Applications of the fusion technique

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Motivation

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Measures in forcing extensions

Let $\mathbb{P} \in V$ be a forcing notion, $G \subseteq \mathbb{P}$ generic over V, and assume that κ is a measurable cardinal in $V[G]$.
Let $W \in V[G]$ be a κ-complete ultrafilter on κ, and denote by $j_W \colon V[G] \to M[H]$ its ultrapower embedding. j_W is determined from:

1. the embedding $j_W \restriction V \colon V \to M$.
2. the $j_W(\mathbb{P})$-generic (over M) set $H = j_W(G)$.

Thus, in order to characterize measures in forcing extensions, we need to analyze the two above components.
In the recent years, a large body of work was invested in developing methods to limit, restrict and control each of the components; this naturally leads to a simple measure structure in the generic extension.

Classifying ultrafilters in forcing extensions - continued

For the analysis of $j_W \restriction V$, we mention two useful tools:

- Schindler's theorem about iterates of the core model (2006): assuming that $V = \mathcal{K}$ is the core model, $j_W \restriction V \colon V \to M$ is an iterate of it (along the main branch of an iteration tree).
- Hamkins' Gap-forcing theorem (1998): if $\delta < \kappa$ and $\mathbb{P} = \mathbb{P}_0 * \dot{\mathbb{P}}_1$ where $\mathbb{P}_0$ is nontrivial, $|\mathbb{P}_0| < \delta$ and $\Vdash_{\mathbb{P}_0}$ "$\dot{\mathbb{P}}_1$ is $(\delta + 1)$-strategically-closed" then, for every definable (in $V[G]$, from a parameter) $j \colon V[G] \to M[H]$ with $\mathrm{crit}(j) = \kappa$, $j \restriction V \colon V \to M$ is definable in V.

Once all the possible restrictions $j_W \restriction V \colon V \to M$ for measures $W \in V[G]$ are identified, the focus turns to classifying all possible generics $H \subseteq j_W(\mathbb{P})$ with $j_W[G] \subseteq H$ in $V[G]$.

Examples - Normal measures in forcing extensions

A very partial list of examples for analysis of **normal measures** in forcing extensions:

1. Friedman-Magidor (2008): starting from a measurable cardinal, produced a model with exactly two normal measures (or any $0 \leq \tau \leq \kappa^{++}$, where κ is the measurable cardinal, and, say, GCH holds). Similar results, starting from stronger assumptions, were obtained with the violation of GCH on a measurable cardinal.
2. Ben-Neria (2016): any well-founded order can be realized as the Mitchell order on a measurable cardinal.
3. Apter-Cummings (2023): a model in which GCH is violated on a strong cardinal, and the Mitchell order on it is linear.
4. Ben-Neria (2014), and later K. (2024): analysis of all normal measures after performing the Magidor iteration of Prikry forcings.

Examples - Non-normal measures in forcing extensions

Examples of analysis of **non-normal measures** in forcing extension:

1. Hayut-Poveda (2022): analysis of all κ-complete ultrafilters on κ after a nonstationary support iteration of (tree) Prikry forcings over $L[\vec{U}]$.
2. Benhamou-Goldberg (2024): analysis of all lifts of sums of normal measures after forcing a discrete Magidor iteration. Forcing the weak Ultrapower Axiom with the negation of the Ultrapower Axiom.
3. Ben-Neria, K. (2024): forcing the violation of GCH on a measurable cardinal κ with a single normal measure U, where every σ-complete ultrafilter is equivalent to U^n for some $n < \omega$.

Main example - the Friedman-Magidor forcing

Theorem (Friedman-Magidor)

Consistently from a measurable cardinal, there exists a forcing extension with exactly two normal measures.

To achieve this, an iterated forcing $\mathbb{P}_{\kappa+1} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha < \kappa + 1 \rangle$ was performed over $V = L[U]$ (where κ is the measurable).
For each inaccessible $\alpha \leq \kappa$, the forcing $\mathbb{Q}_\alpha$ was a two step iteration:

- the first one adds a (Sacks) subset to α.
- the second one ensures that $\mathbb{Q}_\alpha$ is 'self coding': it codes information by ruining / preserving stationary sets from a pre-chosen list $\langle \mathcal{S}^\alpha_i \colon i < \alpha^+ \rangle$ of pairwise disjoint stationary subsets of α^+.

The goal of the above components of $\mathbb{Q}_\alpha$ is to limit the possibilities for generics H over M_U (in the above notations); we omit the details here, since we would like to focus on a different aspect of the forcing, that further restricts possible generics H.

Friedman-Magidor - continued

- If in $\mathbb{P} = \mathbb{P}_\kappa$ the limits are taken with respect to the commonly used Easton support, we encounter the issue that, given $G * g \subseteq \mathbb{P}_{\kappa+1}$ generic over $V = L[U]$, there are multiple $j_U(\mathbb{P})$-generics $H \in V[G * g]$ over M_U, with $H \restriction \kappa + 1 = G * g$. There is no significant restriction on $H \restriction (\kappa, j_U(\kappa))$ when the Easton support iteration is used.
- The solution that Friedman and Magidor found was the use of a different support - the **nonstationary support** (to the best of our knowledge, similar ideas appeared prior to their work in a work of Jensen).
- The nonstationary support has a fusion property that limits the generics $H \restriction (\kappa + 1, j_U(\kappa))$, and also limits the ground model M of $\mathrm{Ult}(V[G, g], W) \simeq M[H]$.

Nonstationary support iterations/products

Framework

Definition

We say that a set of ordinals A is **nonstationary in inaccessibles** if for every inaccessible cardinal λ, $A \cap \lambda$ is a nonstationary subset of λ.

Framework

Let κ be a Mahlo cardinal. Assume that $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha < \kappa \rangle$ is a **nonstationary support** iterated forcing of length κ. That is, for every inaccessible $\alpha \leq \kappa$, $\mathbb{P}_\alpha$ is taken to be the **nonstationary-support a limit** of $\langle \mathbb{P}_\beta \colon \beta < \alpha \rangle$, in which the conditions $p \in \mathbb{P}_\alpha$ are those who satisfy that the set

$$\mathrm{supp}(p) = \alpha \setminus \{\beta < \alpha \colon p \restriction \beta \Vdash p(\beta) = \mathbb{1}_{\dot{\mathbb{Q}}_\beta}\}$$

is nonstationary in inaccessibles.
For every α which is not an inaccessible, $\mathbb{P}_\alpha$ is the inverse limit of its predecessors.

Framework

Framework - continued

We further assume that:

- for every inaccessible $\alpha < \kappa$, rank($\dot{\mathbb{Q}}_\alpha$) is below the next inaccessible after α, and

$$\Vdash_{\mathbb{P}_\alpha} \text{"}\dot{\mathbb{Q}}_\alpha \text{ is an } \alpha\text{-closed forcing notion."}$$

- If α is not an inaccessible, $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is trivial".

The fusion lemma

The fusion lemma

Assume that $\vec{D} = \langle D(\nu) \colon \nu < \kappa \rangle$ is a sequence of dense open subsets of $\mathbb{P}$ and $p \in \mathbb{P}$. Then there exists $p^* \in \mathbb{P}$ extending p and a club $C \subseteq \kappa$ such that, for every $\nu \in C$,

$$\{q \in \mathbb{P}_{\nu+1} \colon q^\frown p^* \setminus (\nu + 1) \in D(\nu)\}$$

is a dense subset of $\mathbb{P}_{\nu+1}$.

Some ideas behind the proof: fusion sequences

Definition

Let $\mathbb{P}$ be a nonstationary support iteration as above. A **fusion sequence** is a sequence of conditions $\langle p_i \colon i < \kappa\rangle$, joint with an increasing, continuous, cofinal in κ sequence of ordinals $\langle \nu_i \colon i < \kappa\rangle$ such that, for every $i \leq j < \kappa$,

- p_j extends p_i.
- $p_j \restriction \nu_i + 1 = p_i \restriction \nu_i + 1$.
- $\nu_j \notin \mathrm{supp}(p_i)$.

- Note that fusion sequences have limits: we can define $p^* = \bigcup_{i<\kappa} p_i \restriction \nu_i + 1$. We obtained a legitimate condition since $\mathrm{supp}(p^*)$ is nonstationary in inaccessibles (specifically, in κ, $\{\nu_i \colon i < \kappa\}$ is a club in κ disjoint to the support of p^*).
- In order to prove the fusion lemma, successor elements in the fusion sequence need to "capture" dense sets: we want the set $\{q \in \mathbb{P}_{\nu_i+1} \colon q^\frown p_{i+1} \in D(\nu_i)\}$ to be a dense subset of $\mathbb{P}_{\nu_i+1}$ for all $i < \kappa$.

Fusion sequences

Construction of Fusion sequences

Let $\vec{D} = \langle D(i) \colon i < \kappa \rangle$ be a sequence of dense open subsets of $\mathbb{P}_\kappa$. We construct a fusion sequence $\langle p_i \colon i < \kappa \rangle$, $\langle \nu_i \colon i < \kappa \rangle$ that "captures" $\vec{D}$. Simultaneously, we construct an inclusion-decreasing sequence of clubs in κ, $\langle C_i \colon i < \kappa \rangle$, each C_i disjoint from $\text{supp}(p_i)$.

1. Start from any condition $p_0 = p \in \mathbb{P}_\kappa$ and $\nu_0 < \kappa$. Let $C_0 \subseteq \kappa$ be a club disjoint from $\text{supp}(p_0)$.
2. **Successor steps:** given p_i, ν_i, C_i, pick any $\nu_{i+1} \in C_i \setminus \nu_i + 1$. Let p_{i+1} be such that:
    - $p_{i+1} \restriction \nu_{i+1} = p_i \restriction \nu_{i+1}$.
    - $\nu_{i+1} \notin \text{supp}(p_{i+1})$.
    - the set $\{q \in \mathbb{P}_{\nu_{i+1}+1} \colon q^\frown p_{i+1} \setminus \nu_{i+1} + 1 \in D(\nu_{i+1})\}$ is dense in $\mathbb{P}_{\nu_{i+1}+1}$ below $p_{i+1} \restriction \nu_{i+1} + 1$.

    Finally let $C_{i+1} \subseteq C_i$ be a club disjoint to $\text{supp}(p_i)$.

Construction of Fusion sequences

1. **Limit steps:** for a limit $i < \kappa$, let $\nu_i = \sup\{\nu_j : j < i\}$. Let p_i be such that:
   - $p_i \restriction \nu_i = \bigcup_{j<i} p_j \restriction \nu_j + 1$.
   - $\nu_i \notin \mathrm{supp}(p_i)$.
   - $p_i \restriction \nu_i + 1 \Vdash$ "$p_i \setminus \nu_i + 1$ extends $p_j \setminus \nu_i + 1$ for all $j < i$."

   As before, let $C_i \subseteq \bigcap_{j<i} C_j$ be a club disjoint from $\mathrm{supp}(p_i)$.

Generating a generic set over an ultrapower

- Let $U \in V$ be a normal measure on κ.
- Assume that we managed to find $g \in V[G]$ which is generic for $j_U(\mathbb{P})(\kappa)$ over $M_U[G]$ (this sometimes requires an additional forcing over $V[G]$).
- The fusion lemma ensures that the set

$$j_U[G] \setminus \kappa + 1 = \{j_U(p) \setminus \kappa + 1 \colon p \in G\}$$

generates a generic for $j_U(\mathbb{P}) \setminus \kappa + 1$ over $M_U[G * g]$.

Generating a generic set over an ultrapower

Proof

Given a $j_U(\mathbb{P}) \restriction \kappa + 1$-name for dense open set $\dot{E} \subseteq j_U(\mathbb{P}) \setminus \kappa + 1$, let

$$\vec{\dot{e}} = \langle \dot{e}(\nu) \colon \nu < \kappa \rangle$$

be a sequence such that $\dot{E} = [\nu \mapsto \dot{e}(\nu)]_U$. We can assume that each $\dot{e}(\nu)$ is a $\mathbb{P}_{\nu+1}$-name for a dense open subset of $\mathbb{P} \setminus \nu + 1$.
The fusion lemma implies that for some condition $p \in G$ and a club $C \subseteq \kappa$, for every $\nu \in C$,

$$\Vdash_{\mathbb{P}_{\nu+1}} p \setminus \nu + 1 \in \dot{e}(\nu).$$

Since U is normal, $C \in U$, and thus, over M_U,

$$\Vdash_{j_U(\mathbb{P}) \restriction \kappa + 1} j_U(p) \setminus \kappa + 1 \in \dot{E}$$

as desired.

Evaluating functions by old ones

An typical consequence of the fusion lemma is the following.

Lemma

Let $f\colon \kappa \to \mathrm{Ord}$ be a function in $V[G]$. Then there exists a club $C \subseteq \kappa$ and a function $F\colon \kappa \to V$ such that $F \in V$ and for every inaccessible $\nu \in C$,

$$f(\nu) \in F(\nu) \text{ and } |F(\nu)| \leq |\mathbb{P}_{\nu+1}|.$$

Example

Let $V[G * g]$ be a generic extension for the Friedman-Magidor forcing over $V = L[U]$. Then for every normal measure $W \in V[G * g]$ on κ, $j_W \restriction V = j_U$.

Proof sketch

The above Lemma, joint with properties of the forcing $\dot{\mathbb{Q}}_\kappa$ we omit here, imply that every $f\colon \kappa \to \kappa$ in $V[G * g]$ is dominated by ground model function $g\colon \kappa \to \kappa$.
Write $j_W \restriction V$ as an iteration of $V = L[U]$. We argue that the length of the iteration is 1. Indeed, assume otherwise, write $j_W \restriction L[U] = k \circ j_U$ for some $k\colon M_U \to M$ with $\mathrm{crit}(k) = j_U(\kappa)$.
Let $f \in V[G * g]$ be a function $f\colon \kappa \to \kappa$ such that $j_U(\kappa) = [f]_W$. Let $g\colon \kappa \to \kappa$ in V be a function that dominates f.
Then:

$$j_U(\kappa) = [f]_W < [g]_W = (k \circ j_U)(g)(\kappa) = k\left(j_U(g)(\kappa)\right) < j_U(\kappa)$$

which is a contradiction.

Applications

A model with exactly two normal measures, with the same ultrapower

Theorem (Ben-Neria, K.)

Let $\mathbb{P} = \prod_{\alpha<\kappa} \mathbb{Q}_\alpha$ be a nonstationary support product, such that, for every inaccessible $\alpha < \kappa$, $\mathbb{Q}_\alpha = \{\mathbb{1}_{\mathbb{Q}_\alpha}, 0, 1\}$ is an atomic forcing where $0, 1$ are incompatible extensions of $\mathbb{1}_{\mathbb{Q}_\alpha}$. Let $G \subseteq \mathbb{P}$ be generic over V. Then in $V[G]$, every normal measure $U \in V$ on κ lifts to exactly two normal measures on κ, which have the same ultrapower.

Corollary

Forcing over $V = L[U]$, the obtained generic extension is a model in which κ is the unique measurable cardinal, it carries exactly two normal measures, and both of them have the same ultrapower.

Proof.

Denote, for every inaccessible $\alpha < \kappa$, by $G(\alpha)$, the generic bit in $\{0, 1\}$ chosen by G at step α. The proof goes in two steps:

1. Fix $i < 2$. The set–

$$U \cup \{\{\alpha < \kappa \colon G(\alpha) = i\}\}$$

   generates a normal, κ-complete ultrafilter U_i^* in $V[G]$. Moreover, U_0^*, U_1^* have the same ultrapower.

2. Let $W \in V[G]$ be a normal measure of κ, and let $U = W \cap V$. By Hamkins' Gap forcing theorem, $U \in V$. Let $i = j_W(G)(\kappa)$. Then $W = U_i^*$ since W contains the relevant generating set.

We just need to justify the first step.

Proof - continued

Let $\dot{X}$ be a $\mathbb{P}$-name for a subset of κ. Apply fusion on the dense sets $\vec{D} = \langle D_\nu \colon \nu < \kappa \rangle$,

$$D_\nu = \{p \in \mathbb{P} \colon p \parallel \check{\nu} \in \dot{X}\}.$$

Find a condition $p \in G$ and a club $C \subseteq \kappa$ such that for every $\nu \in C$,

$$\{q \in \mathbb{P}_{\nu+1} \colon q^\frown p \setminus \nu + 1 \parallel \check{\nu} \in \dot{X}\}$$

is dense in $\mathbb{P}_{\nu+1}$.
Since $C \in U$,

$$\{q \in j_U(\mathbb{P}) \restriction \kappa + 1 \colon q^\frown j_U(p) \setminus \kappa + 1 \parallel \check{\kappa} \in j_U(\dot{X})\}$$

is dense in $j_U(\mathbb{P}) \restriction \kappa + 1$.
In particular, by extending p inside G,

$$j_U(p) \cup \{(\kappa, i)\} \parallel \check{\kappa} \in j_U(\dot{X}).$$

Proof - continued

If, say,

$$j_U(p) \cup \{(\kappa, i)\} \Vdash \check{\kappa} \in j_U(\dot{X})$$

then, in $V[G]$,

$$\{\nu < \kappa \colon p \cup \{(\nu, i)\} \Vdash \check{\nu} \in \dot{X}\} \cap \{\nu < \kappa \colon G(\nu) = i\} \subseteq X$$

and the former set belongs to U.
It follows that–

$$U_i^* = \{(\dot{X})_G \colon \exists p \in G,\ j_U(p) \cup \{(\kappa, i)\} \Vdash \check{\kappa} \in j_U(\dot{X})\}$$

is an ultrafilter in $V[G]$ generated by $U \cup \{\{\nu < \kappa \colon G(\nu) = i\}$. Using a similar argument, it's not hard to verify that U_i^* is a normal measure on κ. Finally, the above analysis shows that $\mathrm{Ult}(V[G], U_i^*)$ has the form

$$M_U[(\cup j_U[G]) \cup \{(\kappa, i)\}]$$

so U_0^*, U_1^* have the same ultrapower.

Non-normal measures and weak UA

Theorem

Let $\mathbb{P}$ be the same forcing as above. Let $G \subseteq \mathbb{P}$ be generic over $V = L[U]$. $V[G]$ satisfies the weak Ultrapower Axiom: given $U, W \in V[G]$ σ-complete ultrafilters, there are $U^* \in M_W \simeq \mathrm{Ult}(V[G], W)$ and $W^* \in M_U \simeq \mathrm{Ult}(V[G], U)$ such that $\mathrm{Ult}(M_W, U^*) \simeq \mathrm{Ult}(M_U, W^*)$. Furthermore, in our $V[G]$ one of U^*, W^* can be taken to be trivial.

Sketch

For every σ-complete ultrafilter $W \in V[G]$,

$$\mathrm{Ult}(V[G], W) \simeq M_{U^n}[(\cup j_{U^n}[G]) \cup f_W]$$

for a function $f_W \colon \{\kappa, j_U(\kappa), \dots, j_{U^{n-1}}(\kappa)\} \to 2$. In particular, the model $\mathrm{Ult}(V[G], W)$ depends only on n, and all the relevant generic extensions of M_{U^n} give rise to the same extension. Now one can easily "weakly compare" two σ-complete ultrafilters of $V[G]$ by internal ultrapowers.

Nonstationary support iterations of Prikry-type forcings

Prikry-type forcings

A forcing notion $\mathbb{Q}$ is a **Prikry-type** forcing notion if there exists a suborder $\leq^*_{\mathbb{Q}} \subseteq \leq_{\mathbb{Q}}$ such that, for every statement σ in the forcing language and a condition $q \in \mathbb{Q}$, there exists a $\leq^*_{\mathbb{Q}}$ extension q^* of q that decides σ.

The sub-order $\leq^*_{\mathbb{Q}}$ is called the direct extension order.
We remark that every forcing notion $\mathbb{Q}$ can be seen as a Prikry-type forcing, by simply setting $\leq^*_{\mathbb{Q}} = \leq_{\mathbb{Q}}$. However, interesting Prikry-type forcings are those in which $\leq^*_{\mathbb{Q}}$ satisfies additional properties, for instance being κ-closed (in which case, we will say that $\langle \mathbb{Q}, \leq_{\mathbb{Q}}, \leq^*_{\mathbb{Q}} \rangle$ is a κ-closed Prikry-type forcing).

Iterations of Prikry-type forcings

- Iterations of Prikry forcings were introduced and originally applied by Magidor in his famous work about Identity crisis.
- Gitik generalized the technique to iterations of Prikry-type forcings (with the full support (Magidor iteration) and Easton support (Gitik iteration)).
- Ben-Neria and Unger were the first to define and use the nonstationary support iteration of Prikry-type forcings.
- The **nonstationary support iteration** $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha < \kappa \rangle$ of Prikry-type forcings is defined similarly to the standard nonstationary support, with an additional requirement that, when extending a condition $p \in \mathbb{P}_\alpha$ (for $\alpha \leq \kappa$), only finitely many coordinates inside $\mathrm{supp}(p)$ can be non-directly extended.

Formal definition of the nonstationary support iteration of Prikry-type forcings

More formally, let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha < \kappa \rangle$ be a nonstationary support iteration of Prikry type forcings $\langle \dot{\mathbb{Q}}_\alpha, \dot{\leq}_{\dot{\mathbb{Q}}_\alpha}, \dot{\leq}^*_{\dot{\mathbb{Q}}_\alpha} \rangle$.
For $p, q \in \mathbb{P}_\alpha$ for some $\alpha \leq \kappa$, we say that q extends p if:

- $\mathrm{supp}(p) \subseteq \mathrm{supp}(q)$.
- for all $\beta \in \mathrm{supp}(p)$, $q \restriction \beta \Vdash$ "$q(\beta)$ extends $p(\beta)$."
- there exists a finite $b \subseteq \alpha$, such that, for every $\beta \in \mathrm{supp}(p) \setminus b$, $q \restriction \beta \Vdash$ "$q(\beta)$ extends $p(\beta)$ in the sense of $\leq^*_{\dot{\mathbb{Q}}_\beta}$ ".

If $b = \emptyset$, we say that q is a **direct extension** of p. This defines a direct extension order $\leq^*_{\mathbb{P}_\alpha}$ on each $\mathbb{P}_\alpha$, which turns $\langle \mathbb{P}_\alpha, \leq_{\mathbb{P}_\alpha}, \leq^*_{\mathbb{P}_\alpha} \rangle$ to a Prikry-type forcing notion.
We maintain the assumption that for all $\alpha < \kappa$, $\langle \dot{\mathbb{Q}}_\alpha, \dot{\leq}_{\dot{\mathbb{Q}}_\alpha}, \dot{\leq}^*_{\dot{\mathbb{Q}}_\alpha} \rangle$ is an α-closed Prikry-type forcing, whose rank is strictly below the next inaccessible above α.

Hamkins' Cover and approximation properties

Definition

Let δ be a regular uncountable cardinal, and let $N \subseteq V$ be a transitive inner model containing the ordinals.

1. N has the δ-**cover property** if for every $A \in V$, $A \subseteq N$ such that $|A| < \delta$, there exists $B \in N$ with $|B|^N < \delta$ such that $A \subseteq B$.
2. N has the δ-**approximation property** if for every $A \in V$, $A \subseteq N$, the following are equivalent:
   1. $A \in N$.
   2. A is δ-approximated in N: that is, for every $X \in N$ with $|X|^N < \delta$, $A \cap X \in N$.

Weak Universality lemma

In the above notations, assume that $N \subseteq V$ has the δ-approximation property. Let $W \in N$ be a δ-complete ultrafilter whose underlying set is some $X \in N$. Then $W \cap N \in N$.

The cover and approximation properties

Theorem (Gitik, K.)

Assume GCH. Let κ be an Mahlo cardinal, and let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha < \kappa \rangle$ be an iteration of Prikry-type forcings (with either Full, Nonstationary or Easton support taken). Assume that $\dot{\mathbb{Q}}_\alpha$ is forced to be trivial, unless $\alpha < \kappa$ is inaccessible, and then–

$$\Vdash_{\mathbb{P}_\alpha} \langle \dot{\mathbb{P}}_\alpha, \leq^*_{\dot{\mathbb{Q}}_\alpha} \rangle \text{ is } \alpha - \text{closed and directed, and } \dot{\mathbb{Q}}_\alpha \in V_\kappa.$$

Let $G \subseteq \mathbb{P}$ be generic over V. Then:

1. The the extension $V \subseteq V[G]$ has the κ-cover and the κ-approximation properties. Consequently:
2. For every κ-complete ultrafilter $W \in V[G]$ whose underlying set is a set $X \in V$, $W \cap V \in V$.

Characterization of normal measures after iterating Prikry-type forcings

Corollary

Let $\mathbb{P} = \mathbb{P}_\kappa$ be a nonstationary support iteration of Prikry-type forcings, satisfying the above properties. Assume, in addition, that $\Delta \subseteq \kappa$ is a stationary set of inaccessibles, and, for every $\alpha < \kappa$ it is forced by $\mathbb{P}_\alpha$ that:

- if $\alpha \notin \Delta$, $\dot{\mathbb{Q}}_\alpha$ is trivial,
- if $\alpha \in \Delta$, $\dot{\mathbb{Q}}_\alpha$ singularizes α.

Then every normal measure $U \in V$ on κ which does not concentrate on Δ generates a normal measure $U^* \in V[G]$ on κ. furthermore, for every normal measure $W \in V[G]$ on κ, $U = W \cap V \in V$, $\Delta \notin U$ and $W = U^*$.

An identity crisis with a unique normal measure

Corollary (Gitik, K.)

Let κ be a supercompact, assume GCH and assume that Mitchell order is linear on κ. Assume also that there are no measurables above κ.
Let $\mathbb{P}_\kappa$ be a nonstationary support iteration of Prikry forcings; that is, for all $\alpha < \kappa$ measurable in V, $\dot{\mathbb{Q}}_\alpha$ is the Prikry forcing with a $\mathbb{P}_\alpha$-name for a normal measure $\dot{W}_\alpha$ on κ in $V^{\mathbb{P}_\alpha}$. Then κ is strongly compact in $V[G]$, it is the only measurable cardinal, and it carries a unique normal measure.

Thank you for your attention!