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Tutorial: Derived models from genericity iterations

Literature

These notes contain text (pictures) from the literature marked with "*".

* John Steel, A stationary-tower-free proof of the Derived Model Theorem, Contemporary Mathematics volume 425, 2007
* John Steel, The Derived Model Theorem, Lecture Notes in Logic, Proceedings of the Logic Colloquium 2006
* Sandra Müller, Grigor Sargsyan, Towards a generic absoluteness theorem for Chang Models, Adv. in Math, 2025

Lukas Koschat, Sandra Müller, Grigor Sargsyan, TBA.

Topics

- Introduction and Background
- Steel's proof of the DMT via flipping functions and genericity iterations
- Müller-Sargsyan Derived Model Representation of $L(\Gamma^\infty, \mathbb{R})$ after collapsing a supercompact cardinal
- Sealing
- Beyond $\mathcal{P}(\mathbb{R})$: the uB-powerset and $\mathcal{C}_\infty$ (M.-Sargsyan)
- Chang-type models and LSA (Koschat-M.-Sargsyan)

Woodin's Derived Model Theorem (and its variants) is the key method to produce models of determinacy

Theorem (Woodin, "Baby" Derived Model Theorem): Let λ be a limit of Woodin cardinals, let $H \subseteq Col(\omega, <\lambda)$ be V-generic. Then, $L(\mathbb{R}^*) \models AD$, where $\mathbb{R}^* = \bigcup_{\alpha<\lambda} \mathbb{R} \cap V[H \cap \alpha]$.

Theorem (Woodin, Derived model theorem): Let λ be a limit of Woodin cardinals, let $H \subseteq Col(\omega, <\lambda)$ be V-generic. Then

(1) $L(\mathbb{R}^*, Hom^*) \models AD^+$,

(2) $Hom^* = \{A \subseteq \mathbb{R}^* \mid A$ is Suslin and co-Suslin in $L(\mathbb{R}^*, Hom^*)\}$

This is the one we are interested in this week. Woodin proved this using stationary tower forcing.

Theorem (Woodin, New Derived Model Theorem) Let λ be a limit of Woodin cardinals, let $H \subseteq Col(\omega, <\lambda)$ be V-generic. Let $V(\mathbb{R}^*) = HOD^{(V[H], V)}_{V \cup \mathbb{R}^* \cup \{\mathbb{R}^*\}}$ and $\mathbb{R}^*$, Hom^* be as usual.

Let $\mathcal{A} = \{B \subseteq \mathbb{R}^* \mid B \in V(\mathbb{R}^*)$ and $L(B, \mathbb{R}^*) \models AD^+\}$

Then, (1) For $B, C \in \mathcal{A}$, either $L(B, \mathbb{R}^*) \subseteq L(C, \mathbb{R}^*)$ or $L(C, \mathbb{R}^*) \subseteq L(B, \mathbb{R}^*)$.

(2) $L(\mathcal{A}, \mathbb{R}^*) \models AD^+$.

Are there derived models of AD that are not of the form $L(P(\mathbb{R}))$?

→ Larson-Sargsyan-Wilson model of AD + all sets are universally Baire

→ we will see examples in this tutorial of the form $L(uB_p)$, $L(y^\omega, uB_p)$, $L(\mathbb{R}, uB)[\mathcal{C}]$ "Chang-type models"

$\mathcal{C}$: club filter on $\mathcal{P}_{\omega_1}(uB)$

→ Nairian models in Grigor's tutorial

Why should we care about such models of determinacy?

→ Models of AD are heavily used in Core Model Induction to climb up the Large Cardinal hierarchy (e.g., to obtain lower bounds for PFA)

A current obstacle in this methods is to construct canonical models of AD with non-trivial sets of sets of reals/ structure above their Θ.
(see Sargsyan-Trang, "The exact consistency strength of Sealing")

→ More recently, models of AD have been very successfully used as ground models for forcing via Woodin's $\mathbb{P}_{max}$

(Larson, Sargsyan) Failure of K^c iterability conjecture below WLW

(Blue, Larson, Sargsyan) Extensions of this via Nairian models, Chang-type models of AD
$L(X_\gamma)$ for $X_\gamma = \bigcup_{\beta<\gamma} (\mathrm{HOD}|\beta)^\omega$ for an ordinal $\gamma < \Theta$

[See Grigor's tutorial]

Again, the key is to construct canonical models of AD with non-trivial sets of sets of reals / structure above their Θ.

One issue here: These Chang-type models of AD and other strong determinacy model, e.g. LSA models, are currently extremely difficult to obtain.
The proof of their existence uses very deep hod mouse theory.

Our arguments will be hod mouse free, they use supercompact cardinals.

Goals in this tutorial

I. "Elementary" proof of DMT (Steel)
(using genericity iterations)

II. "Elementary" proof of Sealing (M.-Sargsyan)
after collapsing a supercompact
(representing $L(\Gamma^\infty, \mathbb{R})$ as a derived model)

III. Using this to construct a model of determinacy
(and Sealing) with non-trivial sets of sets of reals

IV. If time allows, Chang-type extensions and LSA
(see also Koschat's talk)

I. Proving the DMT with genericity iterations

Background / Preliminaries

Idea: A set is homogeneously Suslin in case it is continuously reducible to wellfoundedness of towers of measures.

We start by introducing the measures and homogeneity.

DEFINITION 2.1. For any Z, $\text{meas}_\kappa(Z)$ is the set of all κ-additive measures on $Z^{<\omega}$. We let $\text{meas}(Z) = \text{meas}_{\omega_1}(Z)$.

Clearly, if $\mu \in \text{meas}(Z)$, then there is exactly one $n < \omega$ such that $\mu(Z^n) = 1$. We call n the *dimension* of μ, and write $n = \dim(\mu)$.

If $\mu, \nu \in \text{meas}(Z)$, then we say that μ *projects* to ν iff for some $m \leq n < \omega$, $\dim(\mu) = n$, $\dim(\nu) = m$, and for all $A \subseteq Z^m$

$$\nu(A) = \mu(\{u \mid u \restriction m \in A\}).$$

We say μ and ν are *compatible* if one projects to the other. If μ projects to ν, then there is a natural embedding

$$\pi_{\nu,\mu} \colon \text{Ult}(V, \nu) \to \text{Ult}(V, \mu)$$

given by $\pi([f]_\nu) = [f^*]_\mu$, where $f^*(u) = f(u \restriction m)$ for all $u \in Z^n$.

A *tower of measures* on Z is a sequence $\langle \mu_n \mid n < k \rangle$, where $k \leq \omega$, such that each $\mu_n \in \text{meas}(Z)$, and whenever $m \leq n < k$, then $\dim(\mu_n) = n$ and μ_n projects to μ_m. If $\langle \mu_n \mid n < \omega \rangle$ is an infinite tower of measures, then

$$\text{Ult}(V, \langle \mu_n \mid n < \omega \rangle) = \text{ dir lim}_{n<\omega} \text{Ult}(V, \mu_n),$$

where the direct limit is taken using the natural embeddings π_{μ_n,μ_m}, which commute with one another[1]. We say that the tower $\langle \mu_n \mid n < \omega \rangle$ is *countably complete* just in case whenever $\mu_n(A_n) = 1$ for all $n < \omega$, then $\exists f \; \forall n \; (f \restriction n \in A_n)$. It is easy to show that $\langle \mu_n \mid n < \omega \rangle$ is countably complete if and only if $\text{Ult}(V, \langle \mu_n \mid n < \omega \rangle)$ is wellfounded, and so we shall say that a tower is wellfounded just in case it is countably complete.

DEFINITION 2.2. A *homogeneity system over Y with support Z* is a function

$$\bar{\mu} \colon Y^{<\omega} \to \text{meas}(Z)$$

such that, writing $\mu_s = \bar{\mu}(s)$, we have that for all $s, t \in Y^{<\omega}$,

1. $\dim(\mu_t) = \text{dom}(t)$, and
2. $s \subseteq t \Rightarrow \mu_t$ projects to μ_s.

If $\text{ran}(\bar{\mu}) \subseteq \text{meas}_\kappa(Z)$, then we say that $\bar{\mu}$ is *κ-complete*.

DEFINITION 2.3. If $\bar{\mu}$ is a homogeneity system over Y with support Z, then for each $x \in Y^\omega$, we let $\vec{\mu}_x$ be the tower of measures $\langle \mu_{x \restriction n} \mid n < \omega \rangle$, and set

$$S_{\bar{\mu}} = \{x \in Y^\omega \mid \vec{\mu}_x \text{ is countably complete}\}.$$

DEFINITION 2.4. Let $A \subseteq Y^\omega$; then A is *κ-homogeneous* iff $A = S_{\bar{\mu}}$, for some κ-complete homogeneity system $\bar{\mu}$. We say A is *homogeneous* if it is κ-homogeneous for some κ.

Definition 2.8. A *weak homogeneity system over Y with support Z* is an injective function $\bar{\mu}\colon Y^{<\omega} \to \operatorname{meas}(Z)$ such that for all $s \in Y^{<\omega}$

1. $\dim(\mu_s) \leq \operatorname{dom}(s)$, and
2. if μ_s projects to ν, then $\exists i(\mu_{s\restriction i} = \nu)$.

Definition 2.9. If $\bar{\mu}$ is a (κ-complete) weak homogeneity system over Y, then we set

$$W_{\bar{\mu}} = \Big\{x \in Y^\omega \mid \exists\langle i_k \mid k < \omega\rangle \in \omega^\omega\big(\langle \mu_{x\restriction i_k} \mid k < \omega\rangle \text{ is a wellfounded tower}\big)\Big\},$$

and say that $W_{\bar{\mu}}$ is (κ-)*weakly homogeneous* via $\bar{\mu}$.

So a weak homogeneity system over Y associates continuously to each $x \in Y$ a countable tree of towers of measures, and x is in the set being represented iff at least one of the branches of this tree is a wellfounded tower[2].

Definition 2.15. Let T on $X \times Y$ and U on $X \times Z$ be two trees; then we say T and U are *κ-absolute complements* iff whenever G is $<\kappa$-generic over V

$$V[G] \models p[T] = X^\omega \setminus p[U].$$

We say *T is κ-absolutely complemented* iff $\exists U$ (T and U are κ-absolute complements).

If $p[T] \cap p[U] = \emptyset$ in V, then the same is true in any generic extension of V by the absoluteness of wellfoundedness. We shall use this simple observation again and again. What absolute complementation adds is that T and U are sufficiently "fat" that in the relevant $V[G]$, we have $p[T] \cup p[U] = X^\omega$.

Definition 2.16. (1) A set $A \subseteq X^\omega$ is *κ-universally Baire*, or *κ-absolutely Suslin* iff $A = p[T]$ for some κ-absolutely complemented T.
(2) $UB_\kappa = \{A \subseteq \omega^\omega \mid A \text{ is } \kappa\text{-universally Baire}\}$.

Theorem (Martin-Solovay, Woodin, Martin-Steel):

let $A \subseteq \mathbb{R}$, δ cardinal.

A δ-homogeneous
$\Downarrow$
A δ-weakly homogeneous
$\Downarrow$
A is δ-universally Baire

key ingredient: Martin-Solovay tree

A δ^+-universally Baire, δ Woodin
$\Downarrow$ Woodin gave a stationary tower proof of this but Steel has a stationary tower-free proof
A $<\delta$ weakly homogeneous

A δ^+-weakly homogeneous, δ Woodin
$\Downarrow$
A $<\delta$-homogeneous

Definition 2.17. Let $\bar{\mu}$ be a homogeneity system over Y. For any ordinal θ, we define the Martin-Solovay tree $ms(\bar{\mu}, \theta)$ on $Y \times \mathrm{OR}$ by

$$\big(s, \langle \alpha_n \mid n < e\rangle\big) \in ms(\bar{\mu}, \theta) \Leftrightarrow$$
$$s \in Y^e \wedge \alpha_0 < \theta \wedge \forall n\big(n+1 < e \Rightarrow \pi_{\mu_{s\restriction n}, \mu_{s\restriction(n+1)}}(\alpha_n) > \alpha_{n+1}\big).$$

That is, $ms(\bar{\mu}, \theta)_x$ searches for a proof that $\operatorname{Ult}(V, \bar{\mu}_x)$ is illfounded below the image of θ. (This last restriction makes it a set, rather than a proper class.) It is not hard to see that if $\bar{\mu}$ has support Z and is illfounded, then

↳ Martin-Solovay tree for hom. systems, there is a more complicated version from weak hom. system

Corollary: If λ is a limit of Woodin cardinals,

$\mathrm{Hom}_{<\lambda} = uB_\lambda$.

Recall that for any Z, X and any ordinal γ, $\text{meas}_\gamma(Z)$ denotes the set of all γ-additive measures on $Z^{<\omega}$. We write $\bar{\mu} = (\mu_s \mid s \in X^{<\omega})$ for a γ-complete *homogeneity system* over X with support Z if for each $s \in X^{<\omega}$, $\mu_s \in \text{meas}_\gamma(Z)$. For details on the definition of homogeneity systems we refer the reader to [22].

Definition 2.1. A set $A \subseteq X^\omega$ is γ-homogeneously Suslin if there is a γ-complete homogeneity system $\bar{\mu} = (\mu_s \mid s \in X^{<\omega})$ and a tree $\underbrace{T}_{\text{on } X^{x}\ldots}$ such that $A = p[T]$ and, for all $s \in X^{<\omega}$, $\mu_s(T_s) = 1$. In particular,

$$A = S_{\bar{\mu}} =_{def} \{x \in X^\omega \mid (\mu_{x \upharpoonright n} \mid n < \omega) \text{ is well-founded}\}.$$

i.e, the natural direct limit of $Ult(V, \mu_{x\upharpoonright n})$ for $n<\omega$ is well-founded.

Here $T_s = \{t \mid (s,t) \in T\}$. We write

$$\text{Hom}_\gamma = \{A \subseteq X^\omega \mid A \text{ is } \gamma\text{-homogeneously Suslin}\}$$

and

$$\text{Hom}_{<\eta} = \bigcap_{\gamma<\eta} \text{Hom}_\gamma .$$

For η a limit of Woodin cardinals and $H \subseteq \text{Col}(\omega, {<\eta})$ generic over V, write $H \upharpoonright \alpha = H \cap \text{Col}(\omega, {<\alpha})$. As usual, let

$$\mathbb{R}^* = \mathbb{R}^*_H = \bigcup_{\alpha<\eta} \mathbb{R} \cap V[H \upharpoonright \alpha].$$

Moreover, for any $\alpha < \eta$ and $A \in \text{Hom}_{<\eta}^{V[H\upharpoonright\alpha]}$, write

$$A^* = \bigcup_{\alpha<\beta<\eta} A_{H\upharpoonright\beta}$$

Use here that $A \in \text{Hom}_{<\eta}^{V[H\upharpoonright\alpha]}$ is η-uB in $V[H\upharpoonright\alpha]$.

and

$$\text{Hom}^* = \{A^* \mid \exists \alpha < \eta\, A \in (\text{Hom}_{<\eta})^{V[H\upharpoonright\alpha]}\}.$$

$L(\mathbb{R}^*, \text{Hom}^*)$ is called a derived model of V at η.

We will also say "the" derived model satisfies some formula φ as (by homogeneity of the forcing) the theory of $L(\mathbb{R}^*, \text{Hom}^*)$ does not depend on the choice of H.

The Derived Model Theorem

THEOREM 7.1 (Derived model theorem, Woodin). *Let λ be a limit of Woodin cardinals, and $L(\mathbb{R}^*, \mathrm{Hom}^*)$ be a derived model at λ; then*

(1) $L(\mathbb{R}^*, \mathrm{Hom}^*) \models \mathsf{AD}^+$,
(2) $\mathrm{Hom}^* = \{A \subseteq \mathbb{R}^* \mid A \text{ is Suslin and co-Suslin in } L(\mathbb{R}^*, \mathrm{Hom}^*)\}$.

AD^+ is the theory $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ + Ordinal Determinacy + "all sets of reals are ∞-Borel". These are local consequences of scales[6].

What is the strategy to prove this?
(both in the stationary tower and in the genericity iteration argument)

<u>Theorem</u> (Woodin, Reflection Lemma):
Let G be $\mathrm{Col}(\omega, <\lambda)$-generic over V where λ is a limit of Woodin cardinals. Let $A \in \mathrm{Hom}_{<\lambda}^{V[G \restriction \alpha]}$, where $\alpha < \lambda$.
Let φ be a sentence in the language of set theory with two additional unary predicate symbols and s.t. that
$$\exists B \subseteq \mathbb{R}^* \, (B \in L(\mathbb{R}^*, \mathrm{Hom}^*) \wedge (HC^*, \in, A^*, B) \models \varphi),$$
then
$$\exists B \, (B \in \mathrm{Hom}_{<\lambda}^{V[G\restriction\alpha]} \wedge (HC^{V[G\restriction\alpha]}, \in, A, B) \models \varphi).$$

$L(\mathbb{R}^*, \mathrm{Hom}^*) \models \mathsf{AD}$ follows easily from the Reflection Lemma:

Before proving Lemma 7.4, let us use it to complete the proof of the derived model theorem. So let G be $\mathrm{Col}(\omega, <\lambda)$-generic over V, where λ is a limit of Woodins, and $\mathbb{R}^* = \mathbb{R}^*_G$ and $\mathrm{Hom}^* = \mathrm{Hom}^*_G$. We show first that $L(\mathbb{R}^*, \mathrm{Hom}^*) \models \mathsf{AD}$. For if not, there is a $B \in L(\mathbb{R}^*, \mathrm{Hom}^*)$ such that

$$(\mathrm{HC}^*, \in, B) \models \text{ the game with payoff } B \text{ is not determined.}$$

By Lemma 7.4, we can find $B \in \mathrm{Hom}^V_{<\lambda}$ such that

$$(\mathrm{HC}, \in, B) \models \text{ the game with payoff } B \text{ is not determined.}$$

This contradicts Martin's Theorem 2.7.

The first and most important fact about homogeneously Suslin sets is

THEOREM 2.7 (Martin [4], essentially). *If $A \subseteq Y^\omega$ is $|Y|^+$-homogeneous, then the two-person game of perfect information on Y with payoff set A is determined.*

Proof of the rest of the DMT from the Reflection Lemma:

The remaining axioms of AD^+ are true in $L(\mathbb{R}^*, \mathrm{Hom}^*)$ for similar reasons. In each case the axiom can be expressed in the form "$\forall B \subseteq \mathbb{R}(\mathrm{HC}, \in, B) \models \varphi$", and there are no $\mathrm{Hom}_{<\lambda}$ sets B such that $(\mathrm{HC}^V, \in, B) \models \varphi$. For the axiom $\mathsf{DC}_{\mathbb{R}}$ both parts are obvious. The other two axioms have the form $\forall B \subseteq \mathbb{R}\exists C \subseteq \mathrm{OR} \ldots$, but using the Coding Lemma the quantifier on C can be reduced to a real quantifier over the field of a prewellorder which is projective in B. For Ordinal Determinacy, this is obvious, but for the assertion that B has an infinity-Borel code C, we need a preliminary argument which bounds the least size of such a code by some ordinal projective in B. This can be done[10]. Finally, the fact that there are no $\mathrm{Hom}_{<\lambda}$ counterexamples B to Ordinal Determinacy or the assertion that every set of reals is ∞-Borel follows from the fact that every $\mathrm{Hom}_{<\lambda}$ set has a $\mathrm{Hom}_{<\lambda}$ scale, together with $\mathrm{Hom}_{<\lambda}$- determinacy[11].

To see that all Hom^* sets are Suslin in $L(\mathbb{R}^*, \mathrm{Hom}^*)$, fix C in Hom^*. We then have $A \in \mathrm{Hom}_{<\lambda}^{V[G\restriction\alpha]}$, for some $\alpha < \lambda$, such that $C = A^*$. By Theorem 5.3 there is $B \in \mathrm{Hom}_{<\lambda}^{V[G\restriction\alpha]}$ which codes a scale on A. This fact can be expressed using only real quantifiers, and thus by Lemma 7.3, B^* codes a scale on A^* in $L(\mathbb{R}^*, \mathrm{Hom}^*)$, so C is Suslin in $L(\mathbb{R}^*, \mathrm{Hom}^*)$, as desired. Since Hom^* is closed under complement, all Hom^* sets are co-Suslin in $L(\mathbb{R}^*, \mathrm{Hom}^*)$.

Conversely, suppose A is Suslin and co-Suslin in $L(\mathbb{R}^*, \mathrm{Hom}^*)$, and let T and U be the trees which witness this. We can fix a set $C \in \mathrm{Hom}^*$ such that T and U are ordinal definable over $L(\mathbb{R}^*, \mathrm{Hom}^*)$ from C. (Every set in $L(\mathbb{R}^*, \mathrm{Hom}^*)$ has this form.) We then have $W \in V[G \restriction \alpha]$, where $\alpha < \lambda$, such that $C = p[W] \cap \mathbb{R}^*$. It follows that T and U are definable in $V[G]$ from the parameter $\mathbb{R}^*$ and parameters in $V[G \restriction \alpha]$. But $V[G] = V[G \restriction \alpha][H]$ where H is generic for $\mathrm{Col}(\omega, <\lambda)$, and there is a term τ such that $\tau_H = \mathbb{R}^*$ and $\mathrm{Col}(\omega, <\lambda)$ is homogeneous with respect to τ, in that $\forall p, q \exists \pi(\pi$ is an automorphism of $\mathrm{Col}(\omega, <\lambda)$ and $\pi(p)$ is compatible with q and $\pi\tau = \tau)$. Since T and U are subsets of $V[G \restriction \alpha]$, we have that $T, U \in V[G \restriction \alpha]$. But now T and U project to complements over $\mathbb{R}^*$, and hence in any $V[G \restriction \beta]$ for $\beta < \lambda$. Since the collapse forcing is universal, this implies that T and U are $<\lambda$-absolute complements in $V[G \restriction \alpha]$. Thus $p[T] \in \mathrm{Hom}^*$, as desired. This completes the proof of the derived model theorem, modulo Lemma 7.4.

We will sketch Steel's genericity iteration proof of the Reflection lemma

Steel's stationary-tower-free proof uses the following:

(*) THEOREM 5.3 (Steel). *Let λ be a limit of Woodin cardinals; then every* $\mathrm{Hom}_{<\lambda}$ *set has a* $\mathrm{Hom}_{<\lambda}$ *scale.*

As noticed by Schlutzenberg, the original proof of this theorem also uses the stationary tower. Schlutzenberg-Steel found a new proof of (*) not using the stationary tower.

Another ingredient: Windßus' lemma

Let us say that an iteration tree $\mathcal{T}$ is 2^ω- *closed* iff for all α, $\mathcal{M}_\alpha^{\mathcal{T}} \models$ " $\mathrm{Ult}(V, E_\alpha^{\mathcal{T}})$ is closed under 2^ω-sequences". We say that $\mathcal{T}$ is *above* μ if $\mathrm{crit}(E_\alpha^{\mathcal{T}}) > \mu$ for all α. The following lemma is essentially due to K. Windszus. (See [1].)

Lemma 1.1 *Let $\pi\colon M \to V_\theta$ be elementary, where M is countable and transitive and let $\mu \in M$. Put*

$$W = \{\mathcal{T} \mid \mathcal{T} \text{ is a } 2^\omega\text{-closed iteration tree on } M \text{ of length } \omega+1,\ \mathcal{T} \text{ is above } \mu \text{ and } \mathcal{M}_\omega^{\pi\mathcal{T}} \text{ is wellfounded}\}$$

Then W is $\pi(\mu)$-homogeneously Suslin.

We omit the proof here. It can, for example, be found in Steel's notes on the stationary-tower-free proof of the DMT.

Our first main tool: Flipping Functions

We will need the following slight generalization of *flipping functions*, see [21, Lemma 2.1]. As usual, if $Y \subseteq \text{meas}_\gamma(Z)$ for some γ and Z, we write TW_Y for the set of all towers of measures $\vec{\mu} = (\mu_i \mid i < \omega)$ such that $\mu_i \in Y$ for each $i < \omega$.

Lemma 2.5. *Let δ be a Woodin cardinal and let $Y \subseteq \text{meas}_{\delta^+}(Z)$ be such that $|Y| < \delta$. Then for any $\gamma < \delta$, there is some Z' and $R \subseteq \text{meas}_\gamma(Z')$ as well as a Lipschitz function*

$$f\colon \text{TW}_Y \to \text{TW}_R$$

such that

(1) f is 1-to-1, and

(2) for all $<\gamma$-generics G and all $\vec{\mu} \in (\text{TW}_Y)^{V[G]}$,

$$\vec{\mu} \text{ is well-founded} \iff f(\vec{\mu}) \text{ is ill-founded.}$$

Here recall that a function $f\colon \text{TW}_Y \to \text{TW}_R$ is Lipschitz if the value of $f(\vec{\mu}) \restriction n$ is determined by $\vec{\mu} \restriction n$, for all $\vec{\mu}$ and n. Moreover, since the size of the forcing is small, f induces a map $f\colon (\text{TW}_Y)^{V[G]} \to (\text{TW}_R)^{V[G]}$.

Proof of the Flipping lemma:

Working in V (this is where we need to construct f), pick for each ill-founded tower $\vec{\mu} \in \text{TW}_Y$ sets $A_i^{\vec{\mu}} \in \mu_i$ witnessing the fact that $\vec{\mu}$ is not countably complete (i.e., is ill-founded).
That means, there is no f s.t. $\forall i\, (f \restriction i \in A_i^{\vec{\mu}})$.

For any finite tower ν of Y-measures and $i < \text{lh}(\nu)$, let

$$B_i^\nu = \bigcap \{A_i^{\vec{\mu}} \mid \nu \subseteq \vec{\mu}\}.$$

As $|Y| < \delta$ and Y-measures are δ^+-additive, $B_i^\nu \in \nu_i$ for all i (each $A_i^{\vec{\mu}} \in \mu_i = \nu_i$ for $i < \text{lh}(\nu)$, $\nu \subseteq \vec{\mu}$).

Let $(\nu, t) \in T$ iff $\forall i < k\, (t \restriction i \in B_i^\nu)$ for arbitrary finite towers ν of Y-measures and $t \in Y^{<\omega}$.

Then T is a tree and $p[T]$ is the set of wellfounded $\vec{\mu} \in \text{TW}_Y$, both in V and in generic extensions by forcings of size $< \delta$.

Moreover, T is δ^+-homogeneous (where for each T_ν the homogeneity measure is given by the last measure in ν).

Note that T is a tree on $R \times U$ for some set R of size $< \delta$.

As T is δ^+-homogeneous and δ is Woodin, we can let S be the Martin-Solovay tree projecting to $\text{TW}_Y \setminus p[T]$. By Martin-Steel (proof of PD), S is γ-homogeneous for some $\gamma < \delta$, say this is witnessed by a hom. system

$Y^{<\omega} \to \text{meas}_\gamma(Z')$, $\vec{\nu} \mapsto f(\vec{\nu})$ ("finite Y-towers"). Then f determines the desired Lipschitz function.

Note that we can arrange the Martin-Solovay tree such that f is 1-to-1. □

Our second main tool: Genericity Iterations

The central tool that we will use from coarse inner model theory, besides general iterability results as in [8], is Neeman's genericity iteration. We recall the statement here for the reader's convenience.

Definition 2.3 (Neeman, [11]). Let M be a model of ZFC, x a real, and $\mathbb{P} \in M$ a partial order. An iteration tree $\mathcal{T}$ on M is said to *absorb x to an extension by an image of* $\mathbb{P}$ if for every well-founded cofinal branch b through $\mathcal{T}$, there is a generic extension $M_b^{\mathcal{T}}[g]$ of $M_b^{\mathcal{T}}$, the final model along b, by the partial order $j_{0,b}^{\mathcal{T}}(\mathbb{P})$ so that $x \in M_b^{\mathcal{T}}[g]$.

Theorem 2.4 (Neeman, [10, 11]). *Let M be a model of* ZFC*, let δ be a Woodin cardinal in M such that $\wp^M(\delta)$ is countable in V. Then for every real x there is an iteration tree $\mathcal{T}$ of length ω on M which absorbs x into an extension by an image of* $\mathrm{Col}(\omega, \delta)$.

(There is another genericity iteration due to Woodin which we will not need here.)

Proofs of these genericity iteration results can be found in Neeman's (and Steel's) chapters in the handbook of set theory.

Proof of the Reflection Lemma (sketched)

Proof. We may as well assume $A \in \mathrm{Hom}^V_{<\lambda}$.

First find $B \in L(\mathbb{R}^V, \mathrm{Hom}^V_\lambda)$.

$\boxed{\textbf{Claim 1.}\ \text{For some } B \in L(\mathbb{R}^V, \mathrm{Hom}^V_\lambda),\ (\mathrm{HC}, \in, A, B) \models \varphi.}$

Proof. Fix a $<\lambda$- absolutely complemented pair (S, U) such that $A = p[S]$. Let

$$\pi\colon M \to V_\theta,$$

where θ is sufficiently large and M is countable transitive, with $\pi((\bar{S}, \bar{U}, \bar{\lambda}) = (S, U, \lambda)$. Working in $V^{\mathrm{Col}(\omega,\mathbb{R})}$, we can use the genericity iterations of [4] to form an $\mathbb{R}$-genericity iteration of M, below $\bar{\lambda}$, that is, a sequence

$$I = \langle \mathcal{T}_n \mid n < \omega \rangle$$

such that the $\mathcal{T}_n$ are length $\omega + 1$ iteration trees whose composition

$$\mathcal{T} = \oplus_n \mathcal{T}_n$$

is a normal iteration tree on M, with

$$M_\omega = \lim_n M_n,$$

the direct limit along the main branch of $\mathcal{T}$ (where M_n is the base model of $\mathcal{T}_n$, and the last model of $\mathcal{T}_{n-1}$ if $n > 0$), being such that $\mathbb{R}^V$ is the reals of a symmetric collapse over M_ω below λ_ω, the image of $\bar{\lambda}$. Let

$$i_{n,k}\colon M_n \to M_k$$

be the canonical embedding, for $0 \le n \le k \le \omega$, and $\lambda_k = i_{0,k}(\lambda_0)$. We write

$$\mathrm{Hom}^*_I = \bigcup\{p[T] \cap \mathbb{R}^V \mid \underset{\exists k}{\exists x} \in \mathbb{R}^V (M_\omega^{[x_i \mid i<k]} \models T \text{ is } < \lambda_\omega \text{ absolutely complemented})\},$$

so that $L(\mathbb{R}^V, \mathrm{Hom}^*_I)$ is a derived model of M_ω at λ_ω whose set of reals is $\mathbb{R}^* = \mathbb{R}^V$. Because our individual genericity iterations $\mathcal{T}_n$ have length $\omega + 1$, M is iterable enough that we can do them, realizing the M_n and M_∞ in V_θ in the process. Thus we have realizing maps

$$\sigma_k\colon M_k \to V_\theta,$$

for all $k \le \omega$, such that

$$\sigma_n = \sigma_k \circ i_{n,k}$$

whenever $n \le k \le \omega$. ($\sigma_0 = \pi$.) Finally, we arrange that there is an increasing sequence ordinals δ_k, $k < \omega$, with sup λ_ω, such that

$$\delta_k < \mathrm{crit}(i_{k,\omega}),$$

together with M_k-generic objects g_k for $\mathrm{Col}(\omega, \delta_k)$, such that

$$\mathbb{R}^V = \bigcup_{k<\omega} \mathbb{R} \cap M_k[g_k],$$

and $g_k \in M_n[g_n]$ if $k < n$. If $k \le n \le \omega$, then $i_{k,n}$ lifts to an embedding

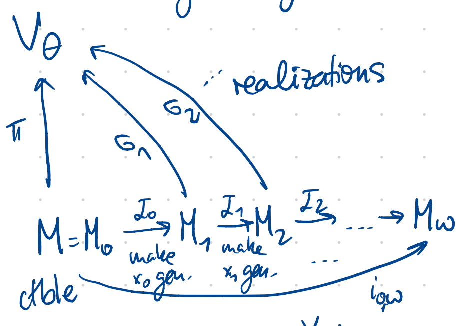
$$i_{k,n}\colon M_k[g_k] \to M_n[g_k],$$

moreover, $\mathbb{R}^V = \bigcup_{k<\omega} M_\omega[g_k]$.

Since $\sigma_\omega \circ i_{0,\omega}((\bar{S}, \bar{U})) = (S, U)$, we easily get that $\mathbb{R}^V \cap p[i_{0,\omega}(\bar{S})] = A$. Written another way, $i_{0,\omega}(\bar{A})^* = A$. The claim will then follow from the elementarity of $i_{0,\omega}$, provided we can show Hom^*_I is a Wadge initial segment of $\mathrm{Hom}^V_{<\lambda}$. Since Hom^*_I is closed downward under Wadge reduction, it suffices to show:

$\mathbb{R}$-genericity iteration:

Fix a generic enumeration $\langle x_i : i<\omega \rangle$ of $\mathbb{R}^V$. Build iteration I via genericity iterations.



$\mathrm{Hom}^*_I = \bigcup\{p[T] \cap \mathbb{R}^V \mid$
$M_\omega \models T$ is $<\lambda_\omega$ absolutely complemented$\}$

Subclaim 1.1. $\mathrm{Hom}^*_I \subseteq \mathrm{Hom}^V_{<\lambda}$.

↳ The proof of this subclaim is when flipping functions get used. We will see flipping functions in action in the proof of the $\mathcal{H}$-Sargsyan derived model representation, so we omit the argument here.

The Subclaim finishes the proof of Claim 1.

Let us write $\mathrm{Hom} = \mathrm{Hom}^V_{<\lambda}$, and $\mathrm{Hom} \restriction \alpha$ for the collection of sets in Hom having Wadge rank $< \alpha$. By Claim 1, we have a lexicographically least pair $\langle \alpha, \beta \rangle\gamma$ such that there is a $B \in L_\beta(\mathrm{Hom} \restriction \alpha)$ such that $(\mathrm{HC}, \in, A, B) \models \varphi$. Let $\langle \alpha_0, \beta_0 \rangle\gamma_0$ be this pair. Let $C \in \mathrm{Hom} \restriction \alpha_0$ be such that some such B is ordinal definable over $L_{\beta_0}(\mathrm{Hom} \restriction \alpha_0)$ from the parameter $(A, C)\gamma_0$. We can eliminate the need for the ordinals by minimizing them, and as a result we can fix B such that $(\mathrm{HC}, \in, A, B) \models \varphi$, and a formula ψ such that

$$x \in B \Leftrightarrow L_{\beta_0}(\mathrm{Hom} \restriction \alpha_0) \models \psi[x, (A, C)].$$

for φ with ordinal parameter γ

Second, minimize ordinal parameters to get $B \in \mathrm{Hom}^V_{<\lambda}$.

Showing that B is $\mathrm{Hom}^V_{<\lambda}$ now completes the proof. $\square$

II. Proving Sealing with genericity iterations

(from a supercompact)

Sealing

Shoenfield's Absoluteness Theorem implies that Σ^1_2-facts are forcing absolute.

<u>Theorem</u> (Steel, Woodin):

Suppose there is a proper class of Woodin cardinals. Let $V[g] \subseteq V[g*h]$ be set-generic extensions of V. Then

(1) $L(\mathbb{R}) \models AD$ and there is an elementary embedding

$$j : L(\underbrace{\mathbb{R}_g}_{= \mathbb{R}^{V[g]}}) \longrightarrow L(\mathbb{R}_{g*h}),$$

(2) for any universally Baire set A, $L(\mathbb{R}, A) \models AD$ and there is an elementary embedding

$$j : L(\mathbb{R}_g, \underbrace{A_g}_{\substack{= \text{the canonical expansion} \\ \text{of } A \text{ from } V \text{ to } V[g], \text{ i.e. } p[T]^{V[g]} \text{ for } A = p[T]^V}}) \longrightarrow L(\mathbb{R}_{g*h}, A_{g*h}).$$

Does the same hold for the model with all uB sets?

<u>Definition</u> (Woodin)

Sealing is the conjunction of

(1) For any g set generic over V,

$$L(\underbrace{\mathbb{R}_g}_{= \mathbb{R}^{V[g]}}, \underbrace{uB_g}_{= (uB)^{V[g]}}) \models AD^+,$$

and $\mathcal{P}(\mathbb{R}_g) \cap L(\mathbb{R}_g, uB_g) = uB_g$, and

(2) for any g, h consecutive set generics over V, there is an elementary embedding

$$j : L(\mathbb{R}_g, uB_g) \longrightarrow L(\mathbb{R}_{g*h}, uB_{g*h})$$

such that $j(A) = A_h$ for any $A \in uB_g$.

<u>Remark</u>: Already (1) is not easy. E.g., (1) implies that Sealing cannot hold in mice [assuming that mice have an $L(\mathbb{R}, uB)$-wellorder of their reals.

The consistency of Sealing

Theorem (Woodin): Suppose there is a proper class of Woodin cardinals. Let κ be supercompact and g $Col(\omega, 2^\kappa)$-generic over V. Then Sealing holds in $V[g]$.

↳ Proved using stationary tower forcing

Our goal for this part of the tutorial:

Stationary-tower-free proof of Woodin's Sealing Theorem (for $g \subseteq Col(\omega, 2^{2^\kappa})$) using flipping functions and genericity iterations, due to M.-Sargsyan.

Theorem (Sargsyan-Trang):

Sealing is consistent from a Woodin limit of Woodin cardinals.

↳ Sargsyan-Trang identified the exact consistency strength of Sealing

↳ Their proof uses heavy hod mouse machinery

Theorem (Sargsyan-Trang):

Sps. there is a proper class of Woodin cardinals and a strong cardinal and suppose self-iterability holds. Then Sealing holds after collapsing the successor of the least strong cardinal to be countable.

Question: Is there a large cardinal that implies Sealing?

Our main goal is to prove the following derived model representation for $L(\Gamma^\infty_{g*h}, \mathbb{R}_{g*h})$

Write $uB = \Gamma^\infty$.

[M.-Sargsyan]

Theorem 1.10. *Let κ be a supercompact cardinal and suppose there is a proper class of Woodin cardinals. Let $g \subseteq \mathrm{Col}(\omega, 2^\kappa)$ be V-generic, h be $V[g]$-generic and $k \subseteq \mathrm{Col}(\omega, 2^\omega)$ be $V[g*h]$-generic. Then, in $V[g*h*k]$, there is $j : V \to M$ such that $j(\kappa) = \omega_1^{V[g*h]}$ and $L(uB_{g*h}, \mathbb{R}_{g*h})$ is a derived model of M, i.e., for some M-generic $G \subseteq \mathrm{Col}(\omega, <\omega_1^{V[g*h]})$,*

$$L(uB_{g*h}, \mathbb{R}_{g*h}) = (L(\mathrm{Hom}^*, \mathbb{R}^*))^{M[G]}.$$

The fact that $L(uB_{g*h}, \mathbb{R}_{g*h}) \vDash AD$ follows from this just as in the derived model theorem.

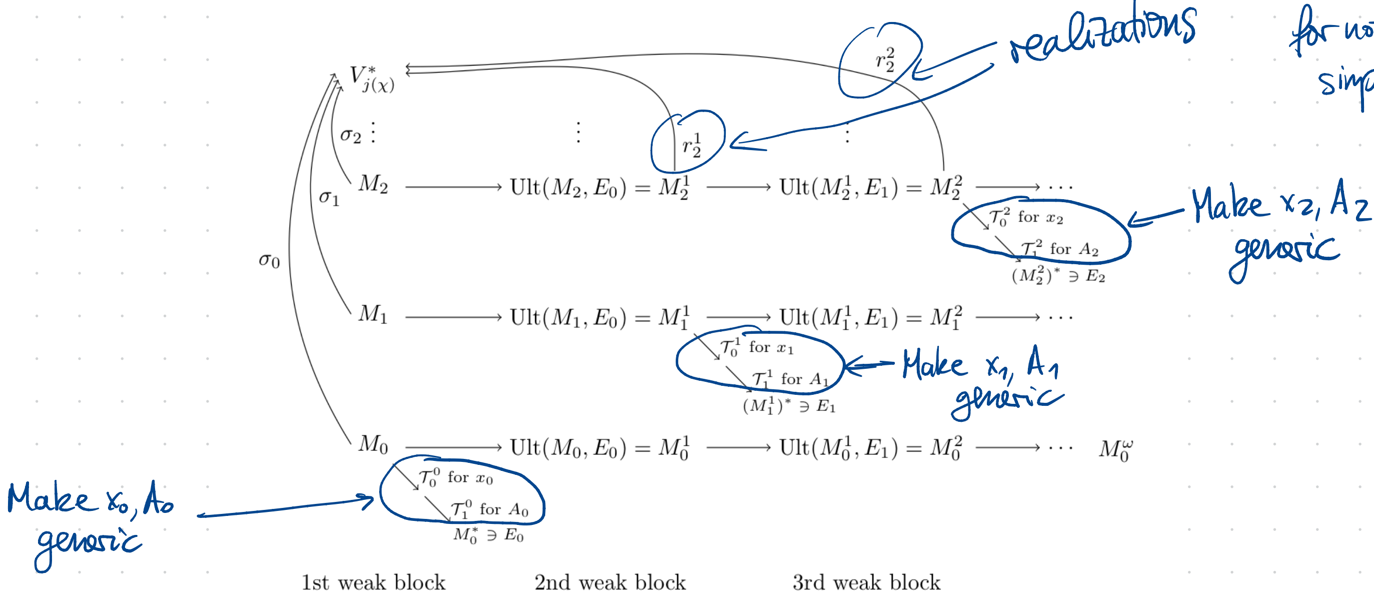
We will give the details how to obtain Sealing from the DMR later.

We now prove the Derived Model Representation (DMR)

We are aiming for the following picture: (inspired by Trang-Sargsyan proof from self-iterability)

Let $g \subseteq Col(\omega, 2^{2^{\kappa}})$.

Fix enumerations $(x_i \mid i<\omega)$ of $\mathbb{R}_g$ and $(A_i \mid i<\omega)$ of uB_g. (We omit the further forcing h for notational simplicity.)



Some points to keep in mind

- Need to ensure $A_i \in M_i$ for all i (in fact, a hom. system $\vec{\mu}_i$ for A_i)
- $V^*_{j(x)}$ need to be sufficiently large that we can keep realizing it to get iterability as in the Martin-Steel iterability proof
- We prove the Derived Model Representation for M_0^ω, can then apply the same extender to V to obtain the theorem as stated
- What does it mean to make a uB-set generic? We want to add a homogeneity system witnessing $A_i \in \mathrm{Hom}^*$, in particular, the homogeneity system needs to "survive" further ultrapowers and its completeness needs to increase with moving up λ.

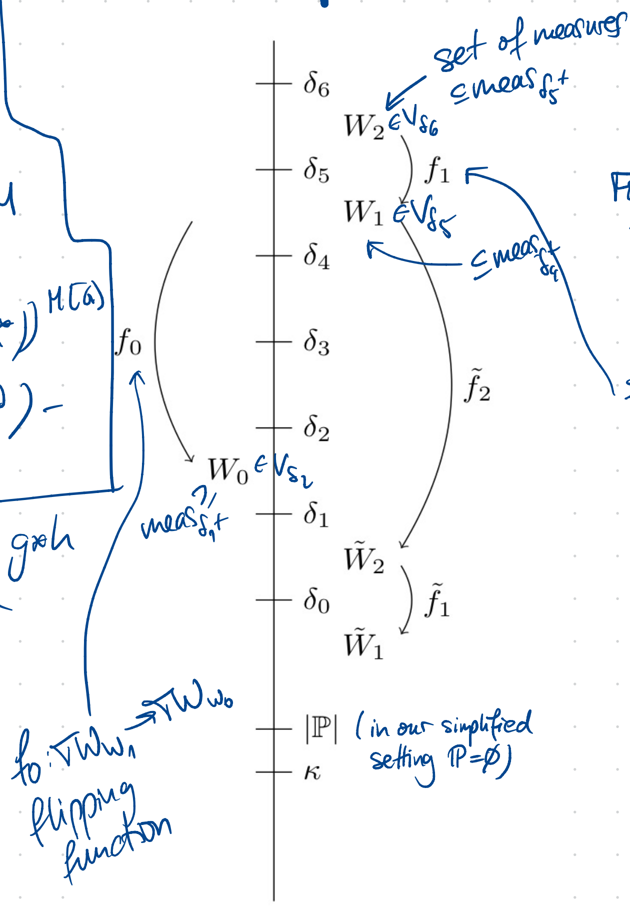
"stretching hom. systems"

The setup for proving the DMR

Let g be $\mathrm{Col}(\omega, 2^\kappa)$-generic over V.
For notational simplicity, we just show that there is an elem. emb. $j: V \to M$ s.t. $j(\kappa) = \omega_1^{V[g]}$ and $L(\mathbb{R}_g, uB_g) = (L(\mathbb{R}^*, \mathrm{Hom}^*))^{M[G]}$ for $G \subseteq \mathrm{Col}(\omega, <\omega_1^{V[g]})$-generic over M.

The general case for $g * h$ is a straightforward generalization.

The setup in V:



δ_i Woodin cardinals

$(x_i \mid i<\omega)$, $(A_i \mid i<\omega)$ enumerations of $\mathbb{R}_g$, uB_g

For each $i<\omega$, fix hom. system $\bar{\mu}_2^{(i)} = (\mu_s^{(i)} \mid s \in \omega^{<\omega}) \subseteq W_2$ s.t. $A_i = S_{\bar{\mu}_2^{(i)}}$.

$f_1: TW_{W_2} \to TW_{W_1}$ flipping function,

$\bar{\mu}_1^{(i)} = f_1'' \bar{\mu}_2^{(i)}$
$\bar{\mu}_0^{(i)} = f_0'' \bar{\mu}_1^{(i)}$,
$\bar{\mu}_2^{\prime,(i)} = \tilde{f}_2'' \bar{\mu}_1^{(i)}$,
$\bar{\mu}_1^{\prime,(i)} = \tilde{f}_1'' \bar{\mu}_1^{\prime,(i)}$.

Reflect the hom. systems to small substructures

Definition 4.1. Let $(\bar{\mu}_2^{(i)})_{i<\omega}$ be a sequence of homogeneity systems for homogeneously Suslin sets in $V[g]$ with measures induced by V-measures and let $f_1, f_0, \tilde{f}_2, \tilde{f}_1 \in V$ be functions on towers of measures as above. Then a $((\bar{\mu}_2^{(i)})_{i<\omega}, f_1, f_0, \tilde{f}_2, \tilde{f}_1)$-*block for* $V[g]$ *at* κ is a sequence

$$(\hat{M}_i, M_i, \pi_i, g_i \mid i < \omega)$$

of models $\hat{M}_i$ and M_i as well as M_i-generics g_i together with elementary embeddings

$$\pi_i : M_i \to V_\chi$$

such that for all $i < \omega$,

(1) $\hat{M}_i = M_i[g_i]$ is countable in $V[g]$,
(2) $\pi_i \in V[g]$,
(3) $\bar{\mu}_2^{(i)} \subseteq \mathrm{rng}(\pi_i)$, $f_1, f_0, \tilde{f}_2, \tilde{f}_1 \in \mathrm{rng}(\pi_0)$,
(4) $\mathrm{rng}(\pi_i) \subseteq \mathrm{rng}(\pi_{i+1})$, and
(5) $\pi_i \upharpoonright \wp(\kappa)^{M_i} = \mathrm{id}$.

identifying the measures in $\bar{\mu}_2^{(i)}$ with the V-measures they are coming from

We aim to "iterate these blocks" pointwise.

BUT we are not assuming iterability for V.

Details of the setup

Let g be $\mathrm{Col}(\omega, 2^\kappa)$-generic over V. Fix some set-sized partial order $\mathbb{P}$ and some $\mathbb{P}$-generic h over $V[g]$. Our aim in this section is to find a model M and an elementary embedding $j\colon V \to M$ such that $\omega_1^{V[g*h]}$ is a limit of Woodin cardinals in M and

$$L(\Gamma^\infty_{g*h}, \mathbb{R}_{g*h}) = L(\mathrm{Hom}^*, \mathbb{R}^*)^{M[G]},$$

where G is $\mathrm{Col}(\omega, {<}\omega_1^{V[g*h]})$-generic over M. To obtain the desired M and j we will first obtain $L(\Gamma^\infty_{g*h}, \mathbb{R}_{g*h})$ as a derived model of a model M_0^ω, where M_0^ω is an iterate of a small substructure of V, see Corollary 4.14 at the end of this section. Theorem 1.10 will be an immediate consequence of the proof of Corollary 4.14.

For notational simplicity, we assume that $\mathbb{P}$ is trivial and omit h from the argument in most of what follows. The more general case works similarly and we will use $\mathbb{P}$ at certain places in the argument below to indicate where some care is needed for the general argument.

Let g' be $\mathrm{Col}(\omega, \Gamma^\infty_g)$-generic over $V[g]$ and let $(A_i \mid i < \omega)$ be an enumeration of Γ^∞_g in $V[g * g']$. Moreover, let $(x_i \mid i < \omega)$ be an enumeration of $\mathbb{R}^{V[g]}$. Let $\delta_0 > \kappa$ be a Woodin cardinal such that $\mathbb{P} \in V_{\delta_0}$ (in case $\mathbb{P}$ is not trivial). Moreover, let

$$\kappa < \delta_0 < \delta_1 < \delta_2 < \delta_3 < \delta_4 < \delta_5 < \delta_6$$

be such that δ_i, $0 \leq i \leq 6$, are Woodin cardinals.

As there is a proper class of Woodin cardinals, each A_i is homogeneously Suslin in $V[g]$. In what follows we identify measures in $V[g]$ with their restrictions to V-measures when the measures are above the size of g. Let $W_2 \in V_{\delta_6}$ be a set of measures with $W_2 \subseteq \mathrm{meas}_{\delta_5^+}(Z_2)$ such that for each $i < \omega$ we can fix a homogeneity system $\bar{\mu}_2^{(i)} = (\mu_s^{(i)} \mid s \in \omega^{<\omega})$ such that for each $s \in \omega^{<\omega}$, $\mu_s^{(i)} \in W_2$, and

$$A_i = \{x \in \omega^\omega \mid (\mu^{(i)}_{x \restriction n} \mid n < \omega) \text{ is well-founded}\}.$$

We can arrange that W_2 is of size $< \delta_0$.

By Lemma 2.5, there is some $W_1 \subseteq \mathrm{meas}_{\delta_4^+}(Z_1)$, $W_1 \in V_{\delta_5}$, as well as a 1-to-1 Lipschitz function

$$f_1\colon \mathrm{TW}_{W_2} \to \mathrm{TW}_{W_1}$$

in V such that for all ${<}\delta_4^+$-generics G over $V[g]$ and all $\vec{\mu} \in (\mathrm{TW}_{W_2})^{V[g][G]}$,

$$\vec{\mu} \text{ is well-founded} \iff f_1(\vec{\mu}) \text{ is ill-founded.}$$

Then we have by Lemma 2.5 again that there is some $W_0 \subseteq \mathrm{meas}_{\delta_1^+}(Z_0)$, $W_0 \in V_{\delta_2}$, as well as a 1-to-1 Lipschitz function

$$f_0\colon \mathrm{TW}_{W_1} \to \mathrm{TW}_{W_0}$$

in V such that for all ${<}\delta_1^+$-generics G over $V[g]$ and all $\vec{\mu} \in (\mathrm{TW}_{W_1})^{V[g][G]}$,

$$\vec{\mu} \text{ is well-founded} \iff f_0(\vec{\mu}) \text{ is ill-founded.}$$

Moreover, there is some $\tilde{W}_2 \subseteq \mathrm{meas}_{\delta_0^+}(\tilde{Z}_2)$, $\tilde{W}_2 \in V_{\delta_1}$ with a 1-to-1 Lipschitz function

$$\tilde{f}_2\colon \mathrm{TW}_{W_1} \to \mathrm{TW}_{\tilde{W}_2}$$

in V such that for all ${<}\delta_0^+$-generics G over $V[g]$ and all $\vec{\mu} \in (\mathrm{TW}_{W_1})^{V[g][G]}$,

$$\vec{\mu} \text{ is well-founded} \iff \tilde{f}_2(\vec{\mu}) \text{ is ill-founded}$$

as well as some $\tilde{W}_1 \subseteq \mathrm{meas}_{|\mathbb{P}|^+}(\tilde{Z}_1)$, $\tilde{W}_1 \in V_{\delta_0}$ with a 1-to-1 Lipschitz function

$$\tilde{f}_1\colon \mathrm{TW}_{\tilde{W}_2} \to \mathrm{TW}_{\tilde{W}_1}$$

in V such that for all ${<}|\mathbb{P}|^+$-generics G over $V[g]$ and all $\vec{\mu} \in (\mathrm{TW}_{\tilde{W}_2})^{V[g][G]}$,

$$\vec{\mu} \text{ is well-founded} \iff \tilde{f}_1(\vec{\mu}) \text{ is ill-founded.}$$

Using these flipping functions[11] we let $\bar{\mu}_1^{(i)} = f_1 \text{"} \bar{\mu}_2^{(i)}$ and $\bar{\mu}_0^{(i)} = f_0 \text{"} \bar{\mu}_1^{(i)}$ as well as $\bar{\mu}_2^{\prime,(i)} = \tilde{f}_2 \text{"} \bar{\mu}_1^{(i)}$ and $\bar{\mu}_1^{\prime,(i)} = \tilde{f}_1 \text{"} \bar{\mu}_2^{\prime,(i)}$ for each $i < \omega$. Figure 1 illustrates the situation.

Use the supercompact to ensure realizability and hence iterability

But recall that we are not assuming any iterability of V. So in order to ensure that we can keep iterating we prove that there are realizations of each iterate into a large model. Let

$$j\colon V \to V^*$$

be a χ-supercompact embedding with critical point κ. So $(V^*)^\chi \subseteq V^*$ and $j(\kappa) > \chi$. In particular, $V_\chi \subseteq V^*$ and g is generic over V^*. Note that $j \circ \pi_i \in V^*[g]$ for all $i < \omega$ as $V^*[g]$ is closed under countable sequences in $V[g]$. Write

$$\sigma_i = j \circ \pi_i \colon M_i \to V^*_{j(\chi)}$$

for all $i < \omega$. Then the sequence $(\hat{M}_i, M_i, \sigma_i, g_i \mid i < \omega)$ forms a weak block, defined as follows:

$$\begin{tikzcd}
V_\chi \arrow[rr, "j"] & & V^*_{j(\chi)} \\
& \vdots & \\
& M_1 \arrow[uul, bend left, "\pi_1"] \arrow[uur, bend right, "\sigma_1"'] & \\
& M_0 \arrow[uuul, bend left, "\pi_0"] \arrow[uuur, bend right, "\sigma_0"'] &
\end{tikzcd}$$

Definition 4.2. Let $(\bar{\nu}^{(i)})_{i<\omega}$ be a sequence of homogeneity systems in $V^*[g]$ and let $f_1^*, f_0^*, \tilde{f}_2^*, \tilde{f}_1^* \in V^*$ be functions on towers of measures. Then a $((\bar{\nu}^{(i)})_{i<\omega}, f_1^*, f_0^*, \tilde{f}_2^*, \tilde{f}_1^*)$-*weak block for* $V^*[g]$ *at* κ is a sequence

$$(\hat{M}_i, M_i, \sigma_i, g_i \mid i < \omega)$$

of models $\hat{M}_i$ and M_i as well as M_i-generics g_i together with elementary embeddings

$$\sigma_i\colon M_i \to V^*_{j(\chi)}$$

such that for all $i < \omega$,

(1) $\hat{M}_i = M_i[g_i]$ is countable in $V^*[g]$,
(2) $\wp(\kappa)^{M_i} = \wp(\kappa)^{M_{i+1}}$,
(3) $\sigma_i \in V^*[g]$,
(4) $\bar{\nu}^{(i)} \subseteq \mathrm{rng}(\sigma_i)$, $f_1^*, f_0^*, \tilde{f}_2^*, \tilde{f}_1^* \in \mathrm{rng}(\sigma_0)$, and
(5) $\mathrm{rng}(\sigma_i) \subseteq \mathrm{rng}(\sigma_{i+1})$.

Difference between blocks and weak blocks:
In a weak block we do not require the critical points of σ_i to be above κ (which is the case for π_i in a block). In fact, in our weak blocks we will have $\mathrm{crit}(\sigma_i) = \kappa$.

Lemma 4.3. *For each $i < \omega$, $j``\bar{\mu}_2^{(i)}$ gives rise to a homogeneity system for A_i in $V^*[g]$.*

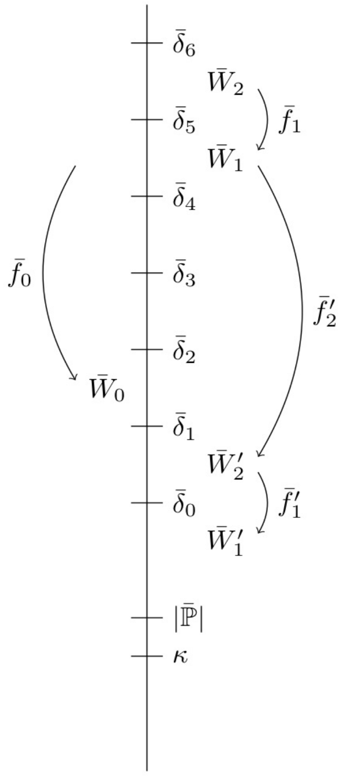
$\hookrightarrow$ So we can use these homogeneity systems for our argument. The proof of this lemma already uses flipping functions.

Why can we realize into $V^*_{j(\chi)}$?
The models M_i and their iterations are hulls of V^* that are countable in $V^*[g]$.
$\sigma_i = j \circ \pi_i \in V^*[g]$ as $(V^*)^\chi \subseteq V^*$. So we can run the Martin-Steel iterability proof, realizing into $V^*[g]$. This works because the iteration copied on V^* will be above κ (as $j(\kappa) > \kappa$, so $\sigma_i(\kappa) > \kappa$ for all $i < \omega$).

Note: This does not work without the supercompact as the π_i-copies of the iterations on V are not above κ; $\sigma_i \in V^*[g]$ uses $(V^*)^\chi \subseteq V^*$.

One step: Making a real and a uB-set generic

The setup in some M_i



Making x and A generic

$$\begin{tikzcd}
V^*_{j(\chi)} & & \\
M \arrow[u, "\sigma_i"] \arrow[r, "i_{\mathcal{T}_0}"] \arrow[rr, bend right, "\tau"'] & \mathcal{M}^{\mathcal{T}_0} \arrow[ul, "\iota"'] \arrow[r, "i_{\mathcal{T}_1}"] & M^* \arrow[ull, "\iota^*"']
\end{tikzcd}$$

4.1. **Making a real generic over a weak block.** Let $\mathcal{T}_0$ be the iteration tree on M_i resulting from making x generic below $\bar\delta_2$ according to Neeman's genericity iteration (see Theorem 2.4) followed by a realizable branch (into $V^*_{j(\chi)}$). Such a realizable branch exists by [8] as the iteration tree resulting from Neeman's genericity iteration is countable in $V[g]$. Write $\iota\colon \mathcal{M}^{\mathcal{T}_0} \to V^*_{j(\chi)}$ for the realization. We may and will assume that the critical points of all extenders used in $\mathcal{T}_0$ are $> (\bar\delta_1^+)^{M_i}$.

4.2. **Making a homogeneity system generic over a weak block.** Let $\mathcal{T}_1$ be the iteration tree on $\mathcal{M}^{\mathcal{T}_0}$ resulting from making $i_{\mathcal{T}_0} \restriction V^M_{\bar\delta_2}$ generic below $i_{\mathcal{T}_0}(\bar\delta_3)$ according to Neeman's genericity iteration (see Theorem 2.4) followed by a realizable branch (into $V^*_{j(\chi)}$). Note that $i_{\mathcal{T}_0} \restriction V^M_{\bar\delta_2}$ can be coded as a real in $V[g]$. As above, a realizable branch exists by [8] as the iteration tree resulting from Neeman's genericity iteration is countable in $V[g]$. Let $\iota^*\colon M^* \to V^*_{j(\chi)}$ be the realization, so, in particular, $\iota^* \circ \tau = \sigma_i$. We may and will assume that $\mathcal{T}_1$ is acting above $i_{\mathcal{T}_0}(\bar\delta_2)$. Let

$$\bar\nu_0^* = i_{\mathcal{T}_0}"(\bar\nu_0).$$

Figure 4 summarizes the genericity iterations.

Write $\tau = i_{\mathcal{T}_1} \circ i_{\mathcal{T}_0}\colon M \to M^*$ and let g^* be $\mathrm{Col}(\omega, \tau(\bar\delta_3))$-generic over M^* with $g^* \in V[g]$. Then $(x, \bar\nu_0^*) \in M^*[g^*]$. As $\tau(\bar f_0)$ and $\tau(\bar f_1)$ are 1-to-1, we can let

$$\bar\nu_1^* = (\tau(\bar f_0))^{-1}"\bar\nu_0^*$$

and

$$\bar\nu_2^* = (\tau(\bar f_1))^{-1}"\bar\nu_1^*.$$

Then $\bar\nu_1^*, \bar\nu_2^* \in M^*[g^*]$. This is why we call this step "making a homogeneity system generic" even though the actual object that is made generic is $i_{\mathcal{T}_0} \restriction V^M_{\bar\delta_2}$.

Let $W_i^* = \tau(\bar W_i)$ for $i \in \{0,1,2\}$. We have $\bar\nu_1^* = (\tau(\bar f_0))^{-1}"(i_{\mathcal{T}_0}"(\bar\nu_0)) = (\tau(\bar f_0))^{-1}"(\tau"(\bar\nu_0))$ as $\bar\nu_0$ is below $\bar\delta_2$ and hence $i_{\mathcal{T}_0}"(\bar\nu_0)$ is below the critical point of $i_{\mathcal{T}_1}$. Hence, as $\bar\nu_0 = \bar f_0"\bar\nu_1$, $\bar\nu_1^* = \tau"\bar\nu_1$. Similarly, $\bar\nu_2^* = \tau"\bar\nu_2$. Moreover, as $\tau(\bar f_1') = \bar f_1'$ as well as $\tau"\bar\nu_2' = \bar\nu_2'$ and $\tau"\bar\nu_1' = \bar\nu_1'$, we have $\bar\nu_2' = \tau(\bar f_2')"\bar\nu_1^*$ and $\bar\nu_1' = \bar f_1'"\bar\nu_2'$ and hence $\bar\nu_2', \bar\nu_1' \in M^*[g^*]$.

The hom. systems that we made generic in fact characterize A

Lemma 4.4. *Let $\gamma \geq \tau(\bar\delta_3)$ be a cardinal in M^* with $\gamma < \tau(\bar\delta_4)$ and let h^* be* $\mathrm{Col}(\omega,\gamma)$*-generic over $M^*[g^*]$ with $h^* \in V[g]$. Then for any $u \in M^*[g^* * h^*] \cap \mathbb{R}_g$,*

$$u \in A \iff ((\bar\nu_2^*)_{u\restriction n} \mid n<\omega) \text{ is well-founded} \iff ((\bar\nu_1^*)_{u\restriction n} \mid n<\omega) \text{ is ill-founded.}$$

Note that $M^*[g^* * h^*]$ is in $V[g]$, so it makes sense to write "$u \in A$" in the statement of Lemma 4.4.

Proof of Lemma 4.4. The second equivalence easily follows from the facts that $\bar\nu_1^* = \tau(\bar f_1)\,``(\bar\nu_2^*)$ and $\tau(\bar f_1)$ is a flipping function for towers of measures in $M^*[g^* * h^*]$. For the first equivalence, recall that $\bar\mu_2 \cup \{f_1^*, f_0^*\} \subseteq \mathrm{rng}(\sigma_i)$ and hence $\bar\mu_2 \cup \{f_1^*, f_0^*\} \subseteq \mathrm{rng}(\iota^*)$. Moreover, $\bar\mu_1 \subseteq \mathrm{rng}(\sigma_i)$ as well as $\bar\mu_1 \subseteq \mathrm{rng}(\iota^*)$. Note that

$$\begin{aligned} ((\iota^*)^{-1}((\bar\mu_2)_{u\restriction n}) \mid n<\omega) \text{ is ill-founded} &\implies (\iota^*((\iota^*)^{-1}((\bar\mu_2)_{u\restriction n})) \mid n<\omega) \text{ is ill-founded} \\ &\iff ((\bar\mu_2)_{u\restriction n} \mid n<\omega) \text{ is ill-founded.} \end{aligned}$$

Therefore, using the fact that $(\iota^*)^{-1}(f_1^*)$ is a flipping function for towers of measures in $M^*[g^* * h^*]$, we have

$$\begin{aligned} u \in A &\iff ((\bar\mu_2)_{u\restriction n} \mid n<\omega) \text{ is well-founded} \\ &\implies ((\iota^*)^{-1}((\bar\mu_2)_{u\restriction n}) \mid n<\omega) \text{ is well-founded} \\ &\iff (\iota^*)^{-1}(f_1^*)(((\iota^*)^{-1}((\bar\mu_2)_{u\restriction n})) \mid n<\omega) \text{ is ill-founded} \\ &\iff (\tau\circ\sigma_i^{-1})(f_1^*)(((\tau\circ\sigma_i^{-1})((\bar\mu_2)_{u\restriction n})) \mid n<\omega) \text{ is ill-founded} \\ &\iff \tau(\bar f_1)((\tau(\bar\nu_2)_{u\restriction n}) \mid n<\omega) \text{ is ill-founded} \\ &\iff (\tau((\bar\nu_1)_{u\restriction n}) \mid n<\omega) \text{ is ill-founded} \\ &\iff ((\bar\nu_1^*)_{u\restriction n} \mid n<\omega) \text{ is ill-founded.} \end{aligned}$$

Moreover,

$$\begin{aligned} u \notin A &\iff ((\bar\mu_2)_{u\restriction n} \mid n<\omega) \text{ is ill-founded} \\ &\iff ((\bar\mu_1)_{u\restriction n} \mid n<\omega) \text{ is well-founded} \\ &\implies ((\iota^*)^{-1}((\bar\mu_1)_{u\restriction n}) \mid n<\omega) \text{ is well-founded} \\ &\iff ((\tau\circ\sigma_i^{-1})((\bar\mu_1)_{u\restriction n}) \mid n<\omega) \text{ is well-founded} \\ &\iff (\tau((\bar\nu_1)_{u\restriction n}) \mid n<\omega) \text{ is well-founded} \\ &\iff ((\bar\nu_1^*)_{u\restriction n} \mid n<\omega) \text{ is well-founded.} \end{aligned}$$

□

To extend this characterization to ultrapowers we use uB-preserving extenders

Our next goal is to extend the characterization in Lemma 4.4 to an ultrapower of M^* by a short extender $E \in M^*$. The key issue for this goal is that we aim for such an extension with systems of measures that are $<\pi_E^{M^*}(\kappa)$-complete, even if $\pi_E^{M^*}(\kappa) > \tau(\bar\delta_6)$. Here $\pi_E^{M^*}: M^* \to \mathrm{Ult}(M^*, E)$ denotes the ultrapower embedding. To achieve the goal, we will use uB-preserving extenders (see Definition 3.1).