Regularity, primeness, and partition results

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We introduce the notions of regularity and primeness for sets in a choiceless context.

Definition (regularity)

Assume ZF. Let X, Y be sets. We say that X is Y-regular if for any $F\colon X \to Y$, there is a $y \in Y$ such that $F^{-1}(y)$ has size $|X|$, that is, we can inject X into $F^{-1}(y)$.

If X, Y are wellordered, this just says $|Y| < \mathrm{cof}(|X|)$.

Definition (primeness)

Assume ZF. We say X is prime if whenever we have an injection $F\colon X \to Y_0 \times Y_1$, then for $i \in \{0,1\}$ we have an injection from X into Y_i.

We say X is strongly prime if whenever $F\colon X \to Y_0 \times Y_1$ is an injection then for some $i \in \{0,1\}$ we have that for some $X' \subseteq X$ of size X (i.e., X injects into X'), we have that $\pi_i \circ F \restriction X'$ is constant.

These notions are related to the ABCD conjecture:

Theorem (Chan, AD^+)

Let α, β, γ, δ be cardinals $< \Theta$. Then α^β injects into γ^δ iff $\alpha \leq \gamma$ and $\beta \leq \delta$.

Chan-J-Trang then got another proof which passes through a result on ∞-Borel sets.

Let $\kappa < \Theta$. We put a topology τ on $\mathcal{P}(\kappa)$ by defining basic open sets about $A \subseteq \kappa$ to be sets of the form

$$N_\sigma(A) = \{B \subseteq \kappa \colon \forall \alpha \in \sigma\ (\alpha \in B \leftrightarrow \alpha \in B)\}$$

Theorem (CJT)

Assume AD^+. *Let* $\kappa < \Theta$. *If* $A \subseteq \mathcal{P}(\kappa)$ *is* τ*-Borel the* A *is* ∞*-Borel.*

Some instances are provable from just AD:

Theorem (AD)

Assume $\kappa \to (\kappa)^{<\kappa}$. *Then* $\kappa^{<\kappa}$ *does not inject into* On^{δ} *for any* $\delta < \kappa$.

Theorem (AD)

$(\omega_n)^\omega$ is prime, ordinal regular, and $\mathcal{P}(\omega_1)$-regular.

Theorem (AD)

$(\omega_\omega)^\omega$ is prime, ordinal regular, and $\mathcal{P}(\alpha)$-regular for all $\alpha < \omega_\omega$.

Theorem (Chan)

Assume AD^+. κ^ω is On-regular for all $\kappa < \Theta$.

Theorem (AD)

$(\omega_n)^{<\omega_2}$, $(\omega_\omega)^{<\omega_2}$ are prime and $\mathcal{P}(\omega_1)$-regular.

We also have the following.

Theorem (AD)

Let κ be a regular limit Suslin cardinal. Then for almost all $f \in \kappa^\kappa$, $(\kappa^+)^{HOD_f}$ is constant.

$(\omega_\omega)^\omega$ is On-regular

We first sketch the proof that $(\omega_\omega)^\omega$ is On-regular.

Fix $\Phi\colon (\omega_\omega)^\omega \to \kappa$.

First assume $\mathrm{cof}(\kappa) < \boldsymbol{\delta}^1_3$. We consider the case $\mathrm{cof}(\kappa) = \omega_2$.

Fix $\rho\colon \omega_2 \to \kappa$ cofinal.

We define a type U. We use lex order on tuples

$$(\alpha, n, \alpha_{n-1}, \ldots, \alpha_0)$$

with $\alpha_0 < \cdots < \alpha_{n-1} < \alpha$.

We let U' be as in U except we have the extra points (α, ω), for $\alpha < \omega_1$ in the domain.

We consider $f\colon U \to \omega_1$ of the correct type. This induces functions $f_m\colon (\omega_1)^{m+1} \to \omega_1$ which represents $[f_m] < \omega_{m+2}$. We let $[f] \in (\omega_\omega)^\omega$ be $[f] = ([f_m])_m$.

Partition P: partition $F\colon U' \to \omega_1$ of the correct type according to whether $\Phi([f]) < \rho([g])$ where $g(\alpha) = F(\alpha, \omega)$.

On the homogeneous side the stated property holds.

Fix $C \subseteq \omega_1$ homogeneous for P, and let $h\colon \omega_1 \to \tilde{C}$ (closure points of C) be of the correct type.

We have a large set, those $p \in (\omega_\omega)^\omega$ represented by f of type U for which $\Phi(p) < \rho([g]) < \kappa$ (will give better argument below).

Suppose now $\mathrm{cof}(\kappa) \geq \boldsymbol{\delta}^1_3$.

Let (P, φ) be a prewellordering of length κ which is $< \mathrm{cof}(\kappa)$-Suslin bounded, so is $\boldsymbol{\Sigma}^1_3$-bounded.

By playing a Martin-type game, there is a strategy τ and a c.u.b. $C \subseteq \omega_1$ such that for any $x \in \omega^\omega$ which is a good code for a function $f_x \colon U \to C$, $\tau(x) \in P$ and $\varphi(\tau(x)) > \Psi([f_x])$.

The set of good codes coding functions from U into C is $\boldsymbol{\Pi}^1_2$. The $\boldsymbol{\Sigma}^1_3$ boundedness of (P, φ) then gives a bound for Ψ on a large set.

$(\omega_\omega)^\omega$ is $\mathcal{P}(\omega_2)$-regular

We next sketch the proof that $(\omega_\omega)^\omega$ is $\mathcal{P}(\omega_2)$-regular.

Fix $\Psi\colon (\omega_\omega)^\omega \to \mathcal{P}(\omega_2)$.

We need to consider certain types of pairs of functions $f, g \in (\omega_\omega)^\omega$.

We will have a canonical ω-sequence of such types.

For $k < \omega$ we define types U_k:

We use lex order on tuples

$$(\alpha, n, \alpha_{n-1}, \dots, \alpha_0, i)$$

with $\alpha_0 < \cdots < \alpha_n$ and where $i = 0$ if $n \leq k$, and $i \in \{0, 1\}$ if $n \geq k$.

An $f \colon U_k \to \omega_1$ of the correct type induces functions f^0, f^1 of type U.

Note that $f^0_m = f^1_m$ for $m \leq k$. For $m \geq k$, f^0, f^1 have the same $m - 1$st invariant, but $[f^0_m] < [f^1_m]$.

Consider also U'_k defined as U_k except we add (α, ω) for $\alpha < \omega_1$ to the domain.

A function $f\colon U'_k \to \omega_1$ induces $f_0, f_1 \in (\omega_\omega)^\omega$ and a $g\colon \omega_1 \to \omega_1$.

Consider the partition $\mathcal{P}_k$: we partition $F\colon U'_k \to \omega_1$ according to whether $\exists \delta < [g]$ with $\Psi(f_0) \restriction \delta \neq \Psi(f_1) \restriction \delta$.

On the homogeneous side the stated property holds.

Fix a c.u.b. $C \subseteq \omega_1$ homogeneous for all of these partitions.

We define a "master function" φ from a certain order-type to ω_1.

Let $<_T$ be lex ordering of tuples of the form

$$(\alpha, n, \alpha_{n-1}, \ldots, \alpha_0, \beta_0, \ldots, \beta_{n-1})$$

where $\alpha_0 < \cdots < \alpha_n < \alpha$, $1 \leq n < \omega$, $\beta_0, \ldots, \beta_n < \alpha$

From φ we define $\Phi\colon (\omega_\omega)^\omega \to (\omega_\omega)^\omega$.

Let $p \in (\omega_\omega)^\omega$ be increasing and assume $p(n) \in (\omega_{n+1}, \omega_{n+2})$. We call these the good sequences.

Let $f_k\colon \omega_{k+1} \to \omega_1$ represent $p(k)$.

Then $\Phi(p) = p'$ where $p'(k) = [g_k]$ is represented by:

$$g_k(\alpha_0, \dots, \alpha_k) = \varphi(\alpha_k, k, \alpha_{k-1}, \dots, \alpha_0, f_0(\alpha_0), f_1(\alpha_0, \alpha_1), \\ \dots, f_{k-1}(\alpha_0, \dots, \alpha_{k-1}))$$

We have:

- Each $p'(n) = [g_n] \in (\omega_{n+1}, \omega_{n+2})$ where g_n is of type $n+1$.
- $p'(n)$ has smaller first invariant than $p'(n+1)$.
- There is a bound on the first invariants of the $\Phi(p)(n)$.

Suppose p, q are good sequences in $(\omega_\omega)^\omega$. Let k be least so that $p(k) \neq q(k)$, and say $p(k) < q(k)$.

- $\Phi(p)(j) = \Phi(q)(j)$ for $j < k$.
- For $j \geq k$, $\phi(p)(j) < \Phi(q)(j)$ and $\Phi(p)(j)$, $\Phi(q)(j)$ have the same jth invariant (next to largest invariant).

So, p', q' are almost everywhere ordered as in type U_k.

Let $A = \Phi(\mathcal{G})$, where $\mathcal{G}$ is the set of good sequences.

So, A has size $(\omega_\omega)^\omega$.

Fix $g \colon \omega_1 \to C$ of the correct type with $g > \sup_n p'(n)(1)$ for any $p' \in A$, which we can do by above properties of Φ.

If $p' \neq q' \in A$ then $\sup_n p'(n)(1) = \sup q'(n)(1)$ is fixed. Also, for some $k \in \omega$, p', q' are represented by $F \colon U_k \to C$ of the correct type by sliding argument.

Furthermore there is an $F' \colon U'_k \to C$ of type U'_k such that $f'(\alpha, \omega) = g'(\alpha)$ where $[g'] = [g]$.

Let $\delta = [g]$.

This shows that for $p', q' \in A$, $\Psi(p') \restriction \delta \neq \Psi(q') \restriction \delta$.

This gives a map Ψ' from a large set $A \subseteq (\omega_\omega)^\omega$ to $\mathcal{P}(\delta) \approx \mathcal{P}(\omega_1)$.

By the result for $\mathcal{P}(\omega_1)$, there is a large set $B \subseteq A$ for which Ψ' is constant. From the definition of δ, Ψ is constant on B.

$(\omega_\omega)^\omega$ is prime

Fix $\Psi\colon (\omega_\omega)^\omega \to X_0 \times X_1$ an injection. Let $\Psi_0 = \pi_0 \circ \Psi$, $\Psi_1 = \pi_1 \circ \Psi$ (π_i is the projection map).

For $k < \omega$ consider partition $\mathcal{P}_k$:
Partition F of type U_k, inducing f_0, f_i representing good p_0, $p_1 \in (\omega_\omega)^\omega$ by

$$\mathcal{P}_k(F) = \begin{cases} 0 & \text{if } \Psi_0(p_0) \neq \Psi_0(p_1), \Psi_1(p_0) = \Psi_1(p_1) \\ 1 & \text{if } \Psi_1(p_0) \neq \Psi_1(p_1), \Psi_0(p_0) = \Psi_0(p_1) \\ 2 & \text{if } \Psi_i(p_0) \neq \Psi_i(p_1) \text{ for } i = 0, 1 \end{cases}$$

Fix $C \subseteq \omega_1$ homogeneous for all of the $\mathcal{P}_k$.

Define $\varphi \colon <_T \to C$ and $\Phi \colon (\omega_\omega)^\omega \to (\omega_\omega)^\omega$ as before. Let $A = \Phi(\mathcal{G})$.

Case 1 For all k, $\mathcal{P}_k$ is homogeneous for 2.

In this case, both Ψ_0 and Ψ_1 are injections when restricted to A.

Case 2 For some k, $\mathcal{P}_k$ is homogeneous for 0 or 1 side.

Let k_0 be least such, and wlog say $\mathcal{P}_{k_0}$ is homogeneous for the 0 side.

Claim

For all $k \geq k_0$, $\mathcal{P}_k$ is homogeneous for the 0 side.

Proof of claim:

Fix $k > k_0$. Let f, g into C be ordered as in U_k. Say f induces the f_n, and g the g_n where f_n, g_n are of type $n+1$.

Let h into C be such that

1. $f_n = g_n = h_n$ for $n < k_0$
2. $f_n = g_n < h_n$ for $k_0 \leq n < k$.
3. $f_n < g_n < h_n$ for $n \geq k$
4. f, h and g, h are of type U_{k_0}.

So, $\Psi_1(f) = \Psi_1(h) = \Psi_1(g)$, and so $\mathcal{P}_k(f, g) = 0$.

Let $B \subseteq A$ have full size and such that for all $p \in B$, $p(0), \ldots, p(k_0)$ are fixed.

Then for $p \neq q \in B$, $\Psi_1(p) = \Psi_1(q)$, and so $\Psi_0(p) \neq \Psi_0(q)$.

This shows that $(\omega_\omega)^\omega$ is prime.

An application to HOD

Theorem (AD)

Let κ be a regular limit Suslin cardinal (so κ has the strong partition property). Then $\exists \theta < \kappa^+$ such that for almost all $f \in \kappa^\kappa$ we have $(\kappa^+)^{HOD_f} = \theta$.

We first show:

Theorem (AD)

Let κ be a regular limit Suslin cardinal, and $\Phi \colon \kappa^\kappa \to \kappa^+$. Then $\exists \theta < \kappa^+ \ \forall^* f \in \kappa^\kappa \ \Phi(f) < \theta$.

Let μ_ω be the ω-cofinal normal measure on κ. We have $[\alpha \to \alpha^+]_{\mu_\omega} = \kappa^+$.

There is a tree T on $\omega \times \kappa$ such that for any block function g, that is, $g(\alpha) < \alpha^+$ for $\mathrm{cof}(\alpha) = \omega$, there is a $z \in \omega^\omega$ with T_z wellfounded and $g(\alpha) < |T_z \upharpoonright \alpha|$ for μ_ω almost all α.

Play the game G_Φ:

$$\begin{array}{c|cccccc}
I & x(0) & & x(1) & & x(2) & \cdots \\
 & & & & & & \\
II & & y(0) & & y(1) & & y(2) \quad \cdots \\
 & & z(0) & & z(1) & & z(2) \quad \cdots
\end{array}$$

where I plays out $x \in \omega^\omega$, and II plays out $y, z \in \omega^\omega$.

Say $\alpha < \kappa$ is good if $f_x(\alpha)$, $f_y(\alpha)$ are defined and $T_z \restriction \alpha$ is wellfounded. Otherwise say α is bad.

If there is a least bad α, the *II* wins iff $T_z \restriction \alpha$ is wellfounded and $f_x(\alpha)$ is undefined.

If all α are good, then f_x, f_y are defined and T_z is wellfounded. Then *II* wins iff $[\alpha \to T_z \restriction \alpha]_{\mu_\omega} > \Phi(f_{x,y})$.

Here $f_{x,y}$ is the usual joint function:

$$f_{x,y}(\alpha) = \sup_n \max\{f_x(\omega \cdot \alpha + n), f_y(\omega \cdot \alpha + n)\}.$$

A boundedness argument shows I cannot have a winning strategy.

Fix a winning strategy τ for II.

For $\alpha < \kappa$ with $\operatorname{cof}(\alpha) = \omega$, let $A_\alpha = \{x \colon \forall \alpha' < \alpha \; f_x(\alpha') < \alpha\}$. $A_\alpha \in \boldsymbol{\Delta}^\alpha_1$.

By boundedness $g(\alpha) = \sup\{T_z \restriction \alpha \colon x \in A_\alpha, \tau(x)_1 = z\}$ is less than κ.

The usual argument give a c.u.b. $C \subseteq \kappa$ fof that for all $f \colon \kappa \to C$ of the correct type we have $\Phi(f) < [g]_{\mu_\omega}$.

To prove the first theorem, start with a $\delta < \kappa^+$ so that $\forall^* f \in \kappa^\kappa\ \Phi(f) < \delta$.

Let $\alpha_f = (\kappa^+)^{\mathrm{HOD}_f}$.

Fix a bijection $\pi \colon \delta \to \kappa$. Let $\beta_f = \pi(\alpha_f) < \kappa$.

Partition $\mathcal{P}$: partition f of the correct type according to whether for all $\gamma < \kappa$ we have $\alpha(f) = \alpha(f_\gamma)$ where $f_\gamma(\eta) = f(\gamma + \eta)$.

By wellfoundedness, on the homogeneous side this holds, so $\alpha(f)$, and $\beta(f)$, is E_0-invariant.

So, on the homogeneous side we must have $\beta(f) < \min(f)$.

This gives that $\beta(f)$, and hence $\alpha(f)$, is constant on some C^κ.

Question

Does the first theorem hold for $\kappa = \omega_1$?