Sandra Müller, June 28, 2025

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Translating between large cardinals and iteration strategies

When I was an undergraduate student in Münster I was interested in attending Ralf's set theory seminar. I was late signing up because I thought I am missing some of the prerequisits for the class. So I went to his office asking whether it would still be possible to attend. Ralf said sure and asked "Do you want an easy topic or an interesting topic?" I picked the interesting one and presented Martin's proof of Borel determinacy.

In the following semester I attended the seminar again and presented the proof of analytic determinacy from sharps.

This was when I fell in love with large cardinals and determinacy and my journey into inner model theory and PhD studies with Ralf started.

Large Cardinals $\qquad$ Determinacy

equivalences

$x^\#$ ex. for every $x \in {}^\omega\omega$ $\xLeftrightarrow{\text{Harrington, Martin}}$ analytic determinacy

$M_n^\#(x)$ ex. for every $x \in {}^\omega\omega$ $\xLeftrightarrow[\text{(my PhD thesis)}]{\text{Martin, Steel, Woodin, Neeman}}$ Σ^1_{n+1} determinacy

$M_\omega^\#$ exists $\xLeftrightarrow{\text{Martin, Steel, Woodin}}$ $AD^{L(\mathbb{R})}$ + $\mathbb{R}^\#$ exists

equiconsistencies

a limit λ of Woodin cardinals & $<\lambda$-strong cardinals $\xrightarrow{\text{Woodin}}$ AD^+ + all sets of reals are Suslin $\leftrightarrow$ $AD_{\mathbb{R}}$

$\xleftarrow[\text{(building on work of Woodin)}]{\text{Steel}}$

a limit of Woodin cardinals and strong cardinals $\xrightarrow{\text{Larson, Sargsyan, Wilson}}$ AD + all set of reals are universally Baire

$\xleftarrow{\text{M.}}$

Key tool: Procedures to translate iteration strategies (of hybrid mice) into strong cardinals.

Translation procedures

Idea: Translate hybrid mice into ordinary mice while preserving their descriptive set theoretic complexity.

- Steel ("Derived models associated to mice", 2008)

In [25], Steel develops a translation procedure that can be used to prove Woodin's result that

$$\mathrm{ZF} + \mathrm{AD}^+ + \theta_0 < \Theta$$

is equiconsistent with

$$\mathrm{ZFC}\ +\ \text{there is a pair of cardinals } \kappa, \lambda \text{ such that}$$
$$\lambda \text{ is a limit of Woodin cardinals and } \kappa \text{ is } {<}\lambda\text{-strong.}$$

Steel's translation procedure in [25] originates from work of Closson, Neeman, and Steel and was later extended by Zhu in [33]. Steel's and Zhu's constructions use AD^+ theory and work in a specific hod mouse, more specifically, they use a real parameter for the translation. Therefore, their constructions have the disadvantage that the translation cannot be done in HOD itself but only in HOD of a real.

- Sargsyan ("Translation procedures in descriptive inner model theory", 2017)

THEOREM 2.8. *Suppose $\mathcal{K}$ is a translatable structure. Then, in $\mathcal{K}$, there is a proper class premouse $\mathcal{M}$ such that $\mathcal{M}$ has a proper class of Woodin cardinals and a strong cardinal.*

QUESTION 3.1 (Wilson). *Assume there are proper class of Woodin cardinals. Suppose the class*

$$S = \{\lambda : \lambda \textit{ is a limit of Woodin cardinals and the derived model at } \lambda \textit{ satisfies } \mathrm{AD}^+ + \theta_0 < \Theta\}$$

is stationary. Is there a transitive model $\mathcal{M}$ satisfying ZFC such that Ord $\subseteq \mathcal{M}$ and $\mathcal{M}$ has a proper class of Woodin cardinals and a strong cardinal?

We show that the answer is yes.

THEOREM 3.2. *Suppose there are proper class of Woodin cardinals. Assume further that the class S above is stationary. Then there is a transitive model $\mathcal{M}$ satisfying ZFC such that Ord $\subseteq \mathcal{M}$ and $\mathcal{M}$ has a proper class of Woodin cardinals and a strong cardinal.*

→ Not clear how to generalize his method to infinitely many strongs

→ Sargsyan's model $\mathcal{M}$ is not iterable

How do we translate into infinitely many strong cardinals?

Definition 3.20. Let $\mathcal{W}$ be a proper class hybrid strategy premouse in the sense of [13, Chapter 1]. Then $\mathcal{W}$ is a *translatable structure* iff there is a sequence of ordinals $(\delta_i \mid i<\omega)$ such that $\mathcal{W}$ satisfies the following conditions.

(1) $\mathcal{W} \models$ "δ_i is a Woodin cardinal and a cutpoint for every $i<\omega$ and these are the only Woodin cardinals".
(2) In $\mathcal{W}$, let $\mathcal{P}^0 = Lp^\infty_\omega(\mathcal{W}|\delta_0)$ and $\mathcal{P}^i = Lp^{\Sigma_{i-1},\infty}_\omega(\mathcal{W}|\delta_i)$ for $i>0$, where Σ_i denotes the strategy for $\mathcal{P}^i$. Then $\mathcal{P}^0$ is suitable at κ_0 and $\mathcal{P}^i$ is Σ_{i-1}-suitable at κ_i for every $i>0$.
(3) Σ_i is a super fullness preserving iteration strategy for $\mathcal{P}^i$ with hull condensation such that $\mathcal{W}$ is generically closed under Σ_i and there is a set of term relations $\vec{\tau}^{(i)} \subseteq \mathcal{P}^i$ such that whenever g is $\mathrm{Col}(\omega,\mathcal{P}^i)$-generic over $\mathcal{W}$ and $(\tau^{(i)}_k \mid k<\omega)$ is a generic enumeration of $\vec{\tau}^{(i)}$ in $\mathcal{W}[g]$, then $(\Sigma_i)^g$ is strongly guided by $(\tau^{(i)}_k \mid k<\omega)$.
(4) $\mathcal{W}$ is internally $(\mathrm{Ord},\mathrm{Ord})$-iterable.
(5) $\mathcal{W} \models$ "there is no inner model with a superstrong cardinal".[12]
(6) Let δ_ω be the limit of δ_i, $i<\omega$. Let G be $\mathrm{Col}(\omega,{<}\delta_\omega)$-generic over $\mathcal{W}$ and let M be the derived model computed in $\mathcal{W}(\mathbb{R}^*)$. Let $\Phi = (\Sigma^{\mathcal{W}})^G \restriction \mathrm{HC}^M$ for $\Sigma^{\mathcal{W}} = \bigoplus_{i<\omega}\Sigma_i$. Then $\Phi \in M$ and in M, $Lp^\infty_\omega(\mathcal{W}|\delta_i)$ is a κ_i-suitable $\emptyset$-iterable Φ-premouse (cf. [14, Definition 2.7] for the definition of $\emptyset$-iterability).

Definition 4.3. Let $\mathcal{W}$ be a translatable structure with Woodin cardinals $(\delta_i \mid i<\omega)$. Let $(\kappa_i \mid i<\omega)$ be a sequence of cardinals such that for each $i<\omega$, κ_i is the least ${<}\delta_{i+1}$-strong cardinal above δ_i in $\mathcal{W}$. We say $(\mathcal{M}_\xi,\mathcal{N}_\xi \mid \xi\leq\Xi)$ is the *mouse construction* (or short $\mathcal{M}$*-construction*) in $\mathcal{W}$ for $\Xi\leq\mathrm{Ord}$ if it is given inductively as follows.

(1) $\mathcal{N}_0$ is the result of a fully backgrounded extender construction in $\mathcal{W}|\delta_1$.
(2) For $\xi+1<\Xi$, if $\mathcal{M}_\xi = (J^{\vec{E}}_\alpha,\in,\vec{E},\emptyset)$ is a passive premouse, we define $\mathcal{N}_{\xi+1}$ as follows.
   (a) If there is an extender F^* on the sequence of $\mathcal{W}$ and an extender F over $\mathcal{M}_\xi$, both with critical point $\kappa\notin\{\kappa_i \mid i<\omega\}$,[14] such that $(J^{\vec{E}}_\alpha,\in,\vec{E},F)$ is a premouse and for some $\nu<\alpha$, $V^{\mathcal{W}}_{\nu+\omega}\subseteq\mathrm{Ult}(\mathcal{W},F^*)$ and
   $$F\restriction\nu = F^*\cap([\nu]^{<\omega}\times J^{\vec{E}}_\alpha),$$
   we let[15]
   $$\mathcal{N}_{\xi+1} = (J^{\vec{E}}_\alpha,\in,\vec{E},F).$$
   (b) If (2a) fails and there is an extender F^* on the sequence of $\mathcal{W}$ and an extender F over $\mathcal{M}_\xi$, both with critical point $\kappa=\kappa_i$ for some $i<\omega$ with $\alpha<\delta_{i+1}$,[16] such that $(J^{\vec{E}}_\alpha,\in,\vec{E},F)$ is a premouse and for some $\nu<\alpha$, $V^{\mathcal{W}}_{\nu+\omega}\subseteq\mathrm{Ult}(\mathcal{W},F^*)$ and
   $$F\restriction\nu = F^*\cap([\nu]^{<\omega}\times J^{\vec{E}}_\alpha),$$
   we let
   $$\mathcal{N}_{\xi+1} = (J^{\vec{E}}_\alpha,\in,\vec{E},F).$$
   (c) If (2a) and (2b) fail and there is a generically countably complete extender F in $\mathcal{W}$ with critical point κ_i for some $i<\omega$ such that $\alpha>\delta_{i+1}$ and $(J^{\vec{E}}_\alpha,\in,\vec{E},F)$ is a premouse, we let
   $$\mathcal{N}_{\xi+1} = (J^{\vec{E}}_\alpha,\in,\vec{E},F).$$
   (d) If (2a), (2b) and (2c) fail, we let
   $$\mathcal{N}_{\xi+1} = (J^{\vec{E}}_{\alpha+1},\in,\vec{E},\emptyset).$$
   In all cases, we let $\mathcal{M}_{\xi+1} = \mathbb{C}_\omega(\mathcal{N}_{\xi+1})$, if it is defined.
(3) For $\xi+1<\Xi$, if $\mathcal{M}_\xi = (J^{\vec{E}}_\alpha,\in,\vec{E},F)$ is active, we let
   $$\mathcal{N}_{\xi+1} = (J^{\vec{E}'}_{\alpha+1},\in,\vec{E}',\emptyset)$$
   for $\vec{E}' = \vec{E}^\frown F$. Moreover, we let $\mathcal{M}_{\xi+1} = \mathbb{C}_\omega(\mathcal{N}_{\xi+1})$, if it is defined.
(4) If $\lambda\leq\Xi$ is a limit ordinal or $\lambda=\mathrm{Ord}$, we let $\mathcal{N}_\lambda = \mathcal{M}_\lambda$ be the unique passive premouse such that for all ordinals β, $\omega\beta\in\mathcal{N}_\lambda\cap\mathrm{Ord}$ iff $J^{\mathcal{N}_\alpha}_\beta$ is defined and eventually constant as α converges to λ, and for all such $\omega\beta\in\mathcal{N}_\lambda\cap\mathrm{Ord}$, $J^{\mathcal{N}_\lambda}_\beta$ is given by the eventual value of $J^{\mathcal{N}_\alpha}_\beta$ as α converges to λ.

We say the $\mathcal{M}$-construction breaks down and stop the construction if $\mathbb{C}_\omega(\mathcal{N}_\xi)$ is not defined for some $\xi\in\mathrm{Ord}$. Otherwise we say the $\mathcal{M}$-construction converges and $\mathcal{M}=\mathcal{M}_{\mathrm{Ord}}$ is the result of the construction.

Theorem 1.2. *Let $\mathcal{W}$ be a translatable structure as in Definition 3.20 and let $\mathcal{M}$ be the result of an $\mathcal{M}$-construction in $\mathcal{W}$ as in Definition 4.3. Then $\mathcal{M}$ is a countably iterable proper class premouse with a cardinal λ that is a limit of Woodin cardinals and a limit of strong cardinals.*

Start with a sufficiently nice hybrid mouse $\mathcal{W}$ (rigidly layered as in Grigor's thesis).

δ_i, $i<\omega$, Woodin cardinals

κ_i, $i<\omega$, the least $<\delta_{i+1}$ strong cardinal (in $\mathcal{W}$ and in what we have already constructed of $\mathcal{M}$).

Fully backgrd construction but add in addition generically ctbly complete extenders with crit. pt. κ_i at stages $\alpha>\delta_{i+1}$.

i.e. extenders that are ctbly complete in any forcing extension of size $<\kappa_i$.

Definition 4.1. We say a (κ,λ)-extender E is *generically countably complete* if it is countably complete in all ${<}\kappa$-generic extensions, i.e., whenever $\mathbb{P}$ is a partial order with $|\mathbb{P}|<\kappa$ and G is $\mathbb{P}$-generic over V, then in $V[G]$, for every sequence $(a_i \mid i<\omega)$ of sets in $[\lambda]^{<\omega}$ and every sequence $(A_i \mid i<\omega)$ of sets $A_i\in E_{a_i}$ there is a function $\tau\colon\bigcup_i a_i\to\kappa$ such that τ"$a_i\in A_i$ for each $i<\omega$.

Idea: Ultrapowers by such extenders (in these generic extensions) can be realized.

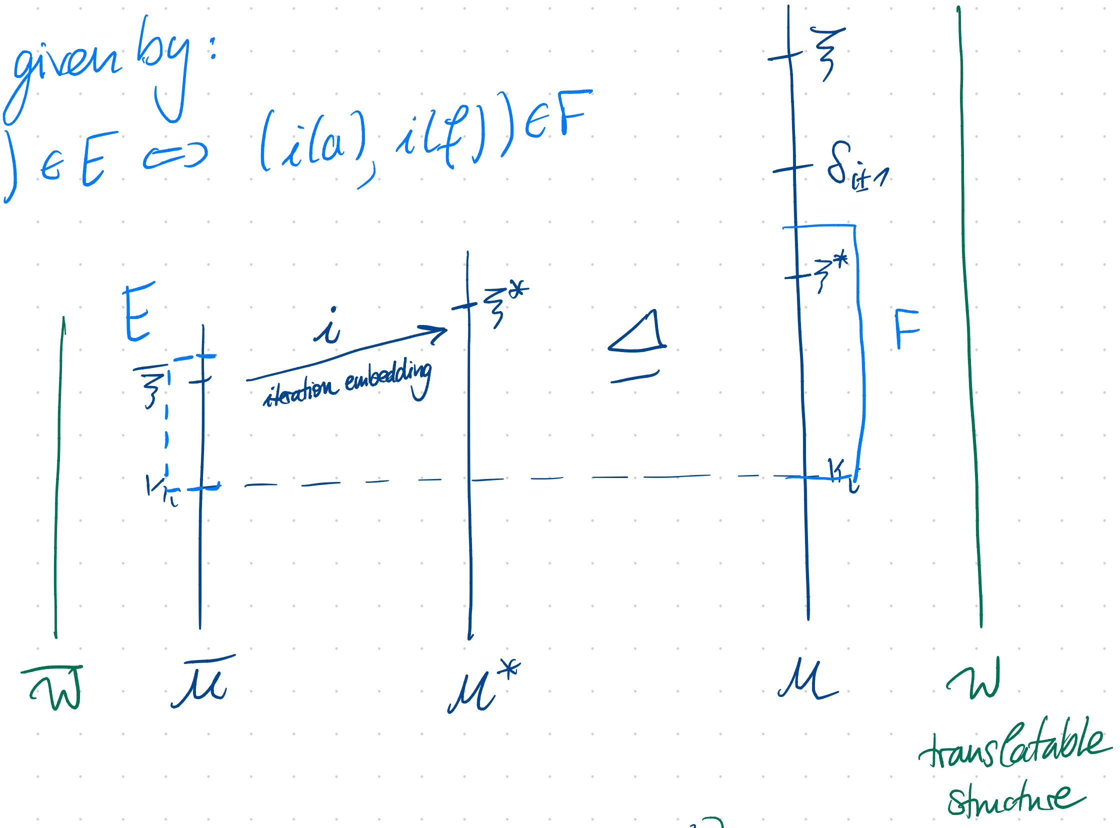
Why are the κ_i strong in $\mathcal{M}$?

We inductively show that the $\mathcal{M}$-construction converges, hulls of the resulting model are sufficiently iterable, and the κ_i are strong in $\mathcal{M}$.

For simplicity, suppose $\mathcal{M}$ converges. Toward a contradiction, suppose i is minimal s.t. κ_i is not ξ-strong for some ξ $(\geq \delta_{i+1})$.

E is given by:

$(a, f) \in E \iff (i(a), i(f)) \in F$



Compare the construction of $\bar{\mathcal{M}}$ against the construction of $\mathcal{M}$. } Need enough iterability to make this work (see next page)

- $\mathcal{M}$-side does not move
- $\bar{\mathcal{M}}$-side only moves above κ_i } This needs a deep understanding of the non-backgrounded extenders that get added by the $\mathcal{M}$-construction
- $\mathcal{M}^*$ iterates into a model of the construction of $\mathcal{M}$
- F overlaps ξ^* in $\mathcal{M}$, witnessing that κ_i is $<\delta_{i+1}$-strong in $\mathcal{M}$
- Argue that $\underbrace{E \in \bar{W}}_{\text{this is non-trivial}}$ and E gets added to $\bar{\mathcal{M}}$ by the $\mathcal{M}$-construction in $\bar{W}$.
  and uses a characterization of the extenders involved

This yields the desired contradiction.

Why is $\bar{\mathcal{M}}$ sufficiently iterable?

Combine the standard lifting to the background model for fully backgrounded constructions (Mitchell-Steel) with almost linearity of the iteration tree (Schindler) when considering extenders not given by a background.

Definition 6.1. Let $\mathcal{N}$ be the result of an $\mathcal{M}$-construction in some translatable structure or an iterate of the result of such a construction. Write $(\kappa_i^{\mathcal{N}}, \delta_i^{\mathcal{N}} \mid i < \omega)$ for the sequence of the specific cardinals κ_i and δ_i, $i < \omega$, chosen in the definition of the $\mathcal{M}$-construction.

(1) An extender E on the $\mathcal{N}$-sequence is called *strongness witness* or *non-backgrounded* iff the critical point of E is $\kappa_i^{\mathcal{N}}$ for some $i < \omega$ and the length of E is above $\delta_{i+1}^{\mathcal{N}}$. Otherwise we say E is *backgrounded*.

(2) An iteration tree $\mathcal{T}$ on $\mathcal{N}$ is called *almost linear on strongness witnesses* iff
  (a) for all $\alpha + 1 < \mathrm{lh}(\mathcal{T})$ such that $E_\alpha^{\mathcal{T}}$ is a strongness witness we have $\mathrm{pred}^*_T(\alpha+1) \in [0, \alpha]_T$, where $\mathrm{pred}^*_T(\alpha+1)$ denotes the unique ordinal $\beta + 1$ such that $\mathcal{M}_{\beta+1}^{\mathcal{T}}$ is obtained as an ultrapower by a strongness witness $E_\beta^{\mathcal{T}}$ and $i_{\beta+1,\mathrm{pred}_T(\alpha+1)} \colon \mathcal{M}_\beta^{*,\mathcal{T}} \to \mathcal{M}^{\mathcal{T}}_{\mathrm{pred}_T(\alpha+1)}$ is an iteration via backgrounded extenders, and
  (b) any $\alpha < \mathrm{lh}(\mathcal{T})$ has only finitely many T-successors $\beta + 1$ such that $E_\beta^{\mathcal{T}}$ is a strongness witness.

The following lemma uses similar ideas as the proof of [21, Lemma 2.4].

Lemma 6.2. *Let $\mathcal{N}$ be the result of an $\mathcal{M}$-construction in some translatable structure. Then any normal iteration tree $\mathcal{T}$ on $\mathcal{N}$ is almost linear on strongness witnesses.*

The precise iterability statement:

Lemma 6.3. *Let $i < \omega$, suppose* $(\mathsf{IH})_i$ *and let*
$$\pi \colon \bar{\mathcal{W}} \to \mathcal{W}|\Omega$$
for some sufficiently large[31] *Ω be such that letting $\mathrm{crit}(\pi) = \nu$, $\mathcal{W}|\kappa_i \subseteq \bar{\mathcal{W}}$, $(\kappa_j, \delta_j \mid j < \omega) \in \mathrm{rng}(\pi)$, the sequence of models of the construction of $\mathcal{M}^{(i+1)}|\Omega$ is in the range of π, $\nu \in (\kappa_i, \delta_{i+1})$ is an inaccessible cardinal in $\mathcal{W}$, and $\pi(\nu) = \delta_{i+1}$. Let $\bar{\mathcal{M}}$ be the collapse of $\mathcal{M}^{(i+1)}|\Omega$. Then $\bar{\mathcal{M}}$ is $(\delta_{i+1}, \delta_{i+1})$-iterable with respect to extenders with critical point above κ_i.*

The picture for one case in the proof:

copy tree

realization

$$\begin{tikzcd}
 & & \mathcal{N} & \\
\mathcal{M} \arrow[r, squiggly, "\iota_0\mathcal{T}_1"] & \mathcal{M}^1 \arrow[ur, squiggly, "\iota_0\mathcal{T}_1"] \arrow[rr, squiggly, "\iota_{\xi+1}\mathcal{T}_2"] & & {} \\
 & & \mathrm{Ult}(\bar{\mathcal{M}}^1, E) \arrow[ul, "\iota_{\xi+1}"'] \arrow[r, squiggly, "\mathcal{T}_2"] & {} \\
\bar{\mathcal{M}} \arrow[uu, "\iota_0"] \arrow[r, squiggly, "\mathcal{T}_1"] & \bar{\mathcal{M}}^1 \arrow[uu, "\iota_\eta"] \arrow[ur, "\pi_E"] \arrow[r, squiggly, "\mathcal{T}_1"] & \bar{\mathcal{N}} \ni E &
\end{tikzcd}$$

iteration via backgrdd. extenders

generically ctbly complete extender

FIGURE 4. Realization into a backgrounded iteration of $\mathcal{M}$ when applying a single non-backgrounded extender E.

The other cases are similar, using that the iteration tree is almost linear on strongness witnesses.

What is next? Other translation procedures?

Work in progress joint with Dominik Adolf, Benny Siskind and Lena Wallner.

Key idea: Translate complexity in derived models into large cardinals in a mouse.

(B/c a long Solovay sequence is essentially a complicated iteration strategy)

<u>Theorem</u>* (Adolf-M-Siskind-Wallner, 2026):
Let M be an iterable canonical inner model with a limit of Woodin cardinals λ. Suppose that $D(M, \lambda)$, the derived model of M at λ, satisfies $\Theta \geq \theta_\omega$.
Then, in M, λ is a limit of $<\lambda$-strong cardinals.

Theorem (AMSW, >2026?)

<u>Conjecture</u> (Adolf-M-Siskind-Wallner): This can be generalized to obtain a converse to Adolf-Sargsyan derived models of mice (for derived models with $AD_{\mathbb{R}} + \Theta \geq \theta_{\aleph_2}$ and even $AD_{\mathbb{R}} + \Theta = \theta_{\aleph_\Theta}$).

This would answer a question of Adolf and Sargsyan.

On the way to the dead sea,

July 2022

With an unplanned tour around Jerusalem b/c of Biden's visit.

Thank you and happy birthday, Ralf!