82)

Assume $\neg\exists 0^\dagger$, but there is an inner model $L[\mu]$. Let $\kappa = \mathrm{crit}(\mu)$. Then for every set $x \subset \mathrm{Ord}$, one of the following holds:

1. Every set $x \subset \mathrm{Ord}$ is covered by a $y \in L[\mu]$, with $|y| = |x| + \omega_1$.
2. $\exists C$, Prikry generic over $L[\mu]$, s.t. every set $x \subset \mathrm{Ord}$ is covered by a $y \in L[\mu][C]$, with $|y| = |x| + \omega_1$.
   Such C is unique up to finite difference.

But this doesn't affect the degree structures at singular cardinals. In canonical models for finitely many measurables, there is only one type of degree structures:

At every singular cardinal of countable cofinality, Γ-degrees are well-ordered on a cone, with $J_\Gamma(A)$ being the successor of A for every A.

Structural properties of almost wellordered Γ-degrees

Corollary (S.)

Assume there is no inner model with infinitely many measurables. The following properties hold on a cone of Γ-degrees at λ, where λ is a strong limit singular cardinal of countable cofinality.

1. There is no infinite descending chains of degrees.
2. There is no incomparable pair of degrees.
3. Both minimal pair property and exact pair property fail.
4. Every degree has exactly one minimal degree.
5. Posner-Robinson fails, i.e. there are $\geqslant\lambda^+$ many subsets of λ which has no solution to the Posner-Robinson equation.
6. Friedberg jump inversion[a] fails.

[a] Which says $\forall \boldsymbol{a}\, \forall \boldsymbol{b} \geqslant J_\Gamma(\boldsymbol{a})\, \exists \boldsymbol{c}$ such that $\boldsymbol{b} = J_\Gamma(\boldsymbol{c})$.

New degree structure appears in $L[\bar{\mu}]$

Theorem (S., 2015)

Assume $V = L[\bar{\mu}]$, where $\bar{\mu} = \langle \mu_n : n < \omega \rangle$, each μ_n is a measure on κ_n.[a] *Let $\kappa_\omega = \sup_n \kappa_n$. Suppose λ is a singular cardinal of countable cofinality.*

1. *If $\lambda \neq \kappa_\omega$, then $(\mathscr{D}^\Gamma_\lambda, <_\Gamma)$ is wellordered on a cone.*
2. *If $\lambda = \kappa_\omega$, consider $(\mathscr{D}^\Gamma_\lambda(\geqslant \bar{\mu}), <_\Gamma)$, Γ-degrees above the degree of $\bar{\mu}$.*
   - *$(\mathscr{D}^\Gamma_\lambda(\geqslant \bar{\mu}), <_\Gamma)$ is prewellordered via their Γ-ordinals.*
   - *Let A_η be a subset of λ that codes the first η many Γ-ordinals, and $\mathcal{C}_\eta$ be the set of $L_{\alpha_\eta}[\bar{\mu}]$-generic Prikry systems for $\mathbb{P}_{\bar{\mu}}$. Then Γ-degrees (above the degree of $\bar{\mu}$) whose Γ-ordinals equal to the η-th Γ-ordinal are exactly the degrees given by*

$$A_\eta \oplus \mathcal{C}_\eta = \{A_\eta \vee C \mid C \in \mathcal{C}_\eta \cup \{\varnothing\}\}.$$

[a] One can replace the assumption by the corresponding anti-large cardinal assumption.

Pictures in $L[\bar{\mu}]$

Corollary (S., 2015)

Assume $V = L[\bar{\mu}]$ and $\lambda = \sup_n \kappa_n$ et al as in the previous theorem. The following properties hold on a cone in $(\mathscr{D}_\lambda^\Gamma, <_\Gamma)$:

1. There are incomparable degrees.[a] (2015)
2. There are infinite descending chains of degrees.
3. There is no Post sequence.
4. Posner-Robinson fails.
5. Friedberg jump inversion fails.

[a] According to Sy Friedman (1981), Solovay showed that there are incomparable $\aleph_\omega$-degrees.

It is not known whether each of (3)-(6) follows from the corresponding anti-large cardinal assumption.

New structural properties

Using Prikry-type forcing to simulate priority arguments, one gets

Theorem 1 (S.)

Assume $\lambda = \sup_n \kappa_n$, each κ_n is measurable. The following hold on a cone in $(\mathscr{D}^\Gamma_\lambda, <_\Gamma)$.

1. *(Zhang, 2023) There is an anti-chain of size 2^λ.*
2. *There is an independent set $\mathcal{A}$ of size 2^λ in $(\mathscr{D}^\Gamma_\lambda, <_\Gamma)$.*[a]
3. *Every degree has a minimal pair, i.e. $\forall \boldsymbol{a}\, \exists \boldsymbol{b}, \boldsymbol{c}$ such that for any $\boldsymbol{d} \leqslant \boldsymbol{b}$ and $\boldsymbol{d} \leqslant \boldsymbol{c} \implies \boldsymbol{d} \leqslant \boldsymbol{a}$.*
4. *For any increasing ω-sequence of degrees $\langle c_i \mid i < \omega \rangle$, there is an exact pair.*

[a] $\mathcal{A}$ is independent if for every $\boldsymbol{a} \in \mathcal{A}$, there is no finite $F \subset \mathcal{A}$ such that $\boldsymbol{a} \leqslant_\Gamma \bigvee F$.

Not known if one can replace ω by λ in the exact pair result.

Proof sketch

- Take an $A \geqslant_\Gamma \bar{\mu}$. Force over M_A with the tree diagonal Prikry forcing $\mathbb{P}_{\bar{\mu}}$
- Enumerate the functionals in M_A:

$$\mathcal{W}_i = \{\Phi_{i,j} \mid j < \kappa_{i-1}\},\ i > 0 \text{ and } \mathcal{W} = \bigcup_i \mathcal{W}_i.$$

  Construct a uniform $\bar{\mu}$-splitting tree T such that

$$\forall f, g \in [T],\ \forall \Phi \in \mathcal{W},\ f \neq \Phi(g) \wedge g \neq \Phi(f)$$

  - Starting with the full tree, build T in ω steps $T = \bigcap_n T_n$.
  - At stage n, for each node $s \in \mathrm{Lev}_n(T_n)$, get direct extension $(T_n)'_s \leqslant^* (T_n)_s$ such that $(T_n)'_s$ decides/computes $\Phi(f) \cap \kappa_n$ for all $\Phi \in \mathcal{W}_{n-1}$ and $f \in [(T_n)_s]$.
  - Shrink further, get $(T_n)''_s \leqslant^* (T_n)'_s$, $s \in \mathrm{Lev}_n(T_n)$, to ensure that for $s \neq t \in \mathrm{Lev}(T_n)$, for every $f \in [(T_n)''_s]$ and $g \in [(T_n)''_t]$, $\Phi(f) \cap \kappa_n \neq g \cap \kappa_n$ for all $\Phi \in \mathcal{W}_{n-1}$.

Consistency strength

By the theory of short core models (Koepke), the assumption that "there is an ω-sequence of measurable cardinals" is optimal for the "new" structure properties.

Corollary 2 (S.)

The cone-version of each of the following statements holding at some/any strong limit cardinals of cofinality ω implies that there is an inner model with an ω-sequence of measurable cardinals.

1. *There are incomparable Γ-degrees.*
2. *Every Γ-degree has a minimal pair.*
3. *There is an anti-chain of size 2^λ in $(\mathscr{D}^\Gamma_\lambda, <_\Gamma)$.*
4. *There are infinite descending chains of Γ-degrees.*
5. *There is an independent set $\mathcal{A}$ of size 2^λ in $(\mathscr{D}^\Gamma_\lambda, <_\Gamma)$.*
6. *For any $<_\Gamma$-increasing ω-sequence of degrees $\langle c_i \mid i < \omega\rangle$, there is an exact pair.*

A similar situation in GDST

Let $\Psi_{\mathrm{PSP}}(\lambda)$ denote the following statement:

> *For every $D \subset \mathcal{P}(\lambda)$ that is definable by a Σ_1-formula with parameters in $H(\lambda) \cup \{\lambda\}$ and of cardinality $>\lambda$, there is a perfect embedding $\pi : \lambda^{\mathrm{cof}(\lambda)} \to \mathcal{P}(\lambda)$ with $\mathrm{ran}(\pi) \subset D$.*

Theorem (Lücke-Müller, 2023)

Let λ be a singular strong limit cardinal.

1. *If λ is an $\mathrm{cof}(\lambda)$-limit of measurables, then $\Psi_{\mathrm{PSP}}(\lambda)$ holds.*
2. *If $\Psi_{\mathrm{PSP}}(\lambda)$ holds, then there is an inner model with an $\mathrm{cof}(\lambda)$-sequence of measurables.*

Picture in $L[\mathcal{U}]$

Move on to higher canonical models.

Theorem (Yang, 2013)

Assume $\langle \kappa_n : n < \omega \rangle$ is a sequence of measurable cardinals such that $\sup_n o(\kappa_n) = \sup_n \kappa_n$. Let $\lambda = \sup_n \kappa_n$. Then there is Prikry-type forcing $\mathbb{P}$ such that every $\mathbb{P}$-generic sequence has minimal Γ-degree with respect to the ground model.

A corollary of the proof gives that

Corollary 3 (S.)

Assume the setting as above. There is a cone $C \subset \mathscr{D}_\lambda^\Gamma$ such that every degree in C has 2^λ many minimal degrees, and hence 2^λ many pairwise incomparable degrees.

Picture in $L[\mathcal{U}]$

Theorem 4 (S.)

Assume $V = L[\mathcal{U}]$, where $\mathcal{U}$ is a coherent sequences of measures witnessing the condition on the previous slide. The following properties hold on a cone in $(\mathscr{D}^{\Gamma}_{\lambda}, <_{\Gamma})$:

1. $(\mathscr{D}^{\Gamma}_{\lambda}, <_{\Gamma})$ *is prewellordered via their Γ-ordinals.*
2. *Incomparable degrees, minimal pair and infinite descending chains of Γ-degrees remain true.*
3. *There is no Post sequence at λ.*
4. *Posner-Robinson fails at λ.*
5. *Friedberg jump inversion fails at λ.*

Again, it's not clear whether each of (3)-(5) follows from the corresponding anti-large cardinal assumption.

Test question for $L[\mathcal{E}]$

- The results in the previous slides rely heavily on the full Covering Lemma. Once pass one strong, only weak covering is available. This is the first challenge, if one wants to go further up.
- Second challenge: Unlike in $L[\bar{\mu}]$-models, $L[\mathcal{E}]$ allows partial extenders in the sequence.

Test question (Woodin)

Assume there is no proper class inner model with one Woodin cardinal. In $L[\mathcal{E}]$, are the generalized degrees at $\aleph_\omega$ wellordered on a cone?

Answer: Yes.

Picture in $L[\mathcal{E}]$

Theorem 5 (Schindler-S.)

Assume that no transitive model of Γ has an inner model with a Woodin cardinal. Let $L[\mathcal{E}]$ be a fully iterable pure extender model. Let λ be a singular cardinal of $L[\mathcal{E}]$.

1. *If λ is not an ω-limit of measurable cardinals in $L[\mathcal{E}]$, then*
$$L[\mathcal{E}] \models \text{“}\Gamma\text{-degrees at } \lambda \text{ are well ordered on a cone”}.$$
2. *If λ is an ω-limit of measurable cardinals in $L[\mathcal{E}]$, then*
$$L[\mathcal{E}] \models \text{“}\forall \boldsymbol{a}\, \exists \text{ incomparable } \Gamma\text{-degrees } \geqslant_\Gamma \boldsymbol{a}\text{”}.$$

Thanks to Mitchell-Schimmerling's paper (2023).

PROOF SKETCH.

- Fix an A s.t. $L[\mathcal{E}]||\lambda \in M_A$, $K^{M_A} \models$ "$\mathrm{cf}(\lambda) = \mathrm{cf}^V(\lambda) < \lambda$".
- Work in M_A. Let $\pi : H \prec M_A|\theta$, θ large, $\mathrm{ran}(\pi) \supset X$, H transitive, ${}^\omega H \subset H$ and $|H| = |X|^{\aleph_0} < \lambda$.
- Take $\overline{K} = \pi^{-1}(K^{M_A}||\lambda)$, iterate it against K^{M_A}, $\overline{K} \trianglelefteq M_\infty^{\mathcal{T}}$
- If the generators of the extenders used on the main branch of $\mathcal{T}$ are bounded in $\bar{\lambda} = \pi^{-1}(\lambda)$, the standard covering argument gives a small covering set. This leads to clause (1).
- Assume otherwise. Let $\alpha \in [0, \infty)_{\mathcal{T}}$ be such that $\bar{\lambda} \in \mathrm{ran}(\pi^{\mathcal{T}}_{\alpha,\infty})$ (no drop). Let $\lambda' = {\pi^{\mathcal{T}}_{\alpha\infty}}^{-1}(\bar{\lambda})$. For $\beta \in [\alpha, \infty)_{\mathcal{T}}$, $\mathrm{crit}(\pi^{\mathcal{T}}_{\beta\infty}) \leqslant \pi^{\mathcal{T}}_{\alpha\beta}(\lambda')$. Two cases:
    - Unboundely often $\mathrm{crit}(\pi^{\mathcal{T}}_{\beta\infty}) < \pi^{\mathcal{T}}_{\alpha\beta}(\lambda')$.
      $\Rightarrow \exists\langle\beta_n\rangle_n$ s.t. $\lambda = \sup_n \pi(\kappa_n)$, where $\kappa_n = \mathrm{crit}(\pi_{\beta_n\infty})$
    - From some point on, $\mathrm{crit}(\pi^{\mathcal{T}}_{\beta\infty}) = \pi^{\mathcal{T}}_{\alpha\beta}(\lambda')$.
      $\Rightarrow \exists\langle\beta_n\rangle_n$ s.t. $\lambda = \sup_n \pi(\kappa_n)$ and $\langle\pi(\kappa_n)\rangle_n$ is Prikry generic over K^{M_A}, thus λ is measurable in K^{M_A}.

More properties of this feature

Several other structural properties can also be used to characterize ω-limit of measurables in $L[\mathcal{E}]$.

Theorem 6 (S.)

The same holds if one replaces the existence of incomparable degree in the second clause by one of the following statements:

- *Every Γ-degree has a minimal pair.*
- *the existence of anti-chain of size 2^λ.*
- *the existence of exact pair (for ω-sequence of increaseing degrees).*
- *the existence of independent set of size 2^λ.*

Degrees outside the cone

- At λ which is singular but not ω-limit of measurable cardinals, incomparable degrees may appear outside of the cone (on which the degrees are wellordered).

Example (Schindler-S.)

In a fully iterable model $L[\mathcal{E}]$ with two measurable cardinals, there is a singular cardinal (below the first measurable) at which there are incomparable Γ-degrees.

Other negative results

Theorem 7 (S.)

Assume as in the previous slide. In $L[\mathcal{E}]$,

1. *There is no Post sequence at λ.*
2. *Posner-Robinson fails at λ.*
3. *Friedberg jump inversion fails at λ.*

Minimal degrees in $L[\mathcal{E}]$

Theorem 8 (Martin-S.)

Assume as in the previous slide. Then If λ is an ω-limit of measurable cardinals $\langle \kappa_n \mid n < \omega \rangle$ satisfying the condition

$$\sup_n \kappa_n = \sup_n o(\kappa_n),\ \forall n$$

then

$$L[\mathcal{E}] \models \text{“}\{\boldsymbol{a} \mid \exists \textit{ minimal degree } \geqslant_\Gamma \boldsymbol{a}\} \textit{ contains a cone”}.$$

Conjecture (Martin-S.)

The failure of the statement occurs at λ such that $\lambda = \sup_n \kappa_n$, each κ_n is measurable, but their Mitchell ranks $o(\kappa_n)$, are bounded in λ.

Picture from the top, I_0

Definition

$I_0(\lambda)$ is the following assertion: There exists an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ such that $\mathrm{crit}(j) < \lambda$.

Theorem (S, 2015)

Assume ZFC $+ I_0(\lambda)$. *Posner-Robinson holds in* $L(V_{\lambda+1})$, *i.e. for co-*λ *many* $X \subset \lambda$,

$$(\exists G \subset \lambda)\,[\boldsymbol{x} \vee \boldsymbol{G} \equiv_{\mathsf{Z}} \boldsymbol{J_\Gamma(G)}].$$

Remarks

- Our proof of Posner-Robinson under $I_0(\lambda)$ uses a perfect set theorem for projective (Σ^1_5) subsets of $\mathcal{P}(\lambda)$.

Question

What is the consistency strength of having the λ-perfect set property for projective subsets of $\mathcal{P}(\lambda)$?

Guess: at least ω many Woodins

Pictures in I_0 models

Corollary 9 (S.)

Assume ZFC $+ I_0(\lambda)$. *In* $(\mathscr{D}^{\Gamma}_{\lambda}, <_{\Gamma})$, *the following statements hold on a cone,*

1. *Every degree has a minimal pair, which are also incomparable.*
2. *Every increasing ω-sequence of degrees has exact pair.*
3. *There are infinite descending sequence of degrees.*
4. *Every degree has λ-perfect set many minimal degrees.*

Conjecture

Assume ZFC $+ I_0(\lambda)$. In $(\mathscr{D}^{\Gamma}_{\lambda}, <_{\Gamma})$, the following statements hold,

1. There is no Post sequences of degrees (of length ω).
2. Friedberg Jump inversion fails at λ.

Thank you!