# Atomicity and splitting above a Reinhardt cardinal

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## 1 Introduction

This talk is on the structure of small large cardinals, like Mahlo and indescribable cardinals, above extremely large cardinals, large cardinals beyond the Axiom of Choice. The main new result is joint work with Nai-Chung Hou, proved in ZF.[1]

**Theorem 1.1** (Goldberg–Hou)**.** *Suppose $\kappa$ is a totally indescribable cardinal and there is a nontrivial elementary embedding from $V_\kappa$ to itself. Then for all $n < \omega$,*

- *If $n$ is even, the $\Pi^1_n$-indescribable ideal on $\kappa$ is atomic.*
- *If $n$ is odd, the $\Pi^1_n$-indescribable ideal on $\kappa$ is atomless.*

Since the $\Pi^1_n$-indescribable ideal is a $\Sigma^1_{n+1}$-complete subset of $V_{\kappa+1}$, Theorem 1.1 leads to the following corollary, under the same assumptions on $\kappa$ as Theorem 1.1:

**Corollary 1.2.** *For all odd numbers $n$, the pointclass $\mathbf{\Sigma}^1_n(V_{\kappa+1})$ has the separation property.*

We follow Schlutzenberg in considering not Reinhardt cardinals but a first-order weakening of them that seems to capture the salient features of a Reinhardt while avoiding some technical complications. A cardinal $\lambda$ is *rank Berkeley* if for all $\alpha < \lambda < \beta$, there is an elementary embedding from $V_\beta$ to $V_\beta$ whose critical point is between $\alpha$ and $\lambda$.

**Exercise 1** (Schlutzenberg, Woodin)**.** Prove that if $j : V \to V$ is an elementary embedding with critical point $\kappa$, then $\lambda = \sup\{\kappa, j(\kappa), j(j(\kappa)), \dots\}$ is rank Berkeley.

**Exercise 2.** Show that if $\kappa$ is $\Pi^1_1$-indescribable, then $V_\kappa$ satisfies that there is a rank Berkeley cardinal if and only if $V_\kappa$ satisfies that there is a Reinhardt cardinal.

In order to get by in ZF alone, the proof of Theorem 1.1 uses techniques from [2] for simulating the Axiom of Choice using rank reflection. For the sake of simplicity and variety, we will not prove Theorem 1.1 itself, but instead make use of a strong choice assumption throughout these notes:

**Background Theory.** ZF + $\lambda$ is a rank Berkeley cardinal + $\mathrm{DC}_\lambda$.

[1] A set $S \subseteq \kappa$ is $\Pi^1_n$-*indescribable* if for all $X \subseteq V_\kappa$ and $\Pi^1_n$-formulas $\varphi$, if $V_\kappa \vDash \varphi(X)$, then there is some $\alpha \in S$ such that $V_\alpha \vDash \varphi(X \cap V_\alpha)$. A cardinal is *totally indescribable* if it is $\Pi^1_n$-indescribable for all $n < \omega$. Finally, the $\Pi^1_n$-*indescribable ideal* consists of all subsets of $\kappa$ that are *not* $\Pi^1_n$-indescribable.

So from now on, the symbol $\lambda$ will denote a rank Berkeley cardinal such that $\mathrm{DC}_\lambda$ holds. By the Kunen inconsistency theorem, $\lambda$ is the least rank Berkeley cardinal.

Using Woodin's technique of forcing choice by iteratively collapsing supercompact cardinals, one can show that the existence of $\lambda$ is consistent relative to the existence of a rank Berkeley cardinal that is a limit of supercompact cardinals.

## 2 Woodin's proof of the Kunen inconsistency

We begin by reviewing Woodin's analysis of the $\gamma$-club filter in the context of choiceless cardinals. The $\gamma$-club filter on $\delta$ is the filter generated by all sets $C \subseteq \delta$ such that for any $\sigma \subseteq C$ with $\mathrm{ot}(\sigma) = \gamma$, $\sup(\sigma) \in C$. This is the same as the club filter restricted to the set $S^\delta_\gamma = \{\alpha < \delta : \mathrm{cf}(\alpha) = \gamma\}$.[2] The results in this section are due to Woodin.

For any function $j$, let

$$\mathrm{Fix}(j) = \{x \in \mathrm{dom}(j) : j(x) = x\}$$

Recall that under AC, Solovay showed that every stationary subset of a regular cardinal $\delta$ can be split into $\delta$ stationary subsets. Above our rank Berkeley cardinal $\lambda$, this fails badly:

**Theorem 2.1.** *If $\gamma < \delta$ are regular, then $S^\delta_\gamma$ cannot be split into $\lambda$ stationary sets.*

*Proof.* Let $j : V_{\delta+1} \to V_{\delta+1}$ be elementary with $\mathrm{crit}(j) < \lambda$ and $j(\gamma) = \gamma$. Let $\kappa = \mathrm{crit}(j)$. Assume towards a contradiction that $\langle S_\alpha\rangle_{\alpha<\kappa}$ are disjoint stationary subsets of $S^\delta_\gamma$. Let $\langle T_\alpha\rangle_{\alpha<j(\kappa)} = j(\langle S_\alpha\rangle_{\alpha<\kappa})$. Since $j(\gamma) = \gamma$, the set $\mathrm{Fix}(j) \cap \delta$ is a $\gamma$-club subset of $\delta$. Since $T_\kappa$ is a stationary subset of $S^\delta_\gamma$, $T_\kappa$ intersects every $\gamma$-club subset of $\delta$. Therefore there is an ordinal $\xi \in T_\kappa \cap \mathrm{Fix}(j)$. By elementarity, $\xi \in \bigcup_{\alpha<\kappa} S_\alpha$, and so there is some $\alpha < \kappa$ such that $\xi \in S_\alpha$. Then $\xi \in T_{j(\alpha)}$. Since $\kappa = \mathrm{crit}(j) \neq j(\alpha)$, this contradicts that the sets $T_\alpha$ are disjoint. $\square$

The set $A$ is an *atom* of $F$ if $A$ cannot be split into two $F$-positive sets, or equivalently if $F \restriction A$ is an ultrafilter.

**Exercise 3.** If $\delta > \lambda$ is regular, then for any regular $\gamma < \delta$, $S^\delta_\gamma$ is the union of fewer than $\lambda$ atoms of the club filter on $\delta$.

*Hint.* The argument is due to Ulam, but one must check that it goes through using just $\mathrm{DC}_\lambda$. Build a binary branching tree of stationary sets with root node $S^\delta_\gamma$, such that the children of each node form a partition of the parent node into two stationary sets. Take intersections at limits. Nodes are terminal if they are either atoms or nonstationary. The leaves of this tree are the desired partition of $S^\delta_\gamma$ into atoms. $\square$

**Corollary 2.2.** *The cardinal $\lambda^+$ is measurable, and this is witnessed by the club filter restricted to a stationary set.*

[2] If $F$ is a filter on $X$, a set $A \subseteq X$ is $F$-*positive*, denoted $A \in F^+$, if $A$ intersects every set in $F$. In this case, the *restriction* of $F$ to $A$, denoted $F \restriction A$, is the filter on $X$ generated by $F \cup \{A\}$.

## 3 Mahlo cardinals

Woodin's argument does not answer the question: suppose $\kappa > \lambda$ is a Mahlo cardinal. Does the set of regular cardinals less than $\kappa$ contain an atom of the club filter? A variant of his proof does answer this question, however:

**Proposition 3.1.** *If $\delta > \lambda$ is Mahlo, then the set of regular non-Mahlo cardinals below $\delta$ is the union of fewer than $\lambda$ atoms of the club filter.*

*Proof.* Let $S$ be the set of regular non-Mahlo cardinals below $\delta$. To run Woodin's argument, it suffices to show that if $j : V_{\delta+1} \to V_{\delta+1}$ is elementary, then the set $\text{Fix}(j) \cap \delta$ belongs the the club filter restricted to $S$. Let $C$ be the club of ordinals $\gamma < \delta$ such that $j[\gamma] \subseteq \gamma$. We claim that $C \cap S \subseteq \text{Fix}(j) \cap \delta$.

Suppose $\gamma \in C \cap S$, and assume towards a contradiction that $j(\gamma) > \gamma$. We claim $S \cap \gamma$ would be stationary in $\gamma$. Indeed, if $E \subseteq \gamma$ is closed unbounded, then $\gamma \in j(E)$ and of course $\gamma \in j(S \cap \gamma) = S \cap j(\gamma)$, so $j(S \cap \gamma) \cap j(E) \neq \emptyset$; by elementarity $S \cap E \neq \emptyset$. $\square$

Recall that the *trace* of a set $S \subseteq \delta$ is the set $\text{Tr}(S) = \{\alpha < \delta : S \cap \alpha \text{ is stationary}\}$. (Our convention is that if $\text{cf}(\alpha) = \omega$, then every subset of $\alpha$ is nonstationary.) A set $S$ is *thin* if it is disjoint from its trace. If $S$ is stationary, then $S \setminus \text{Tr}(S)$ is a thin stationary set.

Many of our theorems are motivated by the following Cabal result:

**Theorem 3.2** (Kechris–Kleinberg–Moschovakis–Woodin [4])**.** *If $\kappa$ is a strong partition cardinal and $S \subseteq \kappa$ is a thin stationary set, then $S$ is an atom of the club filter.* $\square$

The proof of Proposition 3.1 above shows:

**Exercise 4.** If $\delta > \lambda$ is regular and $S$ is a thin *ordinal definable* stationary subset of $\delta$. Then $S$ is the union of fewer than $\lambda$ atoms of the club filter.

In fact, one only needs that $[S]_{\text{NS}} = \{T \subseteq \delta : T \triangle S \in \text{NS}\}$ is ordinal definable. But what about arbitrary stationary sets? We will show:

**Theorem 3.3.** *The club filter on any regular $\delta > \lambda$ is atomic.*

This is an immediate consequence of the following theorem, recalling that every stationary set $S$ contains the thin stationary set $S \setminus \text{Tr}(S)$.

**Theorem 3.4.** *If $\delta > \lambda$ is a regular cardinal and $S \subseteq \delta$ is a thin stationary set, then $S$ is the union of at most $\lambda$ atoms of the club filter.*

The proof of Theorem 3.4 heavily uses the Ketonen order on filters and ultrafilters.

**Definition 3.5.** If $U$ and $W$ are countably complete ultrafilters on $\delta$, then $U <_{\Bbbk} W$ if for $W$-almost all $\alpha$, there is a countably complete ultrafilter $U_\alpha$ on $\alpha$ such that

$$A \in U \iff \{\alpha : A \cap \alpha \in U_\alpha\} \in W$$

In other words, $U = W\text{-}\lim_{\alpha<\delta} U_\alpha$.

**Theorem 3.6** (Goldberg [1, Chapter 3])**.** *The Ketonen order on ultrafilters is well-founded. The Ultrapower Axiom holds iff it is linear.* $\square$

Assuming choiceless cardinals, it is useful to consider a somewhat stronger order than the Ketonen order. If $A, B \subseteq P(\delta)$, set $A \leq_e B$ if there is an elementary embedding $j : V_{\delta+1} \to V_{\delta+1}$ with $A = j^{-1}[B]$.

**Lemma 3.7.** *Suppose $U \leq_e W$ are countably complete ultrafilters on $\delta$. Then $U \leq_{\Bbbk} W$.*

*Proof.* Let $j : V_{\delta+1} \to V_{\delta+1}$ witness $U \leq_e W$. If $\mathrm{Fix}(j) \cap \delta \in W$, then $U = W$. Otherwise, let $Y = \delta \setminus \mathrm{Fix}(j)$. For $\xi \in Y$, let $U_\xi = \{A \subseteq \xi : \xi \in j(A)\}$. Then $U = W\text{-}\lim_{\xi \in Y} U_\xi$ so $U <_{\Bbbk} W$. □

We use the following semilinearity property of the embedding order, which given Theorem 3.6 is a weak version of the Ultrapower Axiom that is provable from choiceless cardinals:

**Theorem 3.8.** *Suppose $Z$ is an antichain in the embedding order on countably complete ultrafilters on $\delta$. Then $|Z| \leq \lambda$.*

An analog of this result is proved without any choice in [2].

**Lemma 3.9.** *Suppose $U$ and $W$ are distinct countably complete ultrafilters extending the club filter. If $U \leq_e W$, then for all $S \in U$, $\mathrm{Tr}(S) \in W$.*

*Proof.* We follow the notation from the proof of Lemma 3.7. Since $U \neq W$, we may assume $\delta \setminus \mathrm{Fix}(j) \in W$.

**Exercise 5.** If $\xi \in \delta \setminus \mathrm{Fix}(j)$ is a closure point of $j$, then $U_\xi$ extends the club filter.

If $S \in U$, then for $W$-almost all $\xi$, $S \cap \xi \in U_\xi$ and $U_\xi$ extends the club filter, and hence $S \cap \xi$ is stationary. □

**Corollary 3.10.** *If $\delta$ is regular and $S \subseteq \delta$ is thin, any two countably complete ultrafilters extending the club filter restricted to $S$ are incomparable in the embedding order.* □

In particular, the set such ultrafilters has size at most $\lambda$.

To conclude Theorem 3.4 from Corollary 3.10, we need a filter extension theorem:

**Theorem 3.11.** *If $F$ is a $\lambda^+$-complete filter on an ordinal, then $F$ extends to a $\lambda^+$-complete ultrafilter.*

With this theorem in hand, it is straightforward to partition thin sets into atoms:

*Proof of Theorem 3.4.* Let $F$ be the club filter restricted to $S$, and let $Z$ be the set of $\lambda^+$-complete ultrafilters extending $F$. By Theorem 3.8 and Corollary 3.10, $|Z| \leq \lambda$. Using $\mathrm{AC}_\lambda$ and $\lambda^+$-completeness, choose sets $\langle A_U : U \in Z \rangle$ such that for all $U \in Z$, $A_U \in U$ and $A_U \notin W$ for all $W \in Z \setminus \{U\}$. We may assume $A_U \subseteq S$.

We claim $S = \bigcup_{U \in Z} A_U$ modulo a nonstationary set. Otherwise, $T = S \setminus \bigcup_{U \in Z} A_U$ is stationary. By Theorem 3.11, there is a $\lambda^+$-complete ultrafilter $U$ extending the club filter restricted to $T$. But then $U \in Z$, so $A_U \cap T$ is nonempty, contrary to the definition of $T = S \setminus \bigcup_{U \in Z} A_U$.

**Exercise 6.** Show, by a similar argument, that each set $A_U$ is an atom of the club filter. □

# 4 The Ketonen order

We now prove the basic results cited above that require the theory of the Ketonen order.

*Proof of Theorem 3.8.* Let $\theta > \delta$ be a limit ordinal greater than the supremum of the Ketonen ranks of elements of $Z$. Let $j : V_\theta \to V_\theta$ be an elementary embedding with $\operatorname{crit}(j) < \lambda$. Recall that the iterates of $j$ are the embeddings $j_0 = j$, $j_1 = j(j)$, $j_2 = j(j_1)$ and so on, and $\kappa_n = \operatorname{crit}(j_n)$. For all $\xi < \theta$, there is some $n < \omega$ such that $j_n(\xi) = \xi$. By Lemma 3.7, for all countably complete ultrafilters $U$ on $\delta$, there is some $n$ such that $j_n(U) = U$.

Let $Z_n = Z \cap \operatorname{Fix}(j_n)$. We claim $|Z_n| < \kappa_n$. Otherwise there is some $W \in j_n(Z_n) \setminus j_n[Z_n]$. We have that $W$ and $j_n(W)$ are distinct ultrafilters in $j_n(j_n(Z_n))$. (They are distinct since $W \notin \operatorname{ran}(j_n)$. And $W \in j_n(j_n(Z_n))$ since $j_n(Z_n) \subseteq \operatorname{Fix}(j_{n+1})$, so $W = j_{n+1}(W) \in j_{n+1}(j_n(Z_n)) = j_n(j_n(Z_n))$.) Since $W \leq_e j_n(W)$, this contradicts that $j_n(j_n(Z_n))$ is an antichain in the embedding order. $\square$

The proof of Theorem 3.11 uses the filter generated by fixed points of elementary embeddings and an induction on the Ketonen order on filters.

**Definition 4.1.** If $E$ and $F$ are countably complete filters on an ordinal $\delta$, then $E <_{\Bbbk} F$ if there are countably complete filters $E_\alpha$ on each nonzero $\alpha < \delta$ such that $E \subseteq F\text{-}\lim_{\alpha<\delta} E_\alpha$.

The Ketonen order on filters is again well-founded, and this is the key to the proof of the filter extension theorem.

*Proof of Theorem 3.11.* Assume towards a contradiction that the theorem is false, and let $F$ be a counterexample of minimal Ketonen rank. So $F$ is a $\lambda^+$-complete filter on an ordinal $\delta$ and $F$ does not extend to a $\lambda^+$-complete ultrafilter.

Let $\theta$ be a sufficiently large ordinal, and let $G$ be the $\lambda^+$-complete filter generated by sets of the form $\operatorname{Fix}(j) \cap \delta$ where $j : V_\theta \to V_\theta$ is an elementary embedding with $F \in \operatorname{ran}(j)$. Then Woodin's proof of the Kunen inconsistency theorem shows that $\delta$ can be partitioned into fewer than $\lambda$ atoms of $G$.

To obtain a contradiction, we will show that $F$ and $G$ are compatible filters. Granting this, let $H$ be the filter generated by $F \cup G$, and note that $H$ is $\lambda^+$-complete. Therefore there must be an $H$-positive atom $A$ of $G$. It follows that $G \restriction A$ is a $\lambda^+$-complete ultrafilter extending $F$, contrary to our choice of $F$.

To show that $F$ and $G$ are compatible, suppose for $\alpha < \lambda$ that $j_\alpha : V_\theta \to V_\theta$ is an elementary embedding with $F \in \operatorname{ran}(j_\alpha)$. We must show that $\bigcap_{\alpha<\lambda} \operatorname{Fix}(j_\alpha) \cap \delta$ is $F$-positive. Suppose not, so that $Y = \bigcup_{\alpha<\lambda} \delta \setminus \operatorname{Fix}(j_\alpha)$ belongs to $F$. Let $F_\alpha = j_\alpha^{-1}[F]$ and let $E = \bigcap_{\alpha<\lambda} F_\alpha$. The assumption that $Y \in F$ implies $E <_{\Bbbk} F$ by the following construction. For each $\xi < \delta$, let $D^\alpha_\xi$ be the ultrafilter on $\xi + 1$ derived from $j_\alpha$ using $\xi$. Then by the proof of Lemma 3.7, $F_\alpha = F\text{-}\lim_{\xi<\delta} D^\alpha_\xi$. Let $D_\xi = \bigcap_{\alpha<\lambda} D^\alpha_\xi$. Then since $F$ is $\lambda^+$-complete, $E = F\text{-}\lim_{\xi<\delta} D_\xi$. If $\xi \in Y$, then $\xi \in D^+_\xi$, and so for such $\xi$, let $E_\xi = D_\xi \restriction \xi$, viewed as a filter on $\xi$. Then $E \subseteq F\text{-}\lim_{\xi<\delta} E_\xi$, which proves $E <_{\Bbbk} F$.

By the minimality of $F$, $E$ extends to a $\lambda^+$-complete ultrafilter $W$. Then for some $\alpha < \lambda$, $F_\alpha \subseteq W$: otherwise choose $A_\alpha \in F_\alpha \setminus W$ and note that by the $\lambda^+$-completeness of $W$, $\bigcup_{\alpha<\lambda} A_\alpha \in E \setminus W$, contrary to the fact that $E \subseteq W$. But $j_\alpha(F_\alpha) = F$, so by elementarity and our choice of $F$, $F_\alpha$ does not extend to a $\lambda^+$-complete ultrafilter. $\square$

An analog of this result is proved without any choice in [2], but this is much more difficult.

Recall that if $S$ and $T$ are stationary sets, then $S < T$ if $T \subseteq \mathrm{Tr}(S)$. This is Jech's *reflection order*, a well-founded partial order of stationary sets.

**Exercise 7.** Use the well-foundedness of the Ketonen order on filters to establish the well-foundedness of the reflection order.

**Exercise 8.** Show that if $\delta$ is regular, then any antichain in the reflection order on atoms of the club filter has cardinality at most $\lambda$.

This should be compared with the following theorem of Steel:

**Theorem 4.2** (Steel [3])**.** *If $\kappa$ is a strong partition cardinal, then the thin subsets of $\kappa$ are prewellordered by the reflection order.* $\square$

**Exercise 9.** Suppose $\delta$ is a regular cardinal above $\lambda$. Show that for every $\xi < o(\delta)$, there is a maximum stationary set with rank $\xi$ in the reflection order.

# 5 Indescribable cardinals

The first large cardinal hypothesis beyond the Mahlo hierarchy is weak compactness, or equivalently $\Pi^1_1$-indescribability. There is a well-known analogy between the club filter and the $\Pi^1_1$-reflection filter, so it makes sense to try to extend our results on stationary sets to $\Pi^1_1$-indescribable sets. But our main theorem is that the structure is completely different.

**Theorem 5.1** (Goldberg–Hou)**.** *The $\Pi^1_n$-indescribable ideal on any $\Pi^1_n$-indescribable cardinal above $\lambda$ is atomic if $n$ is even and atomless if $n$ is odd.*

Recall that $\kappa$ is $\Pi^1_n$-indescribable if for all $X \subseteq V_\kappa$ and all $\Pi^1_n$-formulas $\varphi$, if $V_\kappa \vDash \varphi(X)$, then there is some $\alpha < \kappa$ such that $V_\alpha \vDash \varphi(X \cap V_\alpha)$.

The $\Pi^1_n$-indescribable filter $F_n(\kappa)$ is the filter generated by sets of the form

$$R_{\varphi,X} = \{\alpha < \kappa : V_{\alpha+1} \vDash \varphi(X \cap V_\alpha)\}$$

where $\varphi$ is $\Pi^1_n$ and $V_{\kappa+1} \vDash \varphi(X)$. We denote the family of $F_n(\kappa)$-positive sets by $F^+_n(\kappa)$, the dual ideal by $F^*_n(\kappa)$, and for $S \in F^+_n(\kappa)$, we let $F_n(S) = F_n(\kappa) \restriction S$, etc. So $F^+_n(\kappa)$ is the family of $\Pi^1_n$-indescribable sets, and $F^*_n(\kappa)$ is the $\Pi^1_n$-indescribable ideal.

Let $\mathcal{E}(V_\alpha)$ denote the set of elementary embeddings from $V_\alpha$ to itself and $\mathcal{E}_n(V_\alpha)$ denote the set of $\Sigma^1_n$-elementary embeddings from $V_\alpha$ to itself, so $j \in \mathcal{E}_n(V_\alpha)$ if $j$ extends to a $\Sigma_n$-elementary embedding from $V_{\alpha+1}$ to itself. When $\alpha$ is a limit ordinal, we will often confuse an embedding $j \in \mathcal{E}(V_\alpha)$ with its canonical extension to $V_{\alpha+1}$, defined by $j^+(A) = \bigcup_{x \in V_\alpha} j(A \cap x)$.

It is not hard to show that if $\kappa$ is an inaccessible cardinal, then $\mathcal{E}(V_\kappa) = \mathcal{E}_0(V_\kappa)$. Our theorem rests on the following periodicity result, due to Martin, that can be seen as an extension of this basic fact:

**Theorem 5.2** (Martin)**.** *Suppose $\kappa$ is an inaccessible cardinal and $n$ is odd. Then $\mathcal{E}_n(V_\kappa) = \mathcal{E}_{n+1}(V_\kappa)$. In particular, $\mathcal{E}_{n+1}(V_\kappa)$ is a $\Pi^1_{n+1}$ subset of $V_{\kappa+1}$.*[3] $\square$

[3] Martin proved this theorem in ZFC without the assumption that $\kappa$ is inaccessible.

Our first lemma, which is well-known, generalizes the fact that if $\kappa$ is $\Pi^1_1$-indescribable, then the club filter on $\kappa$ is $\Sigma^1_1$-complete.

**Lemma 5.3.** *Suppose $n < \omega$ and $\kappa$ is $\Pi^1_n$-indescribable. Then for every $\Sigma^1_{n+1}$ formula $\varphi$ and $X \subseteq V_\kappa$,*

- *If $V_\kappa \vDash \varphi(X)$, then $\{\alpha < \kappa : V_\alpha \vDash \varphi(X \cap V_\alpha)\} \in F_n(\kappa)$.*
- *If $\kappa$ is $\Pi^1_{n+1}$-indescribable, the converse implication holds.*

**Lemma 5.4.** *If $\kappa$ is $\Pi^1_n$-indescribable, $j \in \mathcal{E}(V_\kappa)$, and $\{\alpha < \kappa : j \restriction V_\alpha \in \mathcal{E}_n(V_\alpha)\} \in F^+_n(\kappa)$, then $j \in \mathcal{E}_n(V_\kappa)$.*

*Proof.* The statement that $j \in \mathcal{E}_n(V_\kappa)$ is $\Pi^1_{n+1}$, so the lemma follows from Lemma 5.3. □

**Theorem 5.5** (Goldberg–Hou)**.** *Suppose $n$ is odd and $\kappa$ is $\Pi^1_n$-reflecting. Then $F_n(\kappa)$ is atomless.*

*Proof.* We may assume $\kappa > \lambda$, since otherwise the fact that $F_n(\kappa)$ is atomless is an easy consequence of $\mathrm{DC}_\lambda$.

Suppose $A \in F^+_n(\kappa)$. We will show that for any nontrivial $j \in \mathcal{E}(V_\kappa)$ with $A \in \mathrm{ran}(j)$, $\mathrm{Fix}(j) \cap \kappa \notin F_n(A)$. Suppose towards a contradiction that $\gamma$ is the least critical point of some $j \in \mathcal{E}(V_\kappa)$ with $A \in \mathrm{ran}(j)$ and $\mathrm{Fix}(j)\cap\kappa \in F_n(A)$. Note that the statement that there is no smaller such ordinal is $\Pi^1_{n+1}$ in the parameter $A$. Let $i \in \mathcal{E}(V_\kappa)$ witness the defining property of $\gamma$, and note that by Lemma 5.4, $i$ is $\Sigma^1_n$-elementary. So by Theorem 5.2, $i$ is $\Sigma^1_{n+1}$-elementary. Thus by elementarity and the minimality of $\gamma$, there is no $k \in \mathcal{E}(V_\kappa)$ with critical point less than $i(\gamma)$ such that $i(A) \in \mathrm{ran}(k)$ and $\mathrm{Fix}(k)\cap\kappa \in F_n(i(A))$. But $i$ is such an embedding, since $F_n(A) \subseteq F_n(i(A))$. To see this inclusion, note that since $A \in \mathrm{ran}(i)$, $i(A) = i(i)(A)$, and moreover by the elementarity of $i$, $\mathrm{Fix}(i(i)) \cap \kappa \in F_n(i(A))$; finally $\mathrm{Fix}(i(i)) \cap i(A) \subseteq A$, so $A \in F_n(i(A))$.

Now suppose that $A \in F^+_n(\kappa)$, $\theta > \kappa$ is a sufficiently large ordinal, and $j : V_\theta \to V_\theta$ is an elementary embedding with critical point less than $\kappa$ such that $A \in \mathrm{ran}(j)$ and $j(\xi) = \xi$ where $\xi$ is the Ketonen rank of $F_n(A)$. Let $\bar{A} = j^{-1}(A)$, and note that $F_n(\bar{A})$ has Ketonen rank $\xi$. In particular, $F_n(\bar{A}) \not<_{\Bbbk} F_n(A)$, so by the proof of Lemma 3.7, $\mathrm{Fix}(j) \cap \kappa \in F^+_n(A)$. Since $\mathrm{Fix}(j) \cap \kappa \notin F_n(A)$, it follows that $A$ is not an atom of $F_n(\kappa)$. □

The analysis of $F_n(\kappa)$ when $n$ is even generalizes that of the closed unbounded filter from the previous section, which one might think of as the case $n = 0$. We therefore need to extend the Mahlo operation to higher orders of indescribability. For any $S \subseteq \kappa$,

$$\mathrm{Tr}_n(S) = \{\alpha < \kappa : S \cap \alpha \in F^+_n(\alpha)\}$$

A set $S \in F^+_n(\kappa)$ is *$n$-thin* if $S \cap \mathrm{Tr}_n(S)$ is $F_n(\kappa)$-null.

**Lemma 5.6.** *If $S \in F^+_n(\kappa)$, then $S$ contains an $n$-thin set.*

*Proof.* Otherwise $F_n(S) <_{\Bbbk} F_n(S)$, which is impossible. □

**Lemma 5.7.** *Suppose $n$ is even and $F$ and $G$ extend $F_n(\kappa)$. If $F \leq_e G$, either $F \subseteq G$ or for all $S \in F$, for a $G$-positive set of $\alpha < \delta$, $S \cap \alpha \in F^+_n(\alpha)$.*

*Proof.* This is just like Lemma 3.9, with the following adjustment. We use the fact that for $j \in \mathcal{E}_n(V_\kappa)$, $\{\alpha < \kappa : j \restriction V_\alpha \in \mathcal{E}_n(V_\alpha)\}$ belongs to $F_n(\kappa)$. This is because $n$ is even, so the statement that $j \in \mathcal{E}_n(V_\kappa)$ is $\Pi^1_n$. As a consequence, for $F_n(\kappa)$-almost all $\alpha$, if $j(\alpha) > \alpha$, then the ultrafilter on $\alpha$ derived from $j$ using $\alpha$ extends $F_n(\alpha)$. $\square$

**Theorem 5.8** (Goldberg–Hou)**.** *Suppos $n$ is even and $\kappa > \lambda$ is $\Pi^1_n$-reflecting. Then any $n$-thin stationary set $S \subseteq \kappa$ is the union of at most $\lambda$ atoms of $F_n(\kappa)$.*

*Proof.* This is just like Theorem 3.4. $\square$

The idea behind the following theorem comes from joint work with Tom Benhamou on the structure of filters in $L$.

**Theorem 5.9** (Goldberg–Hou)**.** *If $n$ is odd and $\kappa > \lambda$ is $\Pi^1_n$-indescribable, then any pair of disjoint $\mathbf{\Sigma}^1_n$ subsets of $V_{\kappa+1}$ can be separated by a $\mathbf{\Delta}^1_n$ subset of $V_{\kappa+1}$.*

*Proof.* Let $Y_0$ and $Y_1$ be disjoint $\mathbf{\Sigma}^1_n$ subsets of $V_{\kappa+1}$. For $i = 0, 1$, fix $a_i \subseteq V_\kappa$ and a $\Sigma^1_n$-formula $\varphi_i$ such that
$$Y_i = \{y \subseteq V_\kappa : V_\kappa \vDash \varphi_i(y, a_i)\}$$
For $\alpha < \kappa$, let
$$Y_i^\alpha = \{y \subseteq V_\alpha : V_\alpha \vDash \varphi_i(y, a_i \cap V_\alpha)\}$$
Then $S = \{\alpha < \kappa : A_0^\alpha \cap A_1^\alpha = \emptyset\}$ belongs to $F_n$, and in particular, $S \in F^+_{n-1}$.

Let $A \subseteq S$ be an atom of $F_{n-1}$, and for $i = 0, 1$, let $Z_i$ be the set of $y \subseteq V_\kappa$ such that $\{\alpha < \kappa : y \cap V_\alpha \in Y_i^\alpha\} \in F_{n-1}(A)$. Clearly $Z_i$ is $\mathbf{\Sigma}^1_n$ and $Y_i \subseteq Z_i$ for $i = 0, 1$. Since $F_{n-1}(A)$ is an ultrafilter, $Z_0 \cup Z_1 = V_{\kappa+1}$. $\square$

The *reflection order* on $\Pi^1_n$-indescribable subsets of $\kappa$ is defined by $S <_n T$ if $T \subseteq \mathrm{Tr}_n(S)$ modulo $F^*_n(\kappa)$. As in Exercise 8, we have:

**Exercise 10.** Suppose $n$ is even, $\kappa$ is $\Pi^1_n$-indescribable, and $Z$ is an antichain of atoms of $F_n(\kappa)$ in the reflection order. Then $|Z| \leq \lambda$.

**Proposition 5.10.** *Assume $\kappa$ is $\Pi^1_{n+1}$-indescribable and for any $n$-thin sets $S_0, S_1 \subseteq \kappa$, either $S_0 <_n S_1$, $S_1 <_n S_0$, or $S_0 =_n S_1$. Then $\mathbf{\Pi}^1_n(V_{\kappa+1})$ and $\mathbf{\Sigma}^1_{n+1}(V_{\kappa+1})$ have the prewellordering property.*

*Proof.* It is not hard to check that the relation $<_n$ is a $\Pi^1_n$-prewellorder of the set $T$ of thin subsets of $\kappa$. Moreover, $T$ is $\Pi^1_n$-complete. $\square$

Thus by strengthening the conclusions of Theorem 5.8 and Exercise 10 in a natural way, in analogy with the results from strong partition cardinals (Theorem 3.2 and Theorem 4.2), we would obtain that, if $\kappa$ is totally indescribable, then for all $n < \omega$, the pointclasses $\mathbf{\Pi}^1_{2n+1}(V_{\kappa+1})$ and $\mathbf{\Sigma}^1_{2n+2}(V_{\kappa+1})$ have the prewellordering property.

Unfortunately, this strengthening is not provable from choiceless large cardinals:

**Exercise 11.** It is consistent that for all regular $\delta > \lambda$, $S^\delta_\omega$ is not an atom of the club filter.

Actually for more complicated sets, it is open:

**Question 5.11.** Could the hypotheses of Proposition 5.10 follow from choiceless large cardinals for $n > 1$?

Even for $n = 0$, the following is open:

**Question 5.12.** Suppose $S$ is a thin stationary set of ordinals of cofinality greater than $\lambda$. Is $S$ an atom of the club filter?[4]

It seems more likely that this is independent.

**Question 5.13.** What is the structure of indescribable ideals in Nairian models?

The *n-club filter* $\mathcal{C}_n(\kappa)$ on an ordinal $\kappa$ is defined by recursion as follows. First, $\mathcal{C}_{-1}(\kappa)$ is the tail filter on $\kappa$. For $n < \omega$, $\mathcal{C}_n(\kappa)$ is the filter generated by sets $C \in \mathcal{C}^+_{n-1}(\kappa)$ such that for all $\alpha < \kappa$, if $C \cap \alpha \in \mathcal{C}^+_{n-1}(\alpha)$, then $\alpha \in C$. A cardinal $\kappa$ is *$\omega$-stationary* if $\mathcal{C}_n(\kappa)$ is a proper filter for all $n < \omega$.

Note that if $\kappa$ is $\Pi^1_n$-reflecting, then $\mathcal{C}_n(\kappa) = F_n(\kappa)$. Our results perhaps suggest the following conjecture:

**Conjecture 5.14.** *Assume* AD *and* $V = L(\mathbb{R})$. *Then for a stationary set of $\omega$-stationary cardinals $\kappa < \Theta$, the $n$-club filter on $\kappa$ is atomic if $n$ is even and atomless if $n$ is odd.*

The natural place to look is at admissible Suslin cardinals with strong reflection properties.

# References


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[4]Woodin has asked the same question in the context of $L(V_{\lambda+1})$.