Mouse sets in $L(\mathbb{R})$

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1. Mouse sets in $L(\mathbb{R})$
2. Gaps
3. Ladder mice
4. Admissible gaps
5. Ladder mice just beyond admissible gaps

Definition 1.1.

The $\underline{L(\mathbb{R})\text{ language}}$ is language of set theory augmented with a constant $\dot{\mathbb{R}}$ for $\mathbb{R}$.

$\Sigma_n^{\mathcal{J}_\alpha(\mathbb{R})}$ and $\Pi_n^{\mathcal{J}_\alpha(\mathbb{R})}$ always in $L(\mathbb{R})$ language.

Definition 1.2 ($\Sigma_n^{\mathbb{R}}$ hierarchy).

- $\Sigma_1^{\mathbb{R}}$ denotes Σ_1,

For integers $n > 0$:

- $\Pi_n^{\mathbb{R}}$ denotes $\neg\Sigma_n^{\mathbb{R}}$, so $\Pi_1^{\mathbb{R}} = \Pi_1$
- $\Sigma_{n+1}^{\mathbb{R}}$ denotes $\exists^{\mathbb{R}}\Pi_n^{\mathbb{R}}$.

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Definition 1.3.

Let $\alpha > 0$ be an ordinal, and $n > 0$ an integer.

$\mathrm{OD}_{\alpha n}$ denotes the set of $y \in \mathbb{R}$ such that for some $\xi < \omega_1$ and some Σ_n formula φ,

$$y = \text{ unique real } z \text{ such that } \mathcal{J}_\alpha(\mathbb{R}) \models \varphi(w, z),$$

whenever $w \in \mathrm{WO}_\xi$.

Likewise $\mathrm{OD}_{\alpha n}^{\mathbb{R}}$, but with $\Sigma_n^{\mathbb{R}}$ replacing Σ_n.

So $\mathrm{OD}_{\alpha n}^{\mathbb{R}} \subseteq \mathrm{OD}_{\alpha n}$.

In case $\alpha = 1$, $\mathcal{J}_1(\mathbb{R}) = \mathcal{J}(\mathbb{R})$.

Remark 1.4.

For $n \geq 1$, $(\Sigma_n^{\mathbb{R}})^{\mathcal{J}(\mathbb{R})}$ is recursively equivalent to $\Sigma_n^{\mathcal{J}(\mathbb{R})}$, so $\mathrm{OD}_{1n} = \mathrm{OD}_{1n}^{\mathbb{R}}$.

Theorem 1.5 (Woodin, [1], 1990s).

Let $\lambda > 0$ be an ordinal. Then $\mathrm{OD}_{\lambda 1} = \mathrm{OD}^{\mathbb{R}}_{\lambda 1}$ is a mouse set.

Recall from Steel [5]: for $\alpha \leq \beta$,

- $[\alpha, \beta]$ is a gap iff this interval is maximal such that $\mathcal{J}_\alpha(\mathbb{R}) \preccurlyeq_{1,\mathbb{R}} \mathcal{J}_\beta(\mathbb{R})$.
- A gap $[\alpha, \beta]$ is:
  - projective-like iff $\mathcal{J}_\alpha(\mathbb{R}) \not\models$ KP.
  - non-projective-like or admissible iff $\mathcal{J}_\alpha(\mathbb{R}) \models$ KP.
- Admissible gaps are divided into weak and strong.
- A projective-like gap $[\gamma, \gamma]$ is scale-cofinal iff γ is not of form $\beta + 1$, where $[\alpha, \beta]$ is a strong gap.

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- A projective-like gap $[\gamma, \gamma]$ is scale-cofinal iff γ is not of form $\beta + 1$, where $[\alpha, \beta]$ is a strong gap.

Theorem 2.1 (Rudominer, Steel).

Let $[\alpha, \alpha]$ be projective-like with α of uncountable cofinality. Let $n \geq 1$. Then

$$\mathrm{OD}^{\mathbb{R}}_{\alpha n} = \mathrm{OD}_{\alpha n}$$

is a mouse set.

($n = 1$ version already true by Woodin's result.)

Rudominer introduced ladder mouse M_{ld} and admissible ladder mouse M_{adld}.

Definition 3.1.

M-ladder the least mouse M such that there is $\langle\theta_n\rangle_{n<\omega}$ such that:

– θ_n is an M-cardinal,

– $M_n^\#(M|\theta_n) \lhd M$ and $M_n^\#(M|\theta_n) \models$"$\theta_n$ is Woodin".

Write $M_{\mathrm{ld}} = M$.

Theorem 3.2 (Rudominer 1990s).

$\mathbb{R} \cap M_{\mathrm{ld}} \subseteq \mathrm{OD}_{12} \subseteq \mathbb{R} \cap M_{\mathrm{adld}}$.

Theorem 3.3 (Woodin 2018, [2]).

$\mathbb{R} \cap M_{\mathrm{ld}} = \mathrm{OD}_{12}$ *is a mouse set.*

In fact, there is $\gamma < \omega_2^{M_{\mathrm{ld}}}$ *and a recursive function* $\varphi \mapsto \varrho_\varphi$ *such that for all* Σ_2 *formulas* φ *and all* $x \in \mathbb{R}^{M_{\mathrm{ld}}}$,

$$\mathcal{J}(\mathbb{R}) \models \varphi(x) \iff M_{\mathrm{ld}}|\omega_2^{M_{\mathrm{ld}}} \models \varrho_\varphi(x, \gamma).$$

Theorem 3.4 (S., [3], 2024).

Assume ZF + AD + $V = L(\mathbb{R})$. Let $[\alpha, \alpha]$ be a scale-cofinal projective-like gap. Let $n \geq 1$. Then

$$\mathrm{OD}_{\alpha n} = \mathrm{OD}^{\mathbb{R}}_{\alpha n}$$

is a mouse set.

Remarks:

- New proof that $\mathrm{OD}_{12} = \mathbb{R} \cap M_{\mathrm{ld}}$, avoiding stationary tower.
- General case not quite a direct generalization of OD_{12}.
- Also get anti-correctness...

Anti-correctness for $\Pi_2^{\mathcal{J}(\mathbb{R})}$, under AD + $V = L(\mathbb{R})$:

Theorem 3.5 (S., [3], 2024).

Anti-correctness holds for $\Pi_2^{\mathcal{J}(\mathbb{R})}$ *and* $M = M_{\mathrm{ld}}$*. There is a unique* Σ_1*-elementary*

$$\sigma : \mathcal{J}(\mathbb{R}^M) \to \mathcal{J}(\mathbb{R}),$$

and moreover:

- $\Pi_2^{\mathcal{J}(\mathbb{R})}$ *is uniformly* $\Sigma_2^{\mathcal{J}(\mathbb{R}^M)}$*,*
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- $\Pi_2^{\mathcal{J}(\mathbb{R})}$ *is uniformly* $\Sigma_2^{\mathcal{J}(\mathbb{R}^M)}$,
- $\Pi_2^{\mathcal{J}(\mathbb{R}^M)}$ *is uniformly* $\Sigma_2^{\mathcal{J}(\mathbb{R})}$.

Theorem 3.6 (S., [3], 2024).

Let $[\alpha, \alpha]$ be scale-cofinal projective-like. Then for a cone of reals x, there is an x-mouse $M = M_{\mathrm{ld}}^{\alpha}(x)$ analogous to M_{ld}, and there is a unique $\bar{\alpha}$ and

$$\sigma : \mathcal{J}_{\bar{\alpha}}(\mathbb{R}^M) \to \mathcal{J}_{\alpha}(\mathbb{R}),$$

which is cofinal Σ_1-elementary, and moreover:

- $\Pi_2^{\mathcal{J}_\alpha(\mathbb{R})}(\{x\})$ *is uniformly* $\Sigma_2^{\mathcal{J}_{\bar{\alpha}}(\mathbb{R}^M)}(\{x\})$,
- $\Pi_2^{\mathcal{J}_{\bar{\alpha}}(\mathbb{R}^M)}(\{x\})$ *is uniformly* $\Sigma_2^{\mathcal{J}_\alpha(\mathbb{R})}(\{x\})$.

Remark 4.1.

If $[\alpha, \beta]$ is admissible gap, i.e. $\mathcal{J}_\alpha(\mathbb{R}) \models \mathsf{KP}$, then

$$\mathrm{OD}_{\xi n} = \mathrm{OD}_{\alpha 1}$$

for all $\xi \in [\alpha, \beta)$ and $n < \omega$.

Theorem 4.2 (ess. Martin).

Let $[\alpha, \beta]$ be a strong gap. Then $\mathrm{OD}_{\alpha 1} = \mathrm{OD}_{\beta n}$ for every $n \in [1, \omega)$.

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Let $[\alpha, \beta]$ be a strong gap. Then $\mathrm{OD}_{\alpha 1} = \mathrm{OD}_{\beta n}$ for every $n \in [1, \omega)$.

Question 4.3.

Let $[\alpha, \beta]$ be a weak gap and $n \geq 2$. What can we say about

$$\mathrm{OD}_{\beta n} \text{ and } \mathrm{OD}^{\mathbb{R}}_{\beta n}?$$

Similarly, if $[\alpha, \beta]$ is a strong gap, what about

$$\mathrm{OD}_{\beta+1,n} \text{ and } \mathrm{OD}^{\mathbb{R}}_{\beta+1,n}?$$

The following lemma comes from joint work with Steel, relates to methods from core model induction:

Lemma 4.4.

Let α, β, γ *be such that either:*

- $[\alpha, \beta]$ *is a weak gap and* $\beta = \gamma$*, or*
- $[\alpha, \beta]$ *is a strong gap and* $\beta + 1 = \gamma$*.*

Then $\mathcal{J}_\gamma(\mathbb{R})$ *is a "derived model".*

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- $[\alpha, \beta]$ *is a strong gap and* $\beta + 1 = \gamma$.

Then $\mathcal{J}_\gamma(\mathbb{R})$ *is a "derived model".*

More precisely, there is a mouse operator

$$\mathscr{P} : x \mapsto P_x = \mathscr{P}(x),$$

defined for a cone of reals x, such that:

- $\mathscr{P}$ is definable from params over $\mathcal{J}_\gamma(\mathbb{R})$,
- P_x is a sound ω-small x-mouse which projects to ω,
- $P_x \models$ "there are ω Woodin cardinals",
- $\mathcal{J}_\gamma(\mathbb{R})$ is a "derived model" of an $\mathbb{R}$-genericity iterate of P_x
- the fine structure of $\mathcal{J}_\gamma(\mathbb{R})$ corresponds to that of P_x.

Theorem 5.1 (S., [3]).

Let $[\alpha, \gamma]$ be a weak gap, or $\gamma = \beta + 1$ where $[\alpha, \beta]$ is a strong gap.
Then for a cone of reals x, there is a "γ-ladder" x-mouse $M_{\mathrm{ld}}^{\gamma}(x)$ definable from x over $\mathcal{J}_{\gamma}(\mathbb{R})$, analogous to M_{ld} over $\mathcal{J}(\mathbb{R})$.

Theorem 5.1 (S., [3]).

Let $[\alpha, \gamma]$ be a weak gap, or $\gamma = \beta + 1$ where $[\alpha, \beta]$ is a strong gap. Then for a cone of reals x, there is a "γ-ladder" x-mouse $M^{\gamma}_{\mathrm{ld}}(x)$ definable from x over $\mathcal{J}_{\gamma}(\mathbb{R})$, analogous to M_{ld} over $\mathcal{J}(\mathbb{R})$.

Remark 5.2.

- $M^{\gamma}_{\mathrm{ld}}(x)$ has infinitely many Woodins; a "ladder" ascends to its least Woodin δ_0.
- Defined using operator $\mathscr{P}$ associated to $\mathcal{J}_{\gamma}(\mathbb{R})$.
- $M^{\gamma}_{\mathrm{ld}}(x)|\delta_0$ is closed under $\mathscr{P}$,
- $M^{\gamma}_{\mathrm{ld}}(x) = \mathscr{P}(M^{\gamma}_{\mathrm{ld}}(x)|\delta_0)$.
- used in proof of mouse set theorems for $\mathrm{OD}_{\gamma n}$

Theorem 5.3.

Let $[\alpha, \gamma]$ be a weak gap, or $\gamma = \beta + 1$ where $[\alpha, \beta]$ is a strong gap. Let e be least such that $\rho_{e+1}^{\mathcal{J}_\gamma(\mathbb{R})} = \mathbb{R}$. Then $\mathrm{OD}_{\gamma n}$ is a mouse set for:

- $n \leq e + 1$ ($\mathrm{OD}_{\gamma n} = \mathrm{OD}_{\alpha 1}$ *here*),
- $n \geq e + 3$.

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Let $[\alpha, \gamma]$ be a weak gap, or $\gamma = \beta + 1$ where $[\alpha, \beta]$ is a strong gap. Let e be least such that $\rho_{e+1}^{\mathcal{J}_\gamma(\mathbb{R})} = \mathbb{R}$. Then $\mathrm{OD}_{\gamma n}$ is a mouse set for:

- $n \leq e + 1$ ($\mathrm{OD}_{\gamma n} = \mathrm{OD}_{\alpha 1}$ *here*),
- $n \geq e + 3$.

Question 5.4.

What about $n = e + 2$? What about $\mathrm{OD}_{\gamma, e+2}(\{\vec{p}_{e+1}^{\mathcal{J}_\gamma(\mathbb{R})}\})$?

- Probably not quite the right question.
- For such γ, e, a more natural variant of $\mathrm{OD}^{\mathbb{R}}_{\gamma, e+n}$ exists; call it $\mathrm{OD}^{*\mathbb{R}}_{\gamma, e+n}$.

Theorem 5.5 (S., [3], 2024).

Let γ, e be as above. Then:

1. *For $n \geq 3$,*
$$\mathrm{OD}^{*\mathbb{R}}_{\gamma,e+n} = \mathrm{OD}_{\gamma,e+n}$$
is a mouse set.
2. *For a cone of reals x,*
$$\mathrm{OD}^{*\mathbb{R}}_{\gamma,e+2}(x) = \mathbb{R} \cap M^{\gamma}_{\mathrm{ld}}(x)$$
is an x-mouse set.

Question 5.6.

What about $\mathrm{OD}^{*\mathbb{R}}_{\gamma,e+2}$ (lightface)?

Conjecture 5.7.

It is the mouse set $\mathbb{R} \cap N$, where $N =$ output of appropriate $L[\mathbb{E}]$-construction formed inside $M^{\gamma}_{\mathrm{ld}}(x)$ for a cone of x.

Definition 5.8.

Let γ be end of weak gap / successor of strong gap. Let e be least such that $\rho_{e+1}^{\mathcal{J}_\gamma(\mathbb{R})} = \mathbb{R}$. Then for $n \geq 1$ define $\Sigma^{*\mathbb{R}}_{e+n}$ and $\Pi^{*\mathbb{R}}_{e+n}$ as follows:

- $\Sigma^{*\mathbb{R}}_{e+1} = \mathrm{r}\Sigma_{e+1}(\{\vec{p}\})$ where $\vec{p} = \vec{p}_{e+1}^{\mathcal{J}_\gamma(\mathbb{R})}$,
- $\Pi^{*\mathbb{R}}_{e+n} = \neg\Sigma^{*\mathbb{R}}_{e+n}$,
- $\Sigma^{*\mathbb{R}}_{e+n+1} = \exists^{\mathbb{R}}\Pi^{*\mathbb{R}}_{e+n}$.

Define $\mathrm{OD}^{*\mathbb{R}}_{\gamma,e+1+n}$ (for $n \geq 0$) using these classes.

Anti-correctness for M_1: for Π^1_3 formulas φ and $x \in \mathbb{R} \cap M_1$:

$$\varphi(x) \iff M_1 \models \psi_\varphi(x),$$

where $\psi_\varphi(x)$ is the Σ^1_3 formula asserting “there is a Π^1_2-iterable $\varphi(x)$-prewitness”.

Definition 5.9 (Woodin).

Let φ be Π^1_3 and $x \in \mathbb{R}^{M_1}$.
A $\varphi(x)$-prewitness is a premouse N with $x, \delta \in N$ such that:

- $N \models \mathrm{ZF}^- +$“$\delta$ is Woodin”
- $N \models$“the extender algebra at δ forces $\varphi(x)$”.

Theorem 5.10 (Woodin).

Let φ be Π^1_3 and $x \in \mathbb{R}^{M_1}$. Then TFAE:

- $\varphi(x)$
- *There is a $\varphi(x)$-prewitness $N \lhd M_1|\omega_1^{M_1}$*
- *There is a Π^1_2-iterable $\varphi(x)$-prewitness $P \in \mathrm{HC}^{M_1}$.*

We want, for Π_2 formulas φ, a Σ_2 formula ψ_φ such that:

$$\mathcal{J}(\mathbb{R}) \models \varphi(x) \iff \mathcal{J}(\mathbb{R}^{M_{\mathrm{ld}}}) \models \psi_\varphi(x).$$

$\psi_\varphi(x)$ should say "there is a Π_1-iterable $\varphi(x)$-prewitness".

Remark 5.11.

- Π_1-iterability is $\Pi_1^{\mathcal{J}(\mathbb{R})}$.
- M_{ld} is $\Sigma_1^{\mathcal{J}(\mathbb{R})}$-correct.
- Every Π_1-iterable premouse $P \in \mathrm{HC}^{M_{\mathrm{ld}}}$ is iterable.

Fix a Π_1 formula ϱ, and $x \in \mathbb{R}^{M_{\mathrm{ld}}}$.
There is a natural game $\mathscr{G}(\varrho, x)$, in which player 2 tries to prove that

$$\mathcal{J}(\mathbb{R}) \models \exists^{\mathbb{R}} w \; \varrho(x, w),$$

as follows:

- Player 1 plays arbitrary objects in M_{ld}.
- Player 2 tries to build X, w by finite approximation, such that:
  - $X \preccurlyeq_1 M_{\mathrm{ld}}$,
  - X includes all elements played by player 1,
  - $w \in \mathbb{R}$ is extender algebra generic at each θ_n, while
  - for no n does
    $$M_n^{\#}(\overline{M_{\mathrm{ld}}|\theta_n})[w]$$
    verify $\neg\varrho(x, w)$.
- The first n moves are within $M_n^{\#}(M_{\mathrm{ld}}|\theta_n)$, and the game up to there is definable there.

Then:

- $\mathscr{G}_\varrho$ is closed for player 2.
- If player 2 wins, then $\exists w \; \varrho(x, w)$.
- If player 1 wins, the rank analysis (in V) computes a winning strategy.
- $\mathscr{G}^{M_n^{\#}(M_{\mathrm{ld}}|\theta_n)}(\varrho, x, n)$ denotes the restriction of $\mathscr{G}(\varrho, x)$ to first n moves.

Definition 5.12.

An <u>n-partial ladder</u> is a premouse N such that for some $\vec{\theta}$,

- $\vec{\theta} = \langle \theta_i \rangle_{i \leq n}$ is a strictly increasing $(n+1)$-tuple of ordinals of N,
- θ_i is an N-cardinal for all $i \leq n$,
- θ_n^{+N} is the largest cardinal of N,
- N is closed under $M_k^{\#}$-operator, for each $k < \omega$,
- θ_i is Woodin in $M_i^{\#}(N|\theta_i)$, and θ_i is the least such N-cardinal, for each $i \leq n$.

Write $\vec{\theta}^N = \vec{\theta}$.

Let $\varphi(x) = \forall^{\mathbb{R}} w \, \neg\varrho(w, x)$ (recall ϱ is Π_1).
For a $\underline{\varphi(x)\text{-witness}}$, we want roughly:

- an (iterable) 0-partial ladder premouse P_0 with $x \in P_0$

where player 1' wins the following game $\mathscr{G}_x^{P_0}$:

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0.1 Player 1' plays:
  - P_0

0.2 Player 2' plays:
  - A correct tree $\mathcal{T}_0$ on P_0, based on $P_0|\theta_0^{P_0}$; let $P_0' = M_\infty^{\mathcal{T}_0}$ and $\theta_0' = \theta_0^{P_0'}$,
  - a play σ_0 of $\mathscr{G}_{\varrho,x,1}^{M_0^{\#}(P_0'|\theta_0')}$ of length 1, following rules, player 2 has not lost,

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1.1 Player 1' plays:

– A 1-partial ladder P_1 such that $P_0'|(\theta_0')^{+P_0'} \triangleleft P_1 \triangleleft P_0'$,

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let $P_1' = M_\infty^{\mathcal{T}_1}$ and $\theta_1' = \theta_1^{P_1'}$,

– a play σ_1 of $\mathscr{G}_{\varrho,x,2}^{M_1^{\#}(P_1'|\theta_1')}$ of length 2, extending σ_0, following rules, player 2 has not lost,

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For a $\varphi(x)$-witness, we want roughly:

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- a play σ_1 of $\mathscr{G}_{\varrho,x,2}^{M_1^\#(P_1'|\theta_1')}$ of length 2, extending σ_0, following rules, player 2 has not lost,

2.1 Player 1' plays:
- A 2-partial ladder P_2 such that $P_1'|(\theta_1')^{+P_1'} \triangleleft P_2 \triangleleft P_1'$,

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2.1 Player 1' plays:
- A 2-partial ladder P_2 such that $P_1'|(\theta_1')^{+P_1'} \lhd P_2 \lhd P_1'$,

2.2 , 3.1, ... etc...

The first player to break a rule loses; otherwise player 2' wins.

Write $M = M_{\mathrm{ld}}$. For $i < \omega$ write θ_i^M for the ith "rung" of M, and $Q_i^M = M_i^{\#}(M|\theta_i^M)$.

Game $\mathscr{G}^M$, $K = M = M_{\mathrm{ld}}$: Fix a recursive enumeration $\langle\psi_i\rangle_{i<\omega}$ of all formulas in the passive premouse language. Write $\mathbb{B}_i =$ extender algebra of M at θ_i^M. In round $n < \omega$, player 1 first plays some $\vec{a}_n \in (M|\theta_n^M)^{<\omega}$, and then player 2 plays $\vec{b}_n, \vec{x}_n \in (M|\theta_n^M)^{<\omega}$ such that:

1. (Cofinality of X) $\vec{a}_n \cup \vec{b}_n \subseteq \vec{x}_n$, and if $n > 0$ then $\theta_{n-1}^K \in \vec{x}_n$.
2. (Σ_1-elementarity of X): If $n > 0$ then for all $i, j, k < n$, letting $\vec{x}'_\ell = \vec{x}_\ell \cap M|\theta_i^K$, if
$$M|\theta_i^K \models \exists z\ \psi_j(\vec{x}'_0, \ldots, \vec{x}'_{k-1}, z)$$
then there is some $x \in \vec{x}_n$ such that $M|\theta_i^K \models \psi_j(\vec{x}'_0, \ldots, \vec{x}'_{k-1}, x)$.
3. $\vec{b}_n = \langle b_{ni}\rangle_{i\leq n}$ where $b_{ni} \in \mathbb{B}_i$ for each $i \leq n$.
4. If $n > 0$ then $b_{ni} \leq^{\mathbb{B}_i} b_{n-1,i}$ for each $i < n$.
5. For all $i \leq j \leq n$, (noting that $b_{ni} \in \mathbb{B}_j$) for each $j < i$ and $b \in \mathbb{B}_j$, we have $b_{nj} \leq^{\mathbb{B}_j} b_{ni}$.
6. If $n > 0$ then for each $A \in \vec{x}_{n-1}$ and each $i < n$, if $A \in M|\theta_i^M$ is a maximal antichain of $\mathbb{B}_i$ then there is $a \in A$ such that $b_{ni} \leq^{\mathbb{B}_i} a$.
7. For each $i \leq n$, there is no $W \triangleleft Q_i^M$ with $\theta_i^K < \mathrm{OR}^W$ such that
$$Q_i^M \models b_{ni} \text{ forces ``} W \text{ is an above-}\theta_i^K, \neg\psi(x, y_0, \dot{w})\text{-prewitness''},$$
where $\dot{w}$ denotes the $\mathbb{B}_i$-generic real.

Definition 5.13.

Given a transitive swo'd $X \in \mathrm{HC}$, an X-premouse N and $n < \omega$, we say that $(N, \theta_0, \ldots, \theta_n)$ is an <u>n-partial-potential-ladder</u> iff:

- $\theta_0 < \theta_1 < \ldots < \theta_n$,
- θ_i is a cutpoint of N, for each $i \leq n$,
- $N \models$"θ_i is a limit cardinal which is <u>not</u> Woodin and <u>not</u> measurable" for each $i \leq n$,
- $\theta_n^{+N} < \mathrm{OR}^N$ and θ_n^{+N} is the largest cardinal of N.

Definition 5.14.

Suppose $(N, \vec{\theta})$ is an n-partial-potential-ladder. Then $Q_i^{(N,\vec{\theta})} \trianglelefteq N$ denotes the Q-structure for $N|\theta_i$, for $i \leq n$. Given $x, y_0 \in \mathbb{R}^N$ and a Π_1 formula $\psi(u, v, w)$, let

$$\mathscr{G}^{(N,\vec{\theta})} = \mathscr{G}^{(N,\vec{\theta})}_{\exists^{\mathbb{R}}\psi(x,y_0)}$$

denote the set of partial plays $\sigma = \left\langle (\vec{a}_i, \vec{b}_i, \vec{x}_i) \right\rangle_{i<n}$, relative to $\langle(\theta_i, Q_i)\rangle_{i<n}$ where $Q_i = Q_i^{(N,\vec{\theta})}$.

Definition 5.15.

Let $X \in \mathrm{HC}$ be transitive swo'd, N be an X-premouse, $\vec{\theta}, \varrho \in N^{<\omega}$, $x, y_0 \in \mathbb{R}^N$, and $\Delta \in N$. Let $\psi(u, v, w)$ be a Π_1 formula. We say $(N, \vec{\theta}, \Delta)$ is a

$$(\forall^{\mathbb{R}} \neg\psi(x, y_0), \varrho)\text{-prewitness}$$

iff, letting $n = \mathrm{lh}(\varrho)$, then $(N, \vec{\theta})$ is an n-partial-potential-ladder, and letting

$$(N_n, \vec{\theta}_n, \varrho_n, \Delta_n) = (N, \vec{\theta}, \varrho, \Delta),$$

then $\varrho \in \mathscr{G}^{(N,\vec{\theta})}_{\exists^{\mathbb{R}}\psi(x,y_0)}$ and Δ is a non-empty tree whose elements σ have form

$$\sigma = (\sigma_{n+1}, \ldots, \sigma_{n+k})$$

where $k < \omega$ and for $0 \leq i \leq k$, σ_{n+i} has form

$$\sigma_{n+i} = (N_{n+i}, \vec{\theta}_{n+i}, \varrho_{n+i}, \Delta_{n+i})$$

(so $\sigma_n = (N, \vec{\theta}, \varrho, \Delta)$, but σ_n is not actually an element of σ), and moreover, for each $\sigma \in \Delta$, with σ, k, σ_{n+i} as above, the following conditions hold...

1. If $\sigma \neq \emptyset$ then for every $i < k$, we have the following:
   (a) $N_{n+i+1} \lhd N_{n+i}$,
   (b) $(N_{n+i+1}, \vec{\theta}_{n+i+1})$ is an $(n+i+1)$-potential-partial-ladder,
   (c) $\rho_1^{N_{n+i+1}} = \rho_\omega^{N_{n+i+1}} = \theta_{n+i}^{+N_{n+i}}$,
   (d) $\vec{\theta}_{n+i+1} \restriction (n+i+1) = \vec{\theta}_{n+i}$,
   (e) $\varrho_{n+i+1} \in \mathscr{G}^{(N_{n+i+1}, \vec{\theta}_{n+i+1})}_{\exists^{\mathbb{R}}\psi(x,y_0)}$ (so $\mathrm{lh}(\varrho_{n+i+1}) = n+i+1$),
   (f) $\varrho_{n+i} = \varrho_{n+i+1} \restriction (n+i)$,
   (g) $\Delta_{n+i+1} \in N_{n+i+1}$ is a tree,
   (h) $(\sigma_{n+i+2}, \ldots, \sigma_{n+k}) \in \Delta_{n+i+1}$.
2. $\Delta_{n+k} = \{\tau \mid \sigma ^\frown \tau \in \Delta\}$.
3. Letting $\theta_{n+k} = \max(\vec{\theta}_{n+k})$, there is $\vec{a} \in (N_{n+k}|\theta_{n+k})^{<\omega}$ such that for all $\vec{b}, \vec{x} \in (N_{n+k}|\theta_{n+k})^{<\omega}$ such that

$$\varrho' = \varrho_{n+k} {}^\frown \left\langle (\vec{a}, \vec{b}, \vec{x}) \right\rangle \in \mathscr{G}^{(N_{n+k}, \vec{\theta}_{n+k})}_{\exists^{\mathbb{R}}\psi(x,y_0)},$$

   there is $\sigma' \in \Delta$ such that $\sigma' = \sigma {}^\frown \left\langle (N', \vec{\theta}', \varrho', \Delta') \right\rangle$ for some $N', \vec{\theta}', \Delta'$.

A $\underline{\underline{(\forall^{\mathbb{R}} \neg\psi(x, y_0))\text{-prewitness}}}$ is a $(\forall^{\mathbb{R}} \neg\psi(x, y_0), \emptyset)$-prewitness.

Lemma 5.16.

Let $\varphi(x)$ be $\forall^{\mathbb{R}}\Sigma_1$, of form

$$\forall^{\mathbb{R}} w\, \neg\varrho(w, x),$$

where ϱ is Π_1. Let $x \in \mathbb{R}^{M_{\mathrm{ld}}}$. TFAE:

(i) $\mathcal{J}(\mathbb{R}) \models \varphi(x)$

(ii) *Player 1 wins* $\mathscr{G}_{\varrho,x}$

(iii) *there is a $\varphi(x)$-prewitness (P_0, Δ) with $P_0 \triangleleft M_{\mathrm{ld}}|\omega_1^{M_{\mathrm{ld}}}$.*

(iv) *there is a $\varphi(x)$-prewitness $(P_0, \Delta) \in \mathrm{HC}^{M_{\mathrm{ld}}}$ with $\mathcal{J}(\mathbb{R}^{M_{\mathrm{ld}}}) \models$ "P_0 is Π_1-iterable".*

End of weak gap

Example: $[\alpha, \beta]$ is weak, and for $P_{\mathrm{g}}(x)$ the corresponding mouse on a cone of x,

$$\omega = \rho_1^{P_{\mathrm{g}}(x)} < \lambda^{P_{\mathrm{g}}(x)} < \mathrm{OR}^{P_{\mathrm{g}}(x)},$$

$\lambda^P \notin p_1^{P_{\mathrm{g}}(x)}$, $(\lambda^P)^{+P} < \mathrm{OR}^P$, and $\Sigma_1^{\mathcal{J}_\beta(\mathbb{R})}$ is μ-reflecting.(see [4]).

Definition 5.17.

For an X-premouse R, say that R is relevant if there is $\delta = \delta_0^R < \mathrm{OR}^R$ such that:

- $R \models$“δ is the least Woodin $> \mathrm{rank}(X)$”,
- $R = P_{\mathrm{g}}(R|\delta)$,
- $R|\delta$ is P_{g}-closed.

Definition 5.18.

For relevant R, let:

- $\langle \alpha_n^R \rangle_{n<\omega}$ be the canonical ω-sequence cofinal in OR^R,
- $\gamma_n^R = \sup(\delta_0^R \cap \mathrm{Hull}_1^{R|\alpha_n^R}(X \cup \{p_1^R\})$,
- $t_n^R = \mathrm{Th}_1^{R|\alpha_n^R}(X \cup \gamma_n^R \cup \{p_1^R\})$.

Definition 5.19 (Ladder mouse at end of weak gap).

For a cone of y, $M_{\mathrm{ld}}^\beta(y)$ is the least relevant y-mouse N such that letting $\delta = \delta_0^N$, for each $n < \omega$, there is a relevant $R \triangleleft N|\delta$ with $t_n^R = t_n^N$ (after substituting p_1^R for p_1^N).

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