Happy birthday Ralf!

Towards extending Dehornoy's analysis

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Patrick Dehornoy studied iterated ultrapowers of V by a single measure.

In particular, he analysed intersection models at limit steps of the iteration via Prikry forcing.

- Dehornoy: Iterated ultrapowers and Prikry forcing.
  Annals of Mathematical Logic 15, 2 (1978)

We aim to generalise this analysis to several measurable cardinals.

Joint work with Christopher Henney-Turner, IMPAN. Most results are due to Christopher.

The Bukovský-Dehornoy phenomenon

Suppose that κ is a measurable cardinal and U is a normal measure on κ. Let $\mathbb{P}_U$ denote Prikry forcing with respect to U.

Notation:

- N_α is the αth iterate of V
- $i_{\alpha,\beta} : N_\alpha \to N_\beta$ is the iteration map
- $\kappa_\alpha = i_{0,\alpha}(\kappa)$
- $U_\alpha = i_{0,\alpha}(U)$

So $N_0 = V$, $\kappa_0 = \kappa$, and for every α, $i_{\alpha,\alpha+1} \colon N_\alpha \to N_{\alpha+1}$ is the ultrapower embedding by U_α.

Mathias' criterion (1973): TFAE

1. $(\nu_n)_{n\in\omega}$ is $\mathbb{P}_U$-generic over V
2. for every set $X \in U$, $\nu_n \in X$ for all sufficiently large n

Proposition (Solovay)

The sequence $\vec{\kappa} := (\kappa_n)_{n \in \omega}$ *is* $\mathbb{P}_{U_\omega}$*-generic over* N_ω.

Proof.

By Mathias' criterion. $\square$

Theorem (Bukovský 1977, Dehornoy 1977)

$N_\omega[\vec{\kappa}] = \bigcap_{n<\omega} N_n$

Suppose that $\mathbb{B}$ is a complete Boolean algebra and U is an ultrafilter on $\mathbb{B}$.

- $V^{\mathbb{B}}/U$ is the full Boolean model.
- The subset $\check{V}_U$ of $V^{\mathbb{B}}/U$ given by names σ with $[\![\sigma \in \check{V}]\!] = 1$ is the Boolean ultrapower

Theorem (Bukovsky 1977)

The ω-th iterate N_ω of V on κ is the Boolean ultrapower of V for the Boolean completion $\mathbb{B}$ of Prikry forcing. Moreover, the embedding $i_{0,\omega}$ equals the Boolean ultrapower embedding $V \to \check{V}_U$.

Fuchs and Hamkins (2017) found conditions for Bukovsky-Dehornoy phenomena for Boolean ultrapowers for $\mathbb{B}$. But there are limitations:

Theorem (Fuchs, Hamkins 2017)

Suppose that κ is measurable and there exists no measurable $<\kappa$ in any generic extension. Then for any $\alpha > \omega$, $i_{0,\alpha} : V \to N_\alpha$ is not a Boolean ultrapower embedding.

Sakai discovered that for iterations of generic ultrapowers of length ω, the critical sequence is generic.

Theorem (Sakai 2005)

Suppose that I is a normal precipitous ideal on ω_2 and $V = N_0 \to N_1 \to \ldots$ is a generic iteration of I via the inverse limit of the forcings.
Then the critical sequence is Namba generic over the direct limit.

In this situation, it is not quite clear how one could sensibly define an intersection model.

Do the previous results hold for longer iterations?

Long iterations

Suppose that $\mathrm{cof}(\theta) = \omega$.

Lemma (Dehornoy 1977)

Let $\vec{\kappa} := (\kappa_{i_n})_{n<\omega}$ be a strictly increasing cofinal sequence of critical points below κ_θ. Then $\vec{\kappa}$ is generic over N_θ.

Proof.

By Mathias' criterion. $\square$

Note. There are many generic extensions of N_θ of this kind.

- Two sequences generate the same model if and only if they are eventually equal.
- This follows from the analysis of the submodels of Prikry extensions by Gitik, Kanovei and Koepke (2010).

All sequences are elements of the intersection model $M_\theta = \bigcap_{\alpha<\theta} N_\alpha$.

Dehornoy's analysis is used to describe the intersection model $M_\theta = \bigcap_{\alpha<\theta} N_\alpha$.

Theorem (Dehornoy 1977)

Suppose that $\mathrm{cof}(\theta) = \omega$.

1. $M_\theta = \bigcup\{N_\theta[\vec{\kappa}] : \vec{\kappa}$ *is as above*$\}$.
2. M_θ *is a model of* ZF, *but not of the axiom of choice.*

Theorem (Dehornoy 1977)

If $N_\alpha \models \mathrm{cof}(\theta) > \omega$ *for some* $\alpha < \theta$, *then* $M_\theta = N_\theta$.

$M_\theta = \bigcap_{\alpha<\theta} N_\alpha$ always denotes the intersection model at a limit step θ.

An application of the analysis:

Theorem (Dehornoy 1977)

$\mathrm{HOD}_{N_\theta}^{M_\theta} = N_\theta$ *for all* $\theta \in \mathrm{Ord}$.

There is a related result for Magidor forcing.

- Dehornoy: An application of ultrapowers to changing cofinality The Journal of Symbolic Logic 48, 2 (1983)

Here $(U_\alpha)_{\alpha<\gamma}$ is a strictly increasing (in the Mitchell order) sequence of ultrafilters on the same κ of length $\gamma < \kappa$.

The intersection model is formed via all models given by finitely many ultrapowers.

However, this does not address the question of several measurable cardinals.

More measurables

Theorem (Fuchs 2005)

Let $\vec{\kappa}$ be a discrete sequence of measurables. Then there is a "product" of Prikry forcings for $\vec{\kappa}$ which admits a Mathias criterion.

- The "product" has finite support for stems and full support for full measure sets.
- For finitely many measurables, product and iteration are equivalent.

Theorem (Fuchs 2005)

Suppose $\vec{\kappa}$ is a discrete sequence of measurables and each measurable is iterated ω many times. Then the combination of critical sequences is generic.

Theorem (Welch 2022)

Suppose $\vec{\kappa}$ is a discrete sequence of measurables. Iterate κ_i in λ_i steps, where $\mathrm{cof}(\lambda_i) = \omega$, and choose a cofinal ω-sequence of critical points below each measurable. Then the combination of these sequences is generic.

Ben-Neria recently generalised Fuchs' result to arbitrary sequences of measurables.

Theorem (Ben-Neria 2024)

Let $\vec{\kappa}$ be a sequence of measurables. The Magidor iteration of Prikry forcings for $\vec{\kappa}$ admits a (more complex) Mathias criterion.

Using this, the previous results can be improved:

Theorem (Henney-Turner, Welch 2024)

Let $\vec{\kappa}$ be any sequence of measurables. Iterate κ_i in λ_i steps, where $\mathrm{cof}(\lambda_i) = \omega$, and choose a cofinal ω-sequence of critical points below each measurable. Then the combination of sequences is generic.

Given these results, we ask:

Problem

Can Dehornoy's analysis be extended to two (or more) measurable cardinals?

We found this problem in an indirect way when working on a proof strategy for a tentative application.

Notation.

- Let $\kappa < \lambda$ be measurable cardinals.
- Let $N_{\alpha,\beta}$ be the result of iterating κ α many times, and then iterating λ β many times.
- Let $i_{\alpha,\alpha',\beta,\beta'} : N_{\alpha,\beta} \to N_{\alpha',\beta'}$ be the iteration map.
- Let $\kappa_\alpha = i_{0,\alpha,0,0}(\kappa) = i_{0,\alpha,0,\beta}(\kappa)$.
- Let $\lambda_{\alpha,\beta} = i_{0,\alpha,0,\beta}(\lambda)$.
- Let $M_{\alpha,\beta} = \bigcap_{i<\alpha,j<\beta} N_{i,j}$ be the intersection model.

Suppose:

- κ is iterated ω many times
- λ is iterated θ many times, where $\mathrm{cof}(\theta) = \omega$.

Suppose $X \in N_{n,\alpha}$ for all $n < \omega$ and $\alpha < \theta$. We can apply Dehornoy's result twice:

- $X \in N_{\omega,\alpha}[\vec{\kappa}]$ for all $\alpha < \theta$.
- $X \in N_{\omega,\theta}[\vec{\kappa}][\vec{\nu}]$ for some cofinal ω-sequence $\vec{\nu}$ in $\lambda_{\omega,\theta}$.

But there is a problem if:

- κ is iterated μ many times, where $\mathrm{cof}(\mu) = \omega$.

Then

- For each $\alpha < \theta$, there exists a cofinal sequence $\vec{\mu}^\alpha = (\mu_n^\alpha)_{n\in\omega}$ in μ such that $X \in N_{\mu,\alpha}[\kappa_{\vec{\mu}^\alpha}]$.
- The sequences $\kappa_{\vec{\mu}^\alpha}$ and models $N_{\mu,\alpha}[\kappa_{\vec{\mu}^\alpha}]$ might all be different.
- We would need $\mu_n^\alpha = \mu_n^{\alpha'}$ for all $\alpha < \alpha' < \theta$.

However, we will show that the latter condition holds, if α and α' are sufficiently nice, by analysing supports.

The support of a set X is a minimal set of critical points generating X in a model $N_{\alpha,\beta}$ over $i_{\alpha,\beta}[N_{0,0}]$.

Theorem (Bukovský 1977, Dehornoy 1977)

$M_\omega = N_\omega[\vec{\kappa}]$, where $\vec{\kappa} = (\kappa_n)_{n\in\omega}$.

Claim

Suppose that $x \in N_\omega$. For all sufficiently large $n < \omega$:

$$i_{n,\omega}(x) = i_{\omega,\omega\cdot 2}(x).$$

Proof of Claim.

Pick $x_n \in N_n$ with $i_{n,\omega}(x_n) = x$. Then

$$N_n \models \text{ the } \omega\text{th iterate sends } x_n \text{ to } x$$

By elementarity of $i_{n,\omega}$

$$N_\omega \models \text{ the } \omega\text{th iterate sends } x = i_{n,\omega}(x_n) \text{ to } i_{n,\omega}(x)$$

Hence $i_{\omega,\omega\cdot 2}(x) = i_{n,\omega}(x)$. □

Theorem (Bukovský 1977, Dehornoy 1977)

$M_\omega = N_\omega[\vec{\kappa}]$, *where* $\vec{\kappa} = (\kappa_n)_{n\in\omega}$.

Proof sketch.

Suppose $X \in M_\omega$ is a subset of γ.

Pick $(g_n)_{n\in\omega}$ with $X = i_{0,n}(g_n)(\vec{\kappa}\restriction n)$. Then $i_{n,\omega}(X) = i_{0,\omega}(g_n)(\vec{\kappa}\restriction n)$.

For each $\alpha < \gamma$ TFAE:

1. $\alpha \in X$
2. $i_{n,\omega}(\alpha) \in i_{0,\omega}(g_n)(\vec{\kappa}\restriction n)$ for some/all/sufficiently large n
3. $i_{\omega,\omega\cdot 2}(\alpha) \in i_{0,\omega}(g_n)(\vec{\kappa}\restriction n)$ for sufficiently large n

3. can be tested in $N_\omega[\vec{\kappa}]$, since $i_{0,\omega}((g_n)_{n\in\omega}) \in N_\omega$. $\square$

There are three changes in Dehornoy's proof for iterations of length θ with $\mathrm{cof}(\theta) = \omega$:

1. $i_{\alpha,\theta}$ might move θ. Therefore the equality in the claim above is replaced by

$$i_{\alpha,\theta}(x) = i_{\theta,\theta+i_{\alpha,\theta}(\theta-\alpha)}(x)$$

for sufficiently large $\alpha < \theta$.

This comes from the commuting diagram:

$$\begin{tikzcd}
N_{\alpha+\gamma} \arrow[r, "i_{\alpha,\beta}"] \arrow[d, "i_{\alpha+\gamma,\alpha+\delta}"'] & N_{\beta+i_{\alpha,\beta(\gamma)}} \arrow[d, "i_{\beta+i_{\alpha,\beta}(\gamma),\beta+i_{\alpha,\beta}(\delta)}"] \\
N_{\alpha+\delta} \arrow[r, "i_{\alpha,\beta}"'] & N_{\beta+i_{\alpha,\beta}(\delta)}
\end{tikzcd}$$

This part is similar in the two measure case.

2. A particularly nice cofinal sequence $(\alpha_n)_{n<\omega}$ in θ is constructed.

Dehornoy's properties state that either

- all $\alpha_n < \kappa$
- all α_n are sufficiently closed
- all α_n are additively indecomposable and $i_{0,\alpha_k}(\alpha_n) = \alpha_{n+k}$ for all k, n

In the two measure case, the last property becomes

- all α_i are additively indecomposable and $i_{0,\alpha_k,0,\alpha_{k-1}}(\alpha_n) = \alpha_{n+k}$ for all k, n

3. We need compatibility of supports.

Pick $(g_n)_{n\in\omega}$ with $X = i_{0,\alpha_n}(g_n)(\vec{\kappa}\restriction S_n)$ for some finite subset S_n of α_n.

We need to rule out that infinitely many ordinals are added below some α_k as n increases.

Then the union of the S_n would not have order type ω.

Lemma (folklore)

Suppose that $x \in N_\alpha$. Then there exists a function $f \in N_0$ and a finite tuple of ordinals $j_0 < \ldots < j_n < \alpha$ such that $x = i_{0,\alpha}(f)(\kappa_{j_0}, \ldots, \kappa_{j_n})$

Proof sketch.

By induction. $\square$

We call such a set $\{j_0, \ldots, j_n\}$ a candidate support of x.

Dehornoy proved that every set x in N_α has a least candidate support, called the support of x.

Notation. $e_\alpha(x)$ denotes the support of $x \in N_\alpha$.

The main point is: we need that the union $E(X) = \bigcup_{n\in\omega} e_\alpha(X)$ is

- finite or
- cofinal in θ with order type ω.

Let $\alpha \leq \alpha'$. Let $x \in N_{\alpha'}$.

Theorem (Dehornoy - upwards compatibility of supports)

Suppose

- $\alpha' - \alpha \in \operatorname{Im} i_{0,\alpha}$
- $\mu - \alpha \in \operatorname{Im} i_{0,\alpha}$ *for all* $\mu \in e^0_{\alpha',\beta'}(x) \setminus \alpha$

Then $e_\alpha(x) \subseteq e_{\alpha'}(x)$.

Theorem (Dehornoy - downwards compatibility of supports)

Suppose that $\alpha' - \alpha \in \operatorname{Im} i_{0,\alpha}$. *Then*

$$e_{\alpha'}(x) \cap \alpha \subseteq e_\alpha(x)$$

Supports for two measures

Lemma (folklore)

Suppose $x \in N_{\alpha,\beta}$. Then there is some function $f \in M_{0,0}$ and some finite tuples of ordinals $j_0 < \ldots < j_n < \alpha$ and $k_0 < \ldots < k_m < \beta$ such that $x = i_{0,\alpha,0,\beta}(f)(\kappa_{j_0}, \ldots, \kappa_{j_n}, \lambda_{\alpha,k_0}, \ldots, \lambda_{\alpha,k_m})$

Proof sketch.

By induction. $\square$

We call such a set $\{j_0, \ldots, j_n, k_0, \ldots, k_m\}$ a candidate support of x.

One can show that every set x in $N_{\alpha,\beta}$ has a least candidate support, called the support of x.

Notation. $e_{\alpha,\beta}(x)$ denotes the support of $x \in N_{\alpha,\beta}$.

Let $\alpha \leq \alpha'$ and $\beta \leq \beta'$. Let $x \in N_{\alpha',\beta'}$.

Theorem (upwards compatibility of supports)

Suppose

- $\alpha' - \alpha, \beta' - \beta \in \operatorname{Im} i_{0,\alpha,0,\beta}$
- $\mu - \alpha \in \operatorname{Im} i_{0,\alpha,0,\beta}$ *for all* $\mu \in e^0_{\alpha',\beta'}(x) \setminus \alpha$
- $\mu - \beta \in \operatorname{Im} i_{0,\alpha,0,\beta}$ *for all* $\mu \in e^1_{\alpha',\beta'}(x) \setminus \alpha'$

Then $e_{\alpha,\beta}(x) \subseteq e_{\alpha',\beta'}(x)$.

Theorem (downwards compatibility of lower supports)

Suppose that $\alpha' - \alpha, \beta' - \beta \in \operatorname{Im} i_{0,\alpha,0,\beta}$. *Then*

$$e^0_{\alpha',\beta'}(x) \cap \alpha \subseteq e^0_{\alpha,\beta}(x)$$

Theorem (downwards compatibility of upper supports)

Suppose that $\alpha' - \alpha, \beta' - \beta \in \operatorname{Im} i_{0,\alpha,0,\beta}$, *and* $\alpha > \beta$. *Then*

$$e^1_{\alpha',\beta'}(x) \cap \beta \subseteq e^1_{\alpha,\beta}(x)$$

Result for two measures

Theorem (Henney-Turner, Schlicht)

Suppose that $\mathrm{cof}(\theta) = \omega$ *and* $X \in M_{\theta,\theta} = \bigcap_{\alpha,\beta<\theta} N_{\alpha,\beta}$.
Then $X \in N_{\theta,\theta}[\vec{\kappa}, \vec{\lambda}]$ *for some* $\vec{\kappa} = (\kappa_{\alpha_n})_{n<\omega}$ *and* $\vec{\lambda} = (\lambda_{\theta,\beta_n})_{n<\omega}$.

Outline of the proof:

- Prove some technical lemmas about the iteration maps
- Define the support $e_{\alpha,\alpha'}(X) = e^0_{\alpha,\alpha'}(X) \sqcup e^1_{\alpha,\alpha'}(X)$ of a set X in $N_{\alpha,\alpha'}$ and show that it is well-defined.
- (upwards compatibility) Show that if $\alpha < \beta$ and $\alpha' < \beta'$ are all sufficiently nice, then $e_{\alpha,\alpha'}(X) \subseteq e_{\beta,\beta'}(X)$.
- (downwards compatibility) Suppose that $\alpha < \alpha'$ and $\beta < \beta'$ are all sufficiently nice. Then:
  - $e^0_{\alpha,\alpha'}(X) \supseteq e^0_{\beta,\beta'}(X) \cap \alpha$.
  - $e^1_{\alpha,\alpha'}(X) \supseteq e^1_{\beta,\beta'}(X) \cap \alpha$ if $\alpha > \beta$.
- Let $E(X)$ be the union of supports $e_{\alpha,\alpha'}(X)$ for nice α, α'. Show that $N_{\theta,\theta}[\kappa_{E(X)}, \lambda_{E(X)}]$ is a Prikry extension of $N_{\theta,\theta}$ containing X.

The last step is similar to the argument for the Bukovsky-Dehornoy result for a single measure:

- Suppose that $X \in N_{\theta,\theta}$ is a set of ordinals.
- Fix a cofinal sequence of nice ordinals $\alpha_0 < \alpha_1 < \ldots$ cofinal below θ.
- Let $E(X) = \bigcup_{n\in\omega} e_{\alpha_{2n+1},\alpha_{2n}}(X)$. By compatibility, $E(X)$ is (at most) a pair of Prikry generic ω-sequences
- In $M_{\theta,\theta}[(\kappa,\lambda)_{E(X)}]$ we can calculate the sequence $(i_{\alpha_{2n+1},\theta,\alpha_{2n},\theta}(X))_{n\in\omega}$.
- There exists ϵ such that for all $x \in N_{\theta,\theta}$, for all large enough α and β, $i_{\alpha,\theta,\beta,\theta}(x) = i_{\theta,\theta+\epsilon,\theta,\theta+\epsilon}(x)$.
- $X = \{x \in N_{\theta,\theta} : \exists m \forall n > m, i_{\theta,\theta+\epsilon,\theta,\theta+\epsilon}(x) \in i_{\alpha_{2n+1},\theta,\alpha_{2n},\theta}(X)\}$

A useful lemma: i and e commute.

Lemma

Let $i_{\alpha,\beta,\alpha',\beta'} i_{0,\gamma,0,\gamma'} = i_{0,\delta,0,\delta'}$ with $\alpha \leq \gamma$, and let $x \in N_{\gamma,\gamma'}$. Then writing $i = i_{\alpha,\beta,\alpha',\beta'}$, we have

$$\kappa_{e^0_{\delta,\delta'}}(i(x)) = i(\kappa_{e^0_{\gamma,\gamma'}}(x))$$

and

$$\lambda_{e^1_{\delta,\delta'}}(i(x)) = i(\lambda_{e^1_{\gamma,\gamma'}}(x))$$

The first step to show downwards compatibility is a criterion that

- holds for $e^0_{\alpha,\alpha'}(x)$ if and only if it holds for $e^0_{\beta,\beta'}(x)$, for nice $\alpha, \alpha', \beta, \beta'$
- holds for $e^0_{\alpha,\alpha'}(x)$, but only holds for $e^0_{\beta,\beta'}(x)$ if downwards compatibility holds at (β, β'), for a specific choice of $\alpha, \alpha', \beta, \beta'$

The criterion is "$\kappa_{e^0} \cup \lambda_{e^1} \in \operatorname{Im} i_{\mu,\mu+1,0,0}$" for a suitable choice of μ.

Lemma

Let $\mu < \Lambda, \theta, \Lambda', \theta' \in$ *Ord and* $z \in N_{\theta,\theta'} \cap N_{\Lambda,\Lambda'}$. *Suppose* $\Lambda - \mu, \theta - \mu, \Lambda', \theta' \in \mathrm{Im}\, i_{\mu,\mu+1,0,0}$. *TFAE:*

1. $\kappa_{e^0_{\Lambda,\Lambda'}}(z) \cup \lambda_{e^1_{\Lambda,\Lambda'}}(z) \subseteq \mathrm{Im}\, i_{\mu,\mu+1,0,0}$
2. $\kappa_{e^0_{\theta,\theta'}}(z) \cup \lambda_{e^1_{\theta,\theta'}}(z) \subseteq \mathrm{Im}\, i_{\mu,\mu+1,0,0}$
3. $i_{\mu,\mu+1,0,0}(z) = i_{\mu+1,\mu+2,0,0}(z)$

Outlook

We are working on extending the analysis to the setting of infinitely many measurable cardinals.

- Henney-Turner, Schlicht: Dehornoy for several measures
  In preparation, 34 pages

In a different direction, Sakai proved the genericity of the critical sequence for generic iterations of a normal precipitous ideal.

Problem

Can Sakai's result be extended to generic iterations of arbitrary length of a single normal precipitous ideal?

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