Equiconsistencies between very large cardinals and strengthenings of PFA

Andreas Lietz
TU Wien

June 24th, 2025

3rd Berkeley Inner Model Theory Conference

Some Thoughts on the Consistency Strength of PFA

The evidence

The evidence

Theorem (Magidor '76, Viale '06)
Relative to a supercompact cardinal it is consistent that κ *is supercompact and*

$$V_\kappa \models \text{“PFA } \textit{fails in all forcing extensions} \text{”}.$$

The evidence

Theorem (Magidor '76, Viale '06)
Relative to a supercompact cardinal it is consistent that κ is supercompact and

$$V_\kappa \models \text{“PFA } \textit{fails in all forcing extensions} \text{”}.$$

Idea: Make SCH fails unboundedly often below a supercompact cardinal.

The evidence

Theorem (Magidor '76, Viale '06)
Relative to a supercompact cardinal it is consistent that κ is supercompact and

$$V_\kappa \models \text{"PFA } \textit{fails in all forcing extensions}\text{"}.$$

Idea: Make SCH fails unboundedly often below a supercompact cardinal.

Theorem (Viale-Weiß)
Suppose κ is inaccessible and a standard forcing iteration $\mathbb{P}$ of length κ forces PFA and $\omega_2 = \kappa$. Then κ is strongly compact. If $\mathbb{P}$ is proper then κ is supercompact.

The evidence

Theorem (Magidor '76, Viale '06)
Relative to a supercompact cardinal it is consistent that κ is supercompact and

$$V_\kappa \models \text{“}\mathrm{PFA} \textit{ fails in all forcing extensions”}.$$

Idea: Make SCH fails unboundedly often below a supercompact cardinal.

Theorem (Viale-Weiß)
Suppose κ is inaccessible and a standard forcing iteration $\mathbb{P}$ of length κ forces PFA and $\omega_2 = \kappa$. Then κ is strongly compact. If $\mathbb{P}$ is proper then κ is supercompact.

But what is this really evidence for?

Practical Consistency Strength

Practical Consistency Strength

Warning informal!

Practical Consistency Strength

Warning informal!

Definition

The **practical consistency strength** of a principle φ is the least large cardinal property $\psi(\kappa)$ so that

$$\mathrm{ZFC} + \exists\kappa\ \psi(\kappa) \vdash \exists\mathbb{P}\ \Vdash_{\mathbb{P}} \varphi.$$

Practical Consistency Strength

Warning informal!

Definition

The **practical consistency strength** of a principle φ is the least large cardinal property $\psi(\kappa)$ so that

$$\mathrm{ZFC} + \exists\kappa\ \psi(\kappa) \vdash \exists\mathbb{P}\ \Vdash_{\mathbb{P}} \varphi.$$

More generally, we may ask that the forcing has some nice properties. E.g. it is natural to ask for for a proper forcing which forces PFA, instead of any.

Practical Consistency Strength

Warning informal!

Definition
The **practical consistency strength** of a principle φ is the least large cardinal property $\psi(\kappa)$ so that

$$\mathrm{ZFC} + \exists\kappa\ \psi(\kappa) \vdash \exists\mathbb{P}\ \Vdash_{\mathbb{P}} \varphi.$$

More generally, we may ask that the forcing has some nice properties. E.g. it is natural to ask for for a proper forcing which forces PFA, instead of any.

Definition
Let $\mathcal{P}$ be a class of forcings. The $\mathcal{P}$-**practical consistency strength** of a principle φ is the least large cardinal property $\psi(\kappa)$ so that

$$\mathrm{ZFC} + \exists\kappa\ \psi(\kappa) \vdash \exists\mathbb{P} \in \mathcal{P}\ \Vdash_{\mathbb{P}} \varphi.$$

Practical Consistency Strength

Warning informal!

Definition

The **practical consistency strength** of a principle φ is the least large cardinal property $\psi(\kappa)$ so that

$$\mathrm{ZFC} + \exists\kappa\ \psi(\kappa) \vdash \exists\mathbb{P}\ \Vdash_{\mathbb{P}} \varphi.$$

More generally, we may ask that the forcing has some nice properties. E.g. it is natural to ask for for a proper forcing which forces PFA, instead of any.

Definition

Let $\mathcal{P}$ be a class of forcings. The $\mathcal{P}$-**practical consistency strength** of a principle φ is the least large cardinal property $\psi(\kappa)$ so that

$$\mathrm{ZFC} + \exists\kappa\ \psi(\kappa) \vdash \exists\mathbb{P} \in \mathcal{P}\ \Vdash_{\mathbb{P}} \varphi.$$

These notions are as rigorous as your belief in (i) the strength of natural theories corresponds to large cardinals and (ii) the large cardinal hierarchy is wellordered.

Some Examples

Statement	Con. Str.	Practical Con. Str.	$\mathcal{P}$-Practical Con. Str.

Some Examples

Statement	**Con. Str.**	**Practical Con. Str.**	**$\mathcal{P}$-Practical Con. Str.**
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.			

Some Examples

Statement	**Con. Str.**	**Practical Con. Str.**	**$\mathcal{P}$-Practical Con. Str.**
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible		

Some Examples

Statement	Con. Str.	Practical Con. Str.	$\mathcal{P}$-Practical Con. Str.
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible	inaccessible	

Some Examples

Statement	Con. Str.	Practical Con. Str.	$\mathcal{P}$-Practical Con. Str.
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible	inaccessible	$\mathcal{P} = ?$

Some Examples

Statement	**Con. Str.**	**Practical Con. Str.**	$\mathcal{P}$-**Practical Con. Str.**
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible	inaccessible	$\mathcal{P} = ?$
BPFA			

Some Examples

Statement	Con. Str.	Practical Con. Str.	$\mathcal{P}$-Practical Con. Str.
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible	inaccessible	$\mathcal{P} = ?$
BPFA	Σ_2-reflecting		

Some Examples

Statement	**Con. Str.**	**Practical Con. Str.**	$\mathcal{P}$**-Practical Con. Str.**
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible	inaccessible	$\mathcal{P} = ?$
BPFA	Σ_2-reflecting	Σ_2-reflecting	

Some Examples

Statement	Con. Str.	Practical Con. Str.	$\mathcal{P}$-Practical Con. Str.
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible	inaccessible	$\mathcal{P} =$?
BPFA	Σ_2-reflecting	Σ_2-reflecting	Σ_2-reflecting ($\mathcal{P} =$ proper)

Some Examples

Statement	Con. Str.	Practical Con. Str.	$\mathcal{P}$-Practical Con. Str.
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible	inaccessible	$\mathcal{P} = ?$
BPFA	Σ_2-reflecting	Σ_2-reflecting	Σ_2-reflecting ($\mathcal{P}$ = proper)
$\neg\square_{\omega_1}$			

Some Examples

Statement	**Con. Str.**	**Practical Con. Str.**	$\mathcal{P}$-**Practical Con. Str.**
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible	inaccessible	$\mathcal{P}$ = ?
BPFA	Σ_2-reflecting	Σ_2-reflecting	Σ_2-reflecting ($\mathcal{P}$ = proper)
$\neg\square_{\omega_1}$	Mahlo		

Some Examples

Statement	Con. Str.	Practical Con. Str.	$\mathcal{P}$-Practical Con. Str.
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible	inaccessible	$\mathcal{P} = ?$
BPFA	Σ_2-reflecting	Σ_2-reflecting	Σ_2-reflecting ($\mathcal{P}$ = proper)
$\neg\square_{\omega_1}$	Mahlo	Mahlo	

Some Examples

Statement	Con. Str.	Practical Con. Str.	$\mathcal{P}$-Practical Con. Str.
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible	inaccessible	$\mathcal{P} = ?$
BPFA	Σ_2-reflecting	Σ_2-reflecting	Σ_2-reflecting ($\mathcal{P}$ = proper)
$\neg\square_{\omega_1}$	Mahlo	Mahlo	Mahlo ($\mathcal{P} = \sigma$-closed)

Some Examples

Statement	**Con. Str.**	**Practical Con. Str.**	**$\mathcal{P}$-Practical Con. Str.**
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible	inaccessible	$\mathcal{P} = ?$
BPFA	Σ_2-reflecting	Σ_2-reflecting	Σ_2-reflecting ($\mathcal{P} =$ proper)
$\neg\square_{\omega_1}$	Mahlo	Mahlo	Mahlo ($\mathcal{P} = \sigma$-closed)
Π^1_1-determinacy			

Some Examples

Statement	Con. Str.	Practical Con. Str.	$\mathcal{P}$-Practical Con. Str.
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible	inaccessible	$\mathcal{P} =$?
BPFA	Σ_2-reflecting	Σ_2-reflecting	Σ_2-reflecting ($\mathcal{P} =$ proper)
$\neg\Box_{\omega_1}$	Mahlo	Mahlo	Mahlo ($\mathcal{P} = \sigma$-closed)
Π^1_1-determinacy	$0^\sharp$ exists		

Some Examples

Statement	Con. Str.	Practical Con. Str.	$\mathcal{P}$-Practical Con. Str.
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible	inaccessible	$\mathcal{P}$ = ?
BPFA	Σ_2-reflecting	Σ_2-reflecting	Σ_2-reflecting ($\mathcal{P}$ = proper)
$\neg\Box_{\omega_1}$	Mahlo	Mahlo	Mahlo ($\mathcal{P}$ = σ-closed)
Π^1_1-determinacy	$0^\sharp$ exists	$0^\sharp$ exists	

Some Examples

Statement	**Con. Str.**	**Practical Con. Str.**	**$\mathcal{P}$-Practical Con. Str.**
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible	inaccessible	$\mathcal{P} = ?$
BPFA	Σ_2-reflecting	Σ_2-reflecting	Σ_2-reflecting ($\mathcal{P}$ = proper)
$\neg\square_{\omega_1}$	Mahlo	Mahlo	Mahlo ($\mathcal{P} = \sigma$-closed)
Π^1_1-determinacy	$0^\sharp$ exists	$0^\sharp$ exists	$0^\sharp$ exists ($\mathcal{P} = \{\{\mathbb{1}\}\}$)

Some Examples

Statement	**Con. Str.**	**Practical Con. Str.**	**$\mathcal{P}$-Practical Con. Str.**
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible	inaccessible	$\mathcal{P} = ?$
BPFA	Σ_2-reflecting	Σ_2-reflecting	Σ_2-reflecting ($\mathcal{P}$ = proper)
$\neg\square_{\omega_1}$	Mahlo	Mahlo	Mahlo ($\mathcal{P} = \sigma$-closed)
Π^1_1-determinacy	$0^\sharp$ exists	$0^\sharp$ exists	$0^\sharp$ exists ($\mathcal{P} = \{\{\mathbb{1}\}\}$)
$\mathfrak{c}$ is real-valued measurable			

Some Examples

Statement	Con. Str.	Practical Con. Str.	$\mathcal{P}$-Practical Con. Str.
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible	inaccessible	$\mathcal{P} = ?$
BPFA	Σ_2-reflecting	Σ_2-reflecting	Σ_2-reflecting ($\mathcal{P}$ = proper)
$\neg\Box_{\omega_1}$	Mahlo	Mahlo	Mahlo ($\mathcal{P} = \sigma$-closed)
Π^1_1-determinacy	$0^\sharp$ exists	$0^\sharp$ exists	$0^\sharp$ exists ($\mathcal{P} = \{\{\mathbb{1}\}\}$)
$\mathfrak{c}$ is real-valued measurable	measurable		

Some Examples

Statement	**Con. Str.**	**Practical Con. Str.**	**$\mathcal{P}$-Practical Con. Str.**
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible	inaccessible	$\mathcal{P}$ = ?
BPFA	Σ_2-reflecting	Σ_2-reflecting	Σ_2-reflecting ($\mathcal{P}$ = proper)
$\neg\Box_{\omega_1}$	Mahlo	Mahlo	Mahlo ($\mathcal{P} = \sigma$-closed)
Π^1_1-determinacy	$0^\sharp$ exists	$0^\sharp$ exists	$0^\sharp$ exists ($\mathcal{P} = \{\{\mathbb{1}\}\}$)
$\mathfrak{c}$ is real-valued measurable	measurable	measurable	

Some Examples

Statement	Con. Str.	Practical Con. Str.	$\mathcal{P}$-Practical Con. Str.
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible	inaccessible	$\mathcal{P}$ = ?
BPFA	Σ_2-reflecting	Σ_2-reflecting	Σ_2-reflecting ($\mathcal{P}$ = proper)
$\neg\square_{\omega_1}$	Mahlo	Mahlo	Mahlo ($\mathcal{P} = \sigma$-closed)
Π^1_1-determinacy	$0^\sharp$ exists	$0^\sharp$ exists	$0^\sharp$ exists ($\mathcal{P} = \{\{\mathbb{1}\}\}$)
$\mathfrak{c}$ is real-valued measurable	measurable	measurable	measurable ($\mathcal{P}$ = c.c.c.)

Some Examples

Statement	Con. Str.	Practical Con. Str.	$\mathcal{P}$-Practical Con. Str.
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible	inaccessible	$\mathcal{P} = ?$
BPFA	Σ_2-reflecting	Σ_2-reflecting	Σ_2-reflecting ($\mathcal{P}$ = proper)
$\neg\square_{\omega_1}$	Mahlo	Mahlo	Mahlo ($\mathcal{P} = \sigma$-closed)
Π^1_1-determinacy	$0^\sharp$ exists	$0^\sharp$ exists	$0^\sharp$ exists ($\mathcal{P} = \{\{\mathbb{1}\}\}$)
$\mathfrak{c}$ is real-valued measurable	measurable	measurable	measurable ($\mathcal{P}$ = c.c.c.)
Club Bounding			

Some Examples

Statement	Con. Str.	Practical Con. Str.	$\mathcal{P}$-Practical Con. Str.
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible	inaccessible	$\mathcal{P} = ?$
BPFA	Σ_2-reflecting	Σ_2-reflecting	Σ_2-reflecting ($\mathcal{P}$ = proper)
$\neg\Box_{\omega_1}$	Mahlo	Mahlo	Mahlo ($\mathcal{P} = \sigma$-closed)
Π^1_1-determinacy	$0^\sharp$ exists	$0^\sharp$ exists	$0^\sharp$ exists ($\mathcal{P} = \{\{\mathbb{1}\}\}$)
$\mathfrak{c}$ is real-valued measurable	measurable	measurable	measurable ($\mathcal{P}$ = c.c.c.)
Club Bounding	inacc. lim. of meas.		

Some Examples

Statement	Con. Str.	Practical Con. Str.	$\mathcal{P}$-Practical Con. Str.
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible	inaccessible	$\mathcal{P} = ?$
BPFA	Σ_2-reflecting	Σ_2-reflecting	Σ_2-reflecting ($\mathcal{P}$ = proper)
$\neg\square_{\omega_1}$	Mahlo	Mahlo	Mahlo ($\mathcal{P} = \sigma$-closed)
Π^1_1-determinacy	$0^\sharp$ exists	$0^\sharp$ exists	$0^\sharp$ exists ($\mathcal{P} = \{\{\mathbb{1}\}\}$)
$\mathfrak{c}$ is real-valued measurable	measurable	measurable	measurable ($\mathcal{P}$ = c.c.c.)
Club Bounding	inacc. lim. of meas.	inacc. lim. of meas.	

Some Examples

Statement	Con. Str.	Practical Con. Str.	$\mathcal{P}$-Practical Con. Str.
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible	inaccessible	$\mathcal{P} = ?$
BPFA	Σ_2-reflecting	Σ_2-reflecting	Σ_2-reflecting ($\mathcal{P} =$ proper)
$\neg\Box_{\omega_1}$	Mahlo	Mahlo	Mahlo ($\mathcal{P} = \sigma$-closed)
Π^1_1-determinacy	$0^\sharp$ exists	$0^\sharp$ exists	$0^\sharp$ exists ($\mathcal{P} = \{\{\mathbb{1}\}\}$)
$\mathfrak{c}$ is real-valued measurable	measurable	measurable	measurable ($\mathcal{P} =$ c.c.c.)
Club Bounding	inacc. lim. of meas.	inacc. lim. of meas.	inacc. lim. of meas. ($\mathcal{P} =$ semi-proper)

Some Examples

Statement	Con. Str.	Practical Con. Str.	$\mathcal{P}$-Practical Con. Str.
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible	inaccessible	$\mathcal{P}$ = ?
BPFA	Σ_2-reflecting	Σ_2-reflecting	Σ_2-reflecting ($\mathcal{P}$ = proper)
$\neg\square_{\omega_1}$	Mahlo	Mahlo	Mahlo ($\mathcal{P} = \sigma$-closed)
Π^1_1-determinacy	$0^\sharp$ exists	$0^\sharp$ exists	$0^\sharp$ exists ($\mathcal{P} = \{\{\mathbb{1}\}\}$)
$\mathfrak{c}$ is real-valued measurable	measurable	measurable	measurable ($\mathcal{P}$ = c.c.c.)
Club Bounding	inacc. lim. of meas.	inacc. lim. of meas.	inacc. lim. of meas. ($\mathcal{P}$ = semi-proper)
NS_{ω_1} is saturated			

Some Examples

Statement	**Con. Str.**	**Practical Con. Str.**	**$\mathcal{P}$-Practical Con. Str.**
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible	inaccessible	$\mathcal{P} = ?$
BPFA	Σ_2-reflecting	Σ_2-reflecting	Σ_2-reflecting ($\mathcal{P}$ = proper)
$\neg\Box_{\omega_1}$	Mahlo	Mahlo	Mahlo ($\mathcal{P} = \sigma$-closed)
Π^1_1-determinacy	$0^\sharp$ exists	$0^\sharp$ exists	$0^\sharp$ exists ($\mathcal{P} = \{\{\mathbb{1}\}\}$)
$\mathfrak{c}$ is real-valued measurable	measurable	measurable	measurable ($\mathcal{P}$ = c.c.c.)
Club Bounding	inacc. lim. of meas.	inacc. lim. of meas.	inacc. lim. of meas. ($\mathcal{P}$ = semi-proper)
NS_{ω_1} is saturated	Woodin		

Some Examples

Statement	**Con. Str.**	**Practical Con. Str.**	**$\mathcal{P}$-Practical Con. Str.**
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible	inaccessible	$\mathcal{P}$ = ?
BPFA	Σ_2-reflecting	Σ_2-reflecting	Σ_2-reflecting ($\mathcal{P}$ = proper)
$\neg\Box_{\omega_1}$	Mahlo	Mahlo	Mahlo ($\mathcal{P} = \sigma$-closed)
Π^1_1-determinacy	$0^\sharp$ exists	$0^\sharp$ exists	$0^\sharp$ exists ($\mathcal{P} = \{\{\mathbb{1}\}\}$)
$\mathfrak{c}$ is real-valued measurable	measurable	measurable	measurable ($\mathcal{P}$ = c.c.c.)
Club Bounding	inacc. lim. of meas.	inacc. lim. of meas.	inacc. lim. of meas. ($\mathcal{P}$ = semi-proper)
NS_{ω_1} is saturated	Woodin	Woodin	

Some Examples

Statement	Con. Str.	Practical Con. Str.	$\mathcal{P}$-Practical Con. Str.
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible	inaccessible	$\mathcal{P} = ?$
BPFA	Σ_2-reflecting	Σ_2-reflecting	Σ_2-reflecting ($\mathcal{P} =$ proper)
$\neg\square_{\omega_1}$	Mahlo	Mahlo	Mahlo ($\mathcal{P} = \sigma$-closed)
Π^1_1-determinacy	$0^\sharp$ exists	$0^\sharp$ exists	$0^\sharp$ exists ($\mathcal{P} = \{\{\mathbb{1}\}\}$)
$\mathfrak{c}$ is real-valued measurable	measurable	measurable	measurable ($\mathcal{P} =$ c.c.c.)
Club Bounding	inacc. lim. of meas.	inacc. lim. of meas.	inacc. lim. of meas. ($\mathcal{P} =$ semi-proper)
NS_{ω_1} is saturated	Woodin	Woodin	Woodin ($\mathcal{P} = \mathrm{SSP}$)

Some Examples

Statement	**Con. Str.**	**Practical Con. Str.**	**$\mathcal{P}$-Practical Con. Str.**
All sets of reals in $L(\mathbb{R})$ are Lebesgue meas.	inaccessible	inaccessible	$\mathcal{P} = ?$
BPFA	Σ_2-reflecting	Σ_2-reflecting	Σ_2-reflecting ($\mathcal{P} =$ proper)
$\neg\Box_{\omega_1}$	Mahlo	Mahlo	Mahlo ($\mathcal{P} = \sigma$-closed)
Π^1_1-determinacy	$0^\sharp$ exists	$0^\sharp$ exists	$0^\sharp$ exists ($\mathcal{P} = \{\{\mathbb{1}\}\}$)
$\mathfrak{c}$ is real-valued measurable	measurable	measurable	measurable ($\mathcal{P} =$ c.c.c.)
Club Bounding	inacc. lim. of meas.	inacc. lim. of meas.	inacc. lim. of meas. ($\mathcal{P} =$ semi-proper)
NS_{ω_1} is saturated	Woodin	Woodin	Woodin ($\mathcal{P} = \mathrm{SSP}$)

So practical consistency strength = standard consistency strength in these cases.

Indeed, everybody knows that this is almost always true.

But its not always always true!

A More Drastic Example

A More Drastic Example

A somewhat more drastic example is $\varphi =$ "NS_{ω_1} is ω_1-dense".

A More Drastic Example

A somewhat more drastic example is $\varphi =$ "NS_{ω_1} is ω_1-dense".
The consistency strength of φ is again infinitely many Woodin cardinals.

A More Drastic Example

A somewhat more drastic example is $\varphi =$ "NS_{ω_1} is ω_1-dense".
The consistency strength of φ is again infinitely many Woodin cardinals.

Theorem (L., Woodin)
The SSP-practical consistency strength of "NS_{ω_1} is ω_1-dense" is between a Woodin limit of Woodin cardinals and an inaccessible κ which is a limit of $<\kappa$-supercompact cardinals.

A More Drastic Example

A somewhat more drastic example is $\varphi =$ "NS_{ω_1} is ω_1-dense".
The consistency strength of φ is again infinitely many Woodin cardinals.

Theorem (L., Woodin)
The SSP-practical consistency strength of "NS_{ω_1} is ω_1-dense" is between a Woodin limit of Woodin cardinals and an inaccessible κ which is a limit of $<\kappa$-supercompact cardinals.

It is totally unclear (to me) where the practical consistency strength and SSP-practical consistency strength of "NS_{ω_1} is ω_1-dense" lies.

The Evidence, again

The Evidence, again

Theorem (Magidor, Viale)
Relative to a supercompact cardinal it is consistent that κ is supercompact and

$$V_\kappa \models \text{“PFA } \textit{fails in all forcing extensions}\text{”}.$$

The Evidence, again

Theorem (Magidor, Viale)
Relative to a supercompact cardinal it is consistent that κ is supercompact and

$$V_\kappa \models \text{“PFA } \textit{fails in all forcing extensions}\text{''}.$$

Theorem (Viale-Weiß)
Suppose κ is inaccessible and a standard forcing iteration $\mathbb{P}$ of length κ forces PFA and $\omega_2 = \kappa$. Then κ is strongly compact. If $\mathbb{P}$ is proper then κ is supercompact.

The Evidence, again

Theorem (Magidor, Viale)
Relative to a supercompact cardinal it is consistent that κ is supercompact and

$$V_\kappa \models \text{“}\mathrm{PFA} \textit{ fails in all forcing extensions”}.$$

Theorem (Viale-Weiß)
Suppose κ is inaccessible and a standard forcing iteration $\mathbb{P}$ of length κ forces PFA and $\omega_2 = \kappa$. Then κ is strongly compact. If $\mathbb{P}$ is proper then κ is supercompact.

These theorems “prove” that the practical consistency strength/proper-practical consistency strength of PFA is a supercompact cardinal. But they (arguably) say less about the consistency strength of PFA.

The Evidence, again

Theorem (Magidor, Viale)
Relative to a supercompact cardinal it is consistent that κ is supercompact and

$$V_\kappa \models \text{“PFA \textit{fails in all forcing extensions}”}.$$

Theorem (Viale-Weiß)
Suppose κ is inaccessible and a standard forcing iteration $\mathbb{P}$ of length κ forces PFA and $\omega_2 = \kappa$. Then κ is strongly compact. If $\mathbb{P}$ is proper then κ is supercompact.

These theorems “prove” that the practical consistency strength/proper-practical consistency strength of PFA is a supercompact cardinal. But they (arguably) say less about the consistency strength of PFA.

Why should consistency strength and practical consistency strength agree in the case of PFA?

Some Questions Regarding Practical Consistency Strength

Let us say that a principle φ is **impractical** if the consistency and practical consistency strength of φ differ.

Some Questions Regarding Practical Consistency Strength

Let us say that a principle φ is **impractical** if the consistency and practical consistency strength of φ differ.

Here are some more candidates, are they all impractical?

Some Questions Regarding Practical Consistency Strength

Let us say that a principle φ is **impractical** if the consistency and practical consistency strength of φ differ.

Here are some more candidates, are they all impractical?

- $\neg\square(\omega_3) + \neg\square(\omega_4)$
- $\mathrm{MM}^{++}(\mathfrak{c})$
- Sealing (for uB-sets)

Some Questions Regarding Practical Consistency Strength

Let us say that a principle φ is **impractical** if the consistency and practical consistency strength of φ differ.

Here are some more candidates, are they all impractical?

- $\neg\Box(\omega_3) + \neg\Box(\omega_4)$
- $\mathrm{MM}^{++}(\mathfrak{c})$
- Sealing (for uB-sets)

Impractical statements seem to be common above Woodin cardinals and rare below.

Some Questions Regarding Practical Consistency Strength

Let us say that a principle φ is **impractical** if the consistency and practical consistency strength of φ differ.

Here are some more candidates, are they all impractical?

- $\neg\Box(\omega_3) + \neg\Box(\omega_4)$
- $\mathrm{MM}^{++}(\mathfrak{c})$
- Sealing (for uB-sets)

Impractical statements seem to be common above Woodin cardinals and rare below.

Question
Is there a natural impractical statement with consistency strength below a Woodin cardinal?

How low can we go?

Some Questions Regarding Practical Consistency Strength

Let us say that a principle φ is **impractical** if the consistency and practical consistency strength of φ differ.

Here are some more candidates, are they all impractical?

- $\neg\square(\omega_3) + \neg\square(\omega_4)$
- $\mathrm{MM}^{++}(\mathfrak{c})$
- Sealing (for uB-sets)

Impractical statements seem to be common above Woodin cardinals and rare below.

Question
Is there a natural impractical statement with consistency strength below a Woodin cardinal? How low can we go?

Question
Can we tell impractical statements from practical statements heuristically?

A Candidate Suggested By Woodin

A Candidate Suggested By Woodin

This candidate for an impractical statement of consistency strength below a Woodin was suggested by Woodin.

A Candidate Suggested By Woodin

This candidate for an impractical statement of consistency strength below a Woodin was suggested by Woodin. It is related to the 12th Delfino problem. Consider

$$\varphi = \text{“All projective sets of reals are Lebesgue measurable, have the property of Baire and are uniformized by projectives sets”.}$$

A Candidate Suggested By Woodin

This candidate for an impractical statement of consistency strength below a Woodin was suggested by Woodin. It is related to the 12th Delfino problem. Consider

$$\varphi = \text{“All projective sets of reals are Lebesgue measurable, have the property of Baire and are uniformized by projectives sets”}.$$

These are all natural consequences of projective determinacy.

A Candidate Suggested By Woodin

This candidate for an impractical statement of consistency strength below a Woodin was suggested by Woodin. It is related to the 12th Delfino problem. Consider

$$\varphi = \text{“All projective sets of reals are Lebesgue measurable, have the property of Baire and are uniformized by projectives sets”.}$$

These are all natural consequences of projective determinacy. The 12th Delfino asks whether φ is equivalent to projective determinacy. This was answered in the negative by Steel.

A Candidate Suggested By Woodin

This candidate for an impractical statement of consistency strength below a Woodin was suggested by Woodin. It is related to the 12th Delfino problem. Consider

$$\varphi = \text{“All projective sets of reals are Lebesgue measurable, have the property of Baire and are uniformized by projectives sets”}.$$

These are all natural consequences of projective determinacy. The 12th Delfino asks whether φ is equivalent to projective determinacy. This was answered in the negative by Steel.

Theorem (Schindler,Steel)
The consistency strength of φ is slightly less than infinitely many strong cardinals.

A Candidate Suggested By Woodin

This candidate for an impractical statement of consistency strength below a Woodin was suggested by Woodin. It is related to the 12th Delfino problem. Consider

$$\varphi = \text{“All projective sets of reals are Lebesgue measurable, have the property of Baire and are uniformized by projectives sets”}.$$

These are all natural consequences of projective determinacy. The 12th Delfino asks whether φ is equivalent to projective determinacy. This was answered in the negative by Steel.

Theorem (Schindler,Steel)
The consistency strength of φ is slightly less than infinitely many strong cardinals.

The only known upper bound for practical consistency strength is the strength of PD.

Maximality Principles and Correct Forcing Axioms

Maxmimality Principles

Maxmimality Principles

For $\mathcal{P}$ a class of forcings, a statement $\varphi(x)$ is **provably $\mathcal{P}$-persistent** if

$$\mathrm{ZFC} \vdash \forall x(\varphi(x) \to \forall \mathbb{P} \in \mathcal{P}\ V^{\mathbb{P}} \models \varphi(x)).$$

Definition (Stavi-Väänänen)
Let Γ be a class of first order formulas, $\mathcal{P}$ a definable class of forcings. Γ-$\mathrm{MP}(\mathcal{P}, A)$ holds if

$$\forall \text{provably } \mathcal{P}\text{-persistent } \varphi \in \Gamma\ \forall a \in A \left(\exists \mathbb{P} \in \mathcal{P}\ V^{\mathbb{P}} \models \varphi(a) \to V \models \varphi(a)\right).$$

Studied by Fuchino, Fuchs, Gappo, Goodman, Hamkins, Ikegami, Minden, Parente, Trang, Woodin...

Maxmimality Principles

For $\mathcal{P}$ a class of forcings, a statement $\varphi(x)$ is **provably $\mathcal{P}$-persistent** if

$$\mathrm{ZFC} \vdash \forall x(\varphi(x) \to \forall \mathbb{P} \in \mathcal{P}\ V^{\mathbb{P}} \models \varphi(x)).$$

Definition (Stavi-Väänänen)
Let Γ be a class of first order formulas, $\mathcal{P}$ a definable class of forcings. Γ-$\mathrm{MP}(\mathcal{P}, A)$ holds if

$$\forall \text{provably } \mathcal{P}\text{-persistent } \varphi \in \Gamma\ \forall a \in A \left(\exists \mathbb{P} \in \mathcal{P}\ V^{\mathbb{P}} \models \varphi(a) \to V \models \varphi(a)\right).$$

Studied by Fuchino, Fuchs, Gappo, Goodman, Hamkins, Ikegami, Minden, Parente, Trang, Woodin...

Theorem (Stavi-Väänänen)
Σ_ω-$\mathrm{MP}(\mathcal{P}, H_{2^\omega})$ *is equiconsistent with a reflecting cardinal for* $\mathcal{P}$ *=c.c.c, all.*

Maxmimality Principles

For $\mathcal{P}$ a class of forcings, a statement $\varphi(x)$ is **provably $\mathcal{P}$-persistent** if

$$\mathrm{ZFC} \vdash \forall x(\varphi(x) \rightarrow \forall \mathbb{P} \in \mathcal{P}\ V^{\mathbb{P}} \models \varphi(x)).$$

Definition (Stavi-Väänänen)
Let Γ be a class of first order formulas, $\mathcal{P}$ a definable class of forcings. Γ-$\mathrm{MP}(\mathcal{P}, A)$ holds if

$$\forall \text{provably } \mathcal{P}\text{-persistent } \varphi \in \Gamma\ \forall a \in A \left(\exists \mathbb{P} \in \mathcal{P}\ V^{\mathbb{P}} \models \varphi(a) \rightarrow V \models \varphi(a)\right).$$

Studied by Fuchino, Fuchs, Gappo, Goodman, Hamkins, Ikegami, Minden, Parente, Trang, Woodin...

Theorem (Stavi-Väänänen)
Σ_ω-$\mathrm{MP}(\mathcal{P}, H_{2^\omega})$ is equiconsistent with a reflecting cardinal for $\mathcal{P}$ =c.c.c, all.

The consistency strength of Σ_ω-$\mathrm{MP}(\mathcal{P}, H_{\omega_2})$ is also a reflecting cardinal for $\mathcal{P} = \sigma$-closed, proper, semi-proper.

Maxmimality Principles

For $\mathcal{P}$ a class of forcings, a statement $\varphi(x)$ is **provably $\mathcal{P}$-persistent** if

$$\mathrm{ZFC} \vdash \forall x(\varphi(x) \to \forall \mathbb{P} \in \mathcal{P}\ V^{\mathbb{P}} \models \varphi(x)).$$

Definition (Stavi-Väänänen)
Let Γ be a class of first order formulas, $\mathcal{P}$ a definable class of forcings. Γ-$\mathrm{MP}(\mathcal{P}, A)$ holds if

$$\forall \text{provably } \mathcal{P}\text{-persistent } \varphi \in \Gamma\ \forall a \in A \left(\exists \mathbb{P} \in \mathcal{P}\ V^{\mathbb{P}} \models \varphi(a) \to V \models \varphi(a)\right).$$

Studied by Fuchino, Fuchs, Gappo, Goodman, Hamkins, Ikegami, Minden, Parente, Trang, Woodin...

Theorem (Stavi-Väänänen)
Σ_ω-$\mathrm{MP}(\mathcal{P}, H_{2^\omega})$ *is equiconsistent with a reflecting cardinal for* $\mathcal{P} =$*c.c.c, all.*

The consistency strength of Σ_ω-$\mathrm{MP}(\mathcal{P}, H_{\omega_2})$ is also a reflecting cardinal for $\mathcal{P} = \sigma$-closed, proper, semi-proper.

Forcing Axioms I

Definition
$\mathrm{FA}_{\omega_1}(\mathcal{P})$ holds if for any $\leqslant \omega_1$-many dense subsets of some $\mathbb{P} \in \mathcal{P}$, there is a filter $g \subseteq \mathbb{P}$ meeting all those dense sets.

Forcing Axioms I

Definition
$\mathrm{FA}_{\omega_1}(\mathcal{P})$ holds if for any $\leqslant \omega_1$-many dense subsets of some $\mathbb{P} \in \mathcal{P}$, there is a filter $g \subseteq \mathbb{P}$ meeting all those dense sets.

Fact (Jensen)
The following are equivalent.

1. $\mathrm{FA}_{\omega_1}(\mathcal{P})$.
2. *Whenever* $\mathbb{P} \in \mathcal{P}$ *and* θ *is sufficiently large and regular,* $\mathbb{P} \in X \subseteq H_\theta$ *is of size* ω_1 *then there is some* $Y \prec H_\theta$ *of size* ω_1, $X \cup \omega_1 \subseteq Y$ *such that if*

$$\pi \colon M \to Y$$

is the anticollapse then there is a M*-generic filter* $g \subseteq \pi^{-1}(\mathbb{P})$.

Forcing Axioms II

Definition

Recall that $\mathrm{FA}^+_{\omega_1}(\mathcal{P})$ holds if for all $\mathbb{P} \in \mathcal{P}$, $\mathcal{D}$ a set of $\leqslant \omega_1$-many dense subsets of $\mathbb{P}$ and $\dot{S}$ a name for a stationary subset of ω_1, there is a $\mathcal{D}$-generic filter g with $\dot{S}^g$ stationary.

Forcing Axioms II

Definition
Recall that $\mathrm{FA}^+_{\omega_1}(\mathcal{P})$ holds if for all $\mathbb{P} \in \mathcal{P}$, $\mathcal{D}$ a set of $\leqslant \omega_1$-many dense subsets of $\mathbb{P}$ and $\dot{S}$ a name for a stationary subset of ω_1, there is a $\mathcal{D}$-generic filter g with $\dot{S}^g$ stationary.

Fact (Jensen)
The following are equivalent.

1. $\mathrm{FA}^+_{\omega_1}(\mathcal{P})$.
2. *Whenever $\mathbb{P} \in \mathcal{P}$, $\dot{S}$ a $\mathbb{P}$-name for a stationary set and θ is sufficiently large and regular, $\mathbb{P}, \dot{S} \in X \subseteq H_\theta$ is of size ω_1 then there is some $Y \prec H_\theta$ of size ω_1, $X \cup \omega_1 \subseteq Y$ such that if*

$$\pi\colon M \to Y$$

is the anticollapse then there is a M-generic filter $g \subseteq \pi^{-1}(\mathbb{P})$ such that $\pi^{-1}(\dot{S})^g$ is stationary.

Correct Forcing Axioms I

Correct Forcing Axioms I

Forcing axioms and maximality principles are roughly “orthogonal but in the same spirit”.

Correct Forcing Axioms I

Forcing axioms and maximality principles are roughly "orthogonal but in the same spirit".

Ben Goodman found a beautiful amalgamation of $\mathrm{FA}_{\omega_1}(\mathcal{P})$ and $\Gamma\text{-}\mathrm{MP}(\mathcal{P}, H_{\omega_2})$.

Correct Forcing Axioms I

Forcing axioms and maximality principles are roughly "orthogonal but in the same spirit".

Ben Goodman found a beautiful amalgamation of $\mathrm{FA}_{\omega_1}(\mathcal{P})$ and Γ-$\mathrm{MP}(\mathcal{P}, H_{\omega_2})$.

Definition (Goodman)
Suppose $\mathcal{P}$ is a definable class of forcings and Γ is a class of first order formulas.

Correct Forcing Axioms I

Forcing axioms and maximality principles are roughly "orthogonal but in the same spirit".
Ben Goodman found a beautiful amalgamation of $\mathrm{FA}_{\omega_1}(\mathcal{P})$ and Γ-$\mathrm{MP}(\mathcal{P}, H_{\omega_2})$.

Definition (Goodman)
Suppose $\mathcal{P}$ is a definable class of forcings and Γ is a class of first order formulas. The **Correct Forcing Axiom** Γ-$\mathrm{CFA}(\mathcal{P})$ holds if whenever $\mathbb{P} \in \mathcal{P}$, θ is sufficiently large and regular, $\dot{a} \in H_\theta$, $\varphi \in \Gamma$ is provably $\mathcal{P}$-persistent and

$$\Vdash_{\mathbb{P}} \varphi(\dot{a})$$

for some $\mathbb{P} \in \mathcal{P}$ then there is some elementary $Y \prec H_\theta$ of size ω_1 with $\omega_1 \cup \{\dot{a}, \mathbb{P}\} \subseteq Y$ such that if

$$\pi \colon M \to Y \prec H_\theta$$

is the anticollapse then there is an M-generic filter $g \subseteq \pi^{-1}(\mathbb{P})$ so that $V \models \varphi(\pi^{-1}(\dot{a})^g)$.

Correct Forcing Axioms II

Fact (Goodman)
$\Gamma\text{-CFA}(\mathcal{P})$ *implies both* $\mathrm{FA}_{\omega_1}(\mathcal{P})$ *and* $\Gamma\text{-MP}(\mathcal{P}, H_{\omega_2})$ *(for nontrivial* Γ*).*

Correct Forcing Axioms II

Fact (Goodman)
$\Gamma\text{-}\mathrm{CFA}(\mathcal{P})$ *implies both* $\mathrm{FA}_{\omega_1}(\mathcal{P})$ *and* $\Gamma\text{-}\mathrm{MP}(\mathcal{P}, H_{\omega_2})$ *(for nontrivial* Γ*).*

I want to focus on the Correct Proper Forcing Axioms.

Correct Forcing Axioms II

Fact (Goodman)
Γ-$\mathrm{CFA}(\mathcal{P})$ *implies both* $\mathrm{FA}_{\omega_1}(\mathcal{P})$ *and* Γ-$\mathrm{MP}(\mathcal{P}, H_{\omega_2})$ *(for nontrivial* Γ*).*

I want to focus on the Correct Proper Forcing Axioms.

Definition (Goodman)
Let Γ be a class of first order formulas. Γ-CPFA is short for Γ-$\mathrm{CFA}(\textit{proper})$.

Correct Forcing Axioms II

Fact (Goodman)
$\Gamma\text{-}\mathrm{CFA}(\mathcal{P})$ *implies both* $\mathrm{FA}_{\omega_1}(\mathcal{P})$ *and* $\Gamma\text{-}\mathrm{MP}(\mathcal{P}, H_{\omega_2})$ *(for nontrivial* Γ*).*

I want to focus on the Correct Proper Forcing Axioms.

Definition (Goodman)
Let Γ be a class of first order formulas. $\Gamma\text{-}\mathrm{CPFA}$ is short for $\Gamma\text{-}\mathrm{CFA}(\textit{proper})$.

Fact (Goodman)

- PFA *is equivalent to* $\Sigma_1\text{-}\mathrm{CPFA}$*.*

Correct Forcing Axioms II

Fact (Goodman)
Γ-$\mathrm{CFA}(\mathcal{P})$ *implies both* $\mathrm{FA}_{\omega_1}(\mathcal{P})$ *and* Γ-$\mathrm{MP}(\mathcal{P}, H_{\omega_2})$ *(for nontrivial* Γ*).*

I want to focus on the Correct Proper Forcing Axioms.

Definition (Goodman)
Let Γ be a class of first order formulas. Γ-CPFA is short for Γ-CFA(*proper*).

Fact (Goodman)

- PFA *is equivalent to* Σ_1-CPFA.
- Σ_2-CPFA *holds in the* **standard model for** PFA *("iterate all proper forcings up to a supercompact cardinal")*

Correct Forcing Axioms II

Fact (Goodman)
Γ-$\mathrm{CFA}(\mathcal{P})$ *implies both* $\mathrm{FA}_{\omega_1}(\mathcal{P})$ *and* Γ-$\mathrm{MP}(\mathcal{P}, H_{\omega_2})$ *(for nontrivial* Γ*).*

I want to focus on the Correct Proper Forcing Axioms.

Definition (Goodman)
Let Γ be a class of first order formulas. Γ-CPFA is short for Γ-$\mathrm{CFA}(\textit{proper})$.

Fact (Goodman)

- PFA *is equivalent to* Σ_1-CPFA.
- Σ_2-CPFA *holds in the* **standard model for** PFA *("iterate all proper forcings up to a supercompact cardinal")*

Note that Σ_2-$\mathrm{CPFA} \Rightarrow \Pi_1$-$\mathrm{CPFA} \Rightarrow \mathrm{PFA}^+ = \{$"$S$ is a stationary subset of ω_1"$\}$-CPFA.

CPFA is consistent

Goodman has shown that Σ_n-CPFA is consistent for all n, relative to large cardinals. Recall that $C^{(n)}$ consists of all κ with $V_\kappa \prec_{\Sigma_n} V$.

Definition

A cardinal κ is **supercompact** for $C^{(n)}$ if for every $\lambda \geqslant \kappa$ there is a λ-supercompactness embedding $j\colon V \to M$ with $\mathrm{crit}(j) = \kappa$ and $j(C^{(n)}) \cap \lambda = C^{(n)} \cap \lambda$.

CPFA is consistent

Goodman has shown that Σ_n-CPFA is consistent for all n, relative to large cardinals. Recall that $C^{(n)}$ consists of all κ with $V_\kappa \prec_{\Sigma_n} V$.

Definition

A cardinal κ is **supercompact** for $C^{(n)}$ if for every $\lambda \geqslant \kappa$ there is a λ-supercompactness embedding $j\colon V \to M$ with $\mathrm{crit}(j) = \kappa$ and $j(C^{(n)}) \cap \lambda = C^{(n)} \cap \lambda$.

This is equivalent to κ is $C^{(n-1)}$-extendible (Goodman).

CPFA is consistent

Goodman has shown that Σ_n-CPFA is consistent for all n, relative to large cardinals. Recall that $C^{(n)}$ consists of all κ with $V_\kappa \prec_{\Sigma_n} V$.

Definition
A cardinal κ is **supercompact** for $C^{(n)}$ if for every $\lambda \geqslant \kappa$ there is a λ-supercompactness embedding $j \colon V \to M$ with $\mathrm{crit}(j) = \kappa$ and $j(C^{(n)}) \cap \lambda = C^{(n)} \cap \lambda$.

This is equivalent to κ is $C^{(n-1)}$-extendible (Goodman).

Theorem (Goodman)
Suppose κ is supercompact for $C^{(n)}$. Then there is a proper forcing $\mathbb{P}$ with

$$V^{\mathbb{P}} \models \Sigma_{n+1}\text{-}\mathrm{CPFA}.$$

The Consistency strength of Π_n-CPFA

The Consistency strength of Π_n-CPFA

- Goodman's consistency proof of Σ_{n+1}-CPFA seems optimal in the same way Baumgartner's consistency proof of PFA seems optimal.

The Consistency strength of Π_n-CPFA

- Goodman's consistency proof of Σ_{n+1}-CPFA seems optimal in the same way Baumgartner's consistency proof of PFA seems optimal.
- If you believe that the consistency strength of PFA is a supercompact cardinal then you should believe that the consistency strength of Σ_{n+1}-CPFA is a supercompact for $C^{(n)}$-cardinal.

The Consistency strength of Π_n-CPFA

- Goodman's consistency proof of Σ_{n+1}-CPFA seems optimal in the same way Baumgartner's consistency proof of PFA seems optimal.
- If you believe that the consistency strength of PFA is a supercompact cardinal then you should believe that the consistency strength of Σ_{n+1}-CPFA is a supercompact for $C^{(n)}$-cardinal.
- Maybe it is easier to get a high lower bound for the consistency of Σ_{n+1}-CPFA for large n.

Set Theoretic Geology

Set Theoretic Geology

Recall the definition of the mantle.

Recall the definition of the mantle.

Definition (Fuchs-Hamkins-Reitz)
The mantle $\mathbb{M}$ is defined as

$$\bigcap\{W \subseteq V \mid W \text{ is a set forcing ground of } V\}.$$

Recall the definition of the mantle.

Definition (Fuchs-Hamkins-Reitz)
The mantle $\mathbb{M}$ is defined as

$$\bigcap\{W \subseteq V \mid W \text{ is a set forcing ground of } V\}.$$

- There was an interesting idea regarding PFA:

Recall the definition of the mantle.

Definition (Fuchs-Hamkins-Reitz)
The mantle $\mathbb{M}$ is defined as

$$\bigcap \{W \subseteq V \mid W \text{ is a set forcing ground of } V\}.$$

- There was an interesting idea regarding PFA:
- Assuming PFA, maybe it is possible to undo the forcing and recover a supercompact in $\mathbb{M}$.

Set Theoretic Geology

Recall the definition of the mantle.

Definition (Fuchs-Hamkins-Reitz)
The mantle $\mathbb{M}$ is defined as

$$\bigcap\{W \subseteq V \mid W \text{ is a set forcing ground of } V\}.$$

- There was an interesting idea regarding PFA:
- Assuming PFA, maybe it is possible to undo the forcing and recover a supercompact in $\mathbb{M}$.
- Unfortunately, this does not work. PFA is consistent with $V = \mathbb{M}$.

Recall the definition of the mantle.

Definition (Fuchs-Hamkins-Reitz)
The mantle $\mathbb{M}$ is defined as

$$\bigcap\{W \subseteq V \mid W \text{ is a set forcing ground of } V\}.$$

- There was an interesting idea regarding PFA:
- Assuming PFA, maybe it is possible to undo the forcing and recover a supercompact in $\mathbb{M}$.
- Unfortunately, this does not work. PFA is consistent with $V = \mathbb{M}$.
- But this does not mean this idea could not work in another context!

First hints

First hints

Fact (Schlutzenberg)
Suppose W is a ground and $n \geqslant 2$. Then Σ_n-truth in W is uniformly $\Sigma_n(r)$ where r is a parameter depending on W.

First hints

Fact (Schlutzenberg)
Suppose W is a ground and $n \geqslant 2$. Then Σ_n-truth in W is uniformly $\Sigma_n(r)$ where r is a parameter depending on W.

Also recall the Bedrock Axiom (BA), stating that the mantle is a ground.

First hints

Fact (Schlutzenberg)
Suppose W is a ground and $n \geqslant 2$. Then Σ_n-truth in W is uniformly $\Sigma_n(r)$ where r is a parameter depending on W.

Also recall the Bedrock Axiom (BA), stating that the mantle is a ground.

Lemma
Suppose $n \geqslant 4$ and Σ_n-CPFA and BA holds. Then $\omega_2 \in (C^{(n)})^{\mathbb{M}}$.

First hints

Fact (Schlutzenberg)
Suppose W is a ground and $n \geqslant 2$. Then Σ_n-truth in W is uniformly $\Sigma_n(r)$ where r is a parameter depending on W.

Also recall the Bedrock Axiom (BA), stating that the mantle is a ground.

Lemma
Suppose $n \geqslant 4$ and Σ_n-CPFA and BA holds. Then $\omega_2 \in (C^{(n)})^{\mathbb{M}}$.

Proof.
Suppose $\mathbb{M} \models \exists x\ \psi(x)$ holds and $\psi \in \Pi_{n-1}$. Then the statement

$$\varphi = \exists x \in V_{\omega_2} \cap \mathbb{M}\ \mathbb{M} \models \psi(x)$$

is Σ_n by Fact above $+$ BA, is provably persistent and forced by $\mathrm{Col}(\omega_1, \theta)$ for sufficiently large θ. Thus φ holds in V. □

First hints

Fact (Schlutzenberg)
Suppose W is a ground and $n \geqslant 2$. Then Σ_n-truth in W is uniformly $\Sigma_n(r)$ where r is a parameter depending on W.

Also recall the Bedrock Axiom (BA), stating that the mantle is a ground.

Lemma
Suppose $n \geqslant 4$ and Σ_n-CPFA and BA holds. Then $\omega_2 \in (C^{(n)})^{\mathbb{M}}$.

Proof.
Suppose $\mathbb{M} \models \exists x\ \psi(x)$ holds and $\psi \in \Pi_{n-1}$. Then the statement

$$\varphi = \exists x \in V_{\omega_2} \cap \mathbb{M}\ \mathbb{M} \models \psi(x)$$

is Σ_n by Fact above $+$ BA, is provably persistent and forced by $\mathrm{Col}(\omega_1, \theta)$ for sufficiently large θ. Thus φ holds in V. □

In particular, ω_2 is inaccessible in $\mathbb{M}$ under Σ_4-CPFA.

First hints

Fact (Schlutzenberg)
Suppose W is a ground and $n \geqslant 2$. Then Σ_n-truth in W is uniformly $\Sigma_n(r)$ where r is a parameter depending on W.

Also recall the Bedrock Axiom (BA), stating that the mantle is a ground.

Lemma
Suppose $n \geqslant 4$ and Σ_n-CPFA and BA holds. Then $\omega_2 \in (C^{(n)})^{\mathbb{M}}$.

Proof.
Suppose $\mathbb{M} \models \exists x\ \psi(x)$ holds and $\psi \in \Pi_{n-1}$. Then the statement

$$\varphi = \exists x \in V_{\omega_2} \cap \mathbb{M}\ \mathbb{M} \models \psi(x)$$

is Σ_n by Fact above $+$ BA, is provably persistent and forced by $\mathrm{Col}(\omega_1, \theta)$ for sufficiently large θ. Thus φ holds in V. □

In particular, ω_2 is inaccessible in $\mathbb{M}$ under Σ_4-CPFA.
This can be used to separate all Σ_n-$\mathrm{CFA}(\mathcal{P})$ for $n \geqslant 3$ and fixed reasonable $\mathcal{P}$.

An Equiconsistency

One can prove much more:

Theorem (L.)
Suppose $n \geqslant 4$, $\Pi_n\text{-CPFA}$ *and* BA *holds. Then*

$$\mathbb{M} \models \text{“}\omega_2^V \textit{ is supercompact for } C^{(n)}\text{”}.$$

An Equiconsistency

One can prove much more:

Theorem (L.)
Suppose $n \geqslant 4$, Π_n-CPFA and BA holds. Then

$$\mathbb{M} \models \text{“}\omega_2^V \textit{ is supercompact for } C^{(n)}\text{”}.$$

Corollary (Goodman, L.)
The following theories are equiconsistent for $n \geqslant 4$:

1. $\mathrm{ZFC} + \exists\kappa$ “κ *is supercompact for* $C^{(n)}$”.
2. $\mathrm{ZFC} + \Sigma_{n+1}\text{-}\mathrm{CPFA} + \mathrm{BA}$.
3. $\mathrm{ZFC} + \Pi_n\text{-}\mathrm{CPFA} + \mathrm{BA}$.

An Equiconsistency

One can prove much more:

Theorem (L.)
Suppose $n \geqslant 4$, Π_n-CPFA *and* BA *holds. Then*

$$\mathbb{M} \models \text{“}\omega_2^V \textit{ is supercompact for } C^{(n)}\text{”}.$$

Corollary (Goodman, L.)
The following theories are equiconsistent for $n \geqslant 4$*:*

1. $\mathrm{ZFC} + \exists\kappa$ “κ *is supercompact for* $C^{(n)}$”.
2. $\mathrm{ZFC} + \Sigma_{n+1}$-$\mathrm{CPFA} + \mathrm{BA}$.
3. $\mathrm{ZFC} + \Pi_n$-$\mathrm{CPFA} + \mathrm{BA}$.

In particular, the practical consistency strength of Σ_{n+1}-CPFA is exactly a supercompact for $C^{(n)}$-cardinal.

The Role of BA

The Role of BA

- It would be nice to eliminate BA from the equiconsistency.

The Role of BA

- It would be nice to eliminate BA from the equiconsistency.
- The large cardinals involved are beyond extendible so imply BA (Usuba).

The Role of BA

- It would be nice to eliminate BA from the equiconsistency.
- The large cardinals involved are beyond extendible so imply BA (Usuba).
- It is plausible that Π_4-CPFA implies BA.

The Role of BA

- It would be nice to eliminate BA from the equiconsistency.
- The large cardinals involved are beyond extendible so imply BA (Usuba).
- It is plausible that Π_4-CPFA implies BA.
- The known methods cannot produce models of Π_4-$\mathrm{CPFA} \wedge \neg\mathrm{BA}$.

The Role of BA

- It would be nice to eliminate BA from the equiconsistency.
- The large cardinals involved are beyond extendible so imply BA (Usuba).
- It is plausible that Π_4-CPFA implies BA.
- The known methods cannot produce models of Π_4-$\mathrm{CPFA} \wedge \neg\mathrm{BA}$.

Question
Does Π_4-CPFA imply BA?

The HOD-Dichotomy

The HOD-Dichotomy

Definition
Suppose N is an inner model of ZFC. A cardinal κ is ω-strongly measurable in N if

The HOD-Dichotomy

Definition
Suppose N is an inner model of ZFC. A cardinal κ is ω-strongly measurable in N if

1. $\kappa \cap \mathrm{cof}(\omega) \in N$,

Definition
Suppose N is an inner model of ZFC. A cardinal κ is ω-strongly measurable in N if

1. $\kappa \cap \mathrm{cof}(\omega) \in N$,
2. $(\mathrm{NS}_\kappa \restriction \mathrm{cof}(\omega)) \cap N \in N$ and

The HOD-Dichotomy

Definition
Suppose N is an inner model of ZFC. A cardinal κ is ω-strongly measurable in N if

1. $\kappa \cap \mathrm{cof}(\omega) \in N$,
2. $(\mathrm{NS}_\kappa \upharpoonright \mathrm{cof}(\omega)) \cap N \in N$ and
3. for some $\lambda < \kappa$ with $(2^\lambda)^N < \kappa$, there is no partition $\langle S_i \mid i < \lambda \rangle$ of $\kappa \cap \mathrm{cof}(\omega)$ into V-stationary sets.

The HOD-Dichotomy

Definition
Suppose N is an inner model of ZFC. A cardinal κ is ω-strongly measurable in N if

1. $\kappa \cap \mathrm{cof}(\omega) \in N$,
2. $(\mathrm{NS}_\kappa \restriction \mathrm{cof}(\omega)) \cap N \in N$ and
3. for some $\lambda < \kappa$ with $(2^\lambda)^N < \kappa$, there is no partition $\langle S_i \mid i < \lambda \rangle$ of $\kappa \cap \mathrm{cof}(\omega)$ into V-stationary sets.

Theorem (Woodin)
Suppose κ is extendible. Exactly one of the following holds.

1. *Every singular $\rho \geqslant \kappa$ is singular in HOD and $\rho^+ = (\rho^+)^{\mathrm{HOD}}$.*
2. *Every regular $\lambda \geqslant \kappa$ is ω-strongly measurable in HOD.*

The $\mathbb{M}$-Dichotomy

The $\mathbb{M}$-Dichotomy

For an inner model $M \subseteq V$, let $N(M)$ arise from M by feeding $\kappa \cap \mathrm{cof}(\omega)$ and $M \cap (\mathrm{NS}_\kappa \restriction \mathrm{cof}(\omega))$ into M, for all κ of uncountable cofinality.

The $\mathbb{M}$-Dichotomy

For an inner model $M \subseteq V$, let $N(M)$ arise from M by feeding $\kappa \cap \mathrm{cof}(\omega)$ and $M \cap (\mathrm{NS}_\kappa \restriction \mathrm{cof}(\omega))$ into M, for all κ of uncountable cofinality.

Theorem (L.)
Suppose Π_4-CPFA holds. Then exactly one of the following holds for $N = N(\mathbb{M})$.

1. *Every singular cardinal ρ is singular in N and $\rho^+ = (\rho^+)^N$.*
2. *Every regular cardinal $\kappa \geqslant \omega_2$ is ω-strongly measurable in N.*

The $\mathbb{M}$-Dichotomy

For an inner model $M \subseteq V$, let $N(M)$ arise from M by feeding $\kappa \cap \mathrm{cof}(\omega)$ and $M \cap (\mathrm{NS}_\kappa \upharpoonright \mathrm{cof}(\omega))$ into M, for all κ of uncountable cofinality.

Theorem (L.)
Suppose Π_4-CPFA holds. Then exactly one of the following holds for $N = N(\mathbb{M})$.

1. *Every singular cardinal ρ is singular in N and $\rho^+ = (\rho^+)^N$.*
2. *Every regular cardinal $\kappa \geqslant \omega_2$ is ω-strongly measurable in N.*

It is tempting to conjecture that 1. always holds.

Wild Speculations

Wild Speculations

Let's speculate on two possible scenarios for $n \geqslant 4$.

Case 1: The consistency strength of Π_n-CPFA is exactly a supercompact for $C^{(n)}$-cardinal. It seems likely that Π_4-CPFA implies BA in this scenario. In particular, case 1. of the dichotomy is provable.

Wild Speculations

Let's speculate on two possible scenarios for $n \geqslant 4$.

Case 1: The consistency strength of Π_n-CPFA is exactly a supercompact for $C^{(n)}$-cardinal. It seems likely that Π_4-CPFA implies BA in this scenario. In particular, case 1. of the dichotomy is provable.

Case 2: The consistency strength of Π_4-CPFA is low. In this scenario, Π_4-CPFA does not prove BA and maybe even option 2. in the dichotomy is possible.

Wild Speculations

Let's speculate on two possible scenarios for $n \geqslant 4$.

Case 1: The consistency strength of Π_n-CPFA is exactly a supercompact for $C^{(n)}$-cardinal. It seems likely that Π_4-CPFA implies BA in this scenario. In particular, case 1. of the dichotomy is provable.

Case 2: The consistency strength of Π_4-CPFA is low. In this scenario, Π_4-CPFA does not prove BA and maybe even option 2. in the dichotomy is possible. Maybe $\mathrm{con}(\mathrm{PFA})$ is below a supercompact as well and the optimal method to produce models of PFA can be used to produce models of Π_4-CPFA. In particular, this method likely produces models of $\neg\mathrm{BA}$.