Determinacy of games of fixed countable length

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June 25, 2025
Berkeley Inner Model Theory Conference

This is joint work with Juan P. Aguilera (TU Wien).

Research supported by Austrian Science Fund (FWF) international grant number I6087.

Games of fixed countable length

The Gale–Stewart game $G_{\omega\cdot\theta}(A)$ on $\mathbb{N}$ of length $\omega \cdot \theta$ with payoff set A is defined as follows: two players take turns choosing natural numbers.

I	n_0		n_2		$\cdots$	
II		n_1		n_3		$\cdots$

I wins the game if and only if $\langle n_\xi \mid \xi < \omega \cdot \theta \rangle \in A$.

We say that a game (or its payoff set) is determined if one of the players has a winning strategy in the game.

Main Theorem (Aguilera–G. and et al.)

For any $\alpha < \omega_1$, the following are equivalent:

1. *For all $x \in \mathbb{R}$ coding α, $M^\sharp_\alpha(x)$ exists and is ω_1-iterable.*
2. *$G_{\omega\cdot(1+\alpha)}(A)$ are determined for all $<\omega^2$-$\boldsymbol{\Pi}^1_1$ sets $A \subseteq \mathbb{N}^{\omega\cdot(1+\alpha)} \simeq \mathbb{R}^{1+\alpha}$.*

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Difference hierarchy

For a countable ordinal α, $A \subseteq \mathbb{R}$ is α-$\mathbf{\Gamma}$ if there is a sequence $\langle A_\beta \mid \beta < \alpha \rangle$ with each $A_\beta \in \mathbf{\Gamma}$ such that for all $x \in \mathbb{R}$,

$$x \in A \iff \text{the least } \beta \text{ such that } x \notin A_\beta \vee \beta = \alpha \text{ is odd.}$$

We write $<\alpha$-$\mathbf{\Gamma}$ for $\bigcup_{\beta<\alpha} \beta$-$\mathbf{\Gamma}$.

Theorem (Martin)

$\mathbf{\Pi}^1_1$ determinacy implies $<\omega^2$-$\mathbf{\Pi}^1_1$ determinacy (for games on $\mathbb{N}$ of length ω).

A small digression: Recently Aguilera obtained the optimal determinacy transfer theorem.

Theorem (Aguilera)

Let $\mathbf{\Gamma} = \mathrm{LU}(\mathbf{\Sigma}^0_2, <\omega^2\text{-}\mathbf{\Pi}^1_1, \mathbf{\Pi}^1_1)$ and let $\mathbf{\Delta} = \{A \subseteq \mathbb{R} \mid A, \mathbb{R} \setminus A \in \mathbf{\Gamma}\}$. Then $\mathbf{\Pi}^1_1$-determinacy implies $\mathbf{\Delta}$-determinacy, but not $\mathbf{\Gamma}$-determinacy (for games on $\mathbb{N}$ of length ω).

Here, $W \in \mathbf{\Gamma}$ if there are $A_n \in \mathbf{\Sigma}^0_2$ that are pairwise disjoint, $B_n \in <\omega^2$-$\mathbf{\Pi}^1_1$ (that are not necessarily pairwise disjoint), and $C \in \mathbf{\Pi}^1_1$ such that

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Additively decomposable cases

Regarding the additively decomposable cases, the following were already known:

Theorem (Aguilera–Müller, "Consistency strength of long projective games")

The following schemata are equiconsistent:

1. ZFC + $\{$*There are* $\omega^\alpha + n$ *Woodin cardinals* $\mid n < \omega\}$.
2. ZFC + $\{G_{\omega^{1+\alpha}}(A)$ *are determined for all* $\mathbf{\Pi}^1_n$ *sets* $A \subseteq \mathbb{R}^\alpha \mid n < \omega\}$.

Theorem (Aguilera–Müller, "Projective games on the reals")

The following are equivalent:

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Games of length $< \omega^3$

In the proof of the main theorem, what we have actually shown the equivalence of three items.

Theorem

For any $\alpha < \omega^2$, the following are equivalent:

1. *For all $x \in \mathbb{R}$, $M^\sharp_\alpha(x)$ exists and is ω_1-iterable.*
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3. *$M^\sharp_{-\omega+\alpha}(\mathbb{R})$ exists, satisfies AD, and is countably ω_1-iterable.*

The proof is by induction on α.

- (1) $\Rightarrow$ (2) is due to Neeman.
- To show (2) $\Rightarrow$ (3), we use the existence of $M^\sharp_{-\omega+\alpha}(x)$ for all $x \in \mathbb{R}$, so the inductive proof would break down if $\alpha \geq \omega^2$, in which case $-\omega + \alpha = \alpha$.

Before explaining the proof of (2) $\Rightarrow$ (3), I'd like to talk about another related result.

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For any $\alpha < \omega^2$, the following are equivalent:

1. *For all $x \in \mathbb{R}$, $M^\sharp_\alpha(x)$ exists and is ω_1-iterable.*
2. *$G_{\omega\cdot(1+\alpha)}(A)$ are determined for all $A \subseteq \mathbb{R}^{1+\alpha}$ that are $<\omega^2$-$\boldsymbol{\Pi}^1_1$.*
3. *$M^\sharp_{-\omega+\alpha}(\mathbb{R})$ exists, satisfies* AD, *and is countably ω_1-iterable.*

The proof is by induction on α.

- (1) $\Rightarrow$ (2) is due to Neeman.
- To show (2) $\Rightarrow$ (3), we use the existence of $M^\sharp_{-\omega+\alpha}(x)$ for all $x \in \mathbb{R}$, so the inductive proof would break down if $\alpha \geq \omega^2$, in which case $-\omega + \alpha = \alpha$.

Before explaining the proof of (2) $\Rightarrow$ (3), I'd like to talk about another related result.

Characterization of $\Game^{(n+1)}(<\omega^2\text{-}\mathbf{\Pi}^1_1)$

For $A \subseteq \mathbb{R} \times \mathbb{R}$ and $x \in \mathbb{R}$, $A_x := \{y \in \mathbb{R} \mid \langle x, y\rangle \in A\}$ and

$$\Game A = \{x \in \mathbb{R} \mid \text{I wins the game } G_\omega(A_x)\}$$

For any pointclass $\mathbf{\Gamma}$, $\Game\mathbf{\Gamma}$ is the pointclass of all $\Game A$ such that $A \in \mathbf{\Gamma}$.
We also write $\Game^{(n)}$ for $\Game \cdots \Game$ (n times).

Theorem

Let $n \in \omega$ and assume $\mathbf{\Pi}^1_{n+1}$-determinacy (of games on $\mathbb{N}$ of length ω).

$$\Game^{(n+1)}(<\omega^2\text{-}\mathbf{\Pi}^1_1) = \bigcup_{m\in\omega} \Gamma_{n,m},$$

where $A \in \Gamma_{n,m}$ if and only if there is a formula $\phi(v_0, \ldots, v_m)$ such that

$$\forall x \in \mathbb{R}\,(x \in A \iff M_n(x) \models \phi[x, \gamma_0, \ldots, \gamma_{m-1}]),$$

where $\gamma_0, \ldots, \gamma_{m-1}$ are Silver indiscernibles for $M_n(x)$.

Martin showed the theorem for $n = 0$ and Neeman showed the $\subseteq$ direction for $n > 0$.

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Theorem (Martin)

Assume that $x^\sharp$ exists for all $x \in \mathbb{R}$. Then for all $x \in \mathbb{R}$ and all formulas $\phi(u, v_0, \ldots, v_{m-1})$, there is a game $G_{x,\phi}$ on $\mathbb{N}$ of length ω with payoff in $<\omega^2\text{-}\Pi^1_1(x)$ such that

1. *if I has a winning strategy in $G_{x,\phi}$, then*
$$L(x) \models \phi[x, \gamma_0, \ldots, \gamma_{m-1}]$$
2. *if II has a winning strategy in $G_{x,\phi}$, then*
$$L(x) \not\models \phi[x, \gamma_0, \ldots, \gamma_{m-1}]$$

Moreover, the definition of $G_{x,\phi}$ is uniform in x and ϕ.

By $<\omega^2\text{-}\mathbf{\Pi}^1_1$ determinacy, the lemma implies $\bigcup_{m<\omega} \Gamma_{1,m} \subseteq \Game(<\omega^2\text{-}\mathbf{\Pi}^1_1)$.

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We generalize Martin's argument:

Theorem

Let $n < \omega$ and assume that $M_n^\sharp(x)$ exists for all $x \in \mathbb{R}$. Then for all $x \in \mathbb{R}$ and all formulas $\phi(u, v_0, \dots, v_{m-1})$, there is a game $G^n_{x,\phi}$ on $\mathbb{N}$ of length $\omega \cdot (n+1)$ with payoff in $<\omega^2\text{-}\Pi^1_1(x)$ such that

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Long versions of Martin's game

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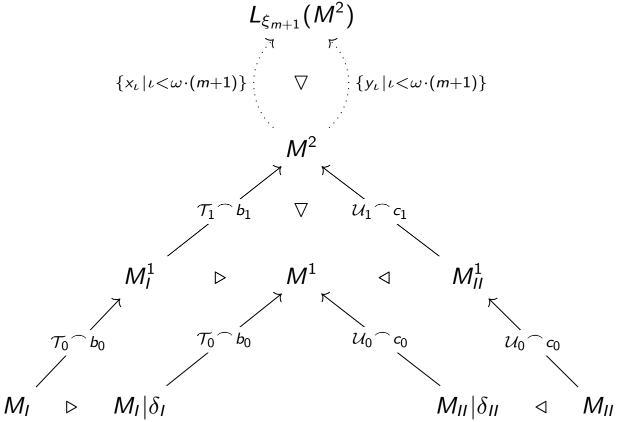
Models played in the game $G^2_{x,\phi}$



Theorem

Let $n < \omega$. Suppose that there is a club $C \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$ such that for all $A \in C$,

- *$M_n^\sharp(A)$ exists and is ω_1-iterable,*
- *$\mathbb{R} \cap M_n(A) = A$.*

Then for all $x \in \mathbb{R}$ and all formulas ϕ, there is a game $G_{x,\phi}^{n,\mathbb{R}}$ on $\mathbb{R}$ of length ω with payoff in $\Game^{(n)}(<\omega^2\text{-}\boldsymbol{\Pi}_1^1)$ such that

1. *if Player I has a winning strategy in $G_{x,\phi}^{n,\mathbb{R}}$, then*

$$\forall^* A \in C\Big(M_n(A) \models \phi[x, A, \gamma_0, \ldots, \gamma_{m-1}]\Big),$$

2. *if Player II has a winning strategy in $G_{x,\phi}^{n,\mathbb{R}}$, then*

$$\forall^* A \in C\Big(M_n(A) \not\models \phi[x, A, \gamma_0, \ldots, \gamma_{m-1}]\Big),$$

where $\gamma_0, \ldots, \gamma_{m-1}$ are Silver indiscernibles of $M_n(A)$. Furthermore, the definition of $G_{x,\phi}^{n,\mathbb{R}}$ is uniform in x and ϕ.

Comments on the proof of (2) ⇒ (3)

Now we combine the previous result with the following theorem:

Theorem (Aguilera–Müller)

Let $\alpha = \omega + n$. Assume that $G_{\omega\cdot(1+\alpha)}(A)$ are determined for all $< \omega^2$-$\mathbf{\Pi}^1_1$ sets $A \subseteq \mathbb{R}^{1+\alpha}$. Then there is a club $C \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$ such that for all $A \in C$,

- *$M_n^\sharp(A)$ exists and is ω_1-iterable,*
- *$\mathbb{R} \cap M_n(A) = A$, and*
- *$M_n(A) \models \mathsf{ZF} + \mathsf{AD}$.*

Aguilera–Müller used weaker determinacy assumption to show this, but the above statement is enough to get the following corollary.

Corollary

Assume that $G_{\omega\cdot(\omega+n)}(A)$ are determined for all $< \omega^2$-$\mathbf{\Pi}^1_1$ sets $A \subseteq \mathbb{R}^{\omega+n}$. Then $M_n^\sharp(\mathbb{R})$ exists, satisfies AD, and is countably ω_1-iterable.

One could generalize this result for games of length $\omega \cdot (m \cdot \omega + n)$.

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More about $M_n^\sharp(\mathbb{R})$

Aguilera–Müller showed the equivalence of projective deteterminacy of games on $\mathbb{R}$ of length ω and the existence of $M_n^\sharp(\mathbb{R})$ for all $n < \omega$. The following theorem is its refinement.

Theorem

For each $n < \omega$, the following are equivalent:

1. *All games on $\mathbb{R}$ of length ω with payoff in $\Game^{(n)}(<\omega^2\text{-}\mathbf{\Pi}^1_1)$ are determined.*
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Also, we extend Martin–Steel's result:

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Suppose that $M_n^\sharp(\mathbb{R})$ exists and is countably ω_1-iterable. Then

$$\wp(\mathbb{R}) \cap M_n(\mathbb{R}) = \Game^{\mathbb{R}}\Game^{(n)}(<\omega^2\text{-}\mathbf{\Pi}^1_1).$$

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Let $\alpha < \omega^2$. Suppose that $M^\sharp_{-\omega+\alpha}(\mathbb{R})$ exists, satisfies AD, and is countably ω_1-iterable. Then for all $x \in \mathbb{R}$, $M^\sharp_\alpha(x)$ exists and is ω_1-iterable.

- The standard argument shows that there are suitable countable premice $\mathcal{P}$ with ω many Woodin cardinals that is iterable in a weak sense.
- One can find $\mathcal{P}$ such that $M_n(\mathcal{P})$ has $\omega + n$ many Woodin cardinals and is ω_1-iterable.

This argument is deeply related to the HOD analysis in $M_n(\mathbb{R})$ (without assuming the existence of $M^\sharp_{\omega+n}$). Indeed, we use some idea in Sargsyan–Müller's work on HOD in $M_n[x, g]$.

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Games of length $< \omega^3$

To deal with determinacy of games of length ω^3, we need to use

$$(L(\mathbb{R}, \mu), \in, \mu) := (L(\mathbb{R})[\mathcal{F}], \in, \mathcal{F} \cap L(\mathbb{R})[\mathcal{F}]),$$

where $\mathcal{F}$ is the club filter on $\wp_{\omega_1}(\mathbb{R})$. This is called the Solovay model (of determiancy).

Theorem (Neeman & Trang–Woodin)

The following are equivalent:

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Based on this result, we can show:

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For any $\omega^2 \leq \alpha < \omega^3$, the following are equivalent:

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Games of length $\geq \omega^3$

To deal with even longer games, we use generalized Solovay models of the form

$$(L(\mathbb{R}, \mu_{<\alpha}), \in, \mu_{<\alpha}) := (L(\mathbb{R})[\mathcal{F}_{<\alpha}], \in, \mathcal{F}_{<\alpha} \cap L(\mathbb{R})[\mathcal{F}_{<\alpha}]),$$

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Using these models, Trang–Woodin obtained the same type of equivalence theorem for games of length $\omega^{1+\alpha}$ for any $\alpha < \omega_1$. Building on their work, we could show the following.

Theorem

For any $\alpha = \omega^\beta + \gamma < \omega_1$, where $\gamma < \omega^\beta$, the following are equivalent:

1. *For all $x \in \mathbb{R}$ coding α, $M^\sharp_\alpha(x)$ exists and is ω_1-iterable.*
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3. *$M^\sharp_\gamma(\mathbb{R}, \mu_{<\beta})$ exists, is countably ω_1-iterable, and satisfies "AD $+ \forall \beta < \alpha\, (\mu_\beta$ is an ulrafilter on $\wp_{\omega_1}(\mathbb{R})^{\omega^\beta})$."*

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For any $\alpha = \omega^\beta + \gamma < \omega_1$, where $\gamma < \omega^\beta$, the following are equivalent:

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2. *$G_{\omega\cdot(1+\alpha)}(A)$ are determined for all $<\omega^2$-$\mathbf{\Pi}^1_1$ sets $A \subseteq \mathbb{N}^{\omega\cdot(1+\alpha)} \simeq \mathbb{R}^{1+\alpha}$.*
3. *$M^\sharp_\gamma(\mathbb{R}, \mu_{<\beta})$ exists, is countably ω_1-iterable, and satisfies "AD $+ \forall \beta < \alpha\, (\mu_\beta$ is an ulrafilter on $\wp_{\omega_1}(\mathbb{R})^{\omega^\beta})$."*

To deal with even longer games, we use generalized Solovay models of the form

$$(L(\mathbb{R}, \mu_{<\alpha}), \in, \mu_{<\alpha}) := (L(\mathbb{R})[\mathcal{F}_{<\alpha}], \in, \mathcal{F}_{<\alpha} \cap L(\mathbb{R})[\mathcal{F}_{<\alpha}]),$$

where $\mathcal{F}_{<\alpha} = \langle \mathcal{F}_\beta \mid \beta < \alpha \rangle$ is a sequence of the club filters $\mathcal{F}_\beta$ on $\wp_{\omega_1}(\mathbb{R})^{\omega^\beta}$. (Here, the club filter $\mathcal{F}_\beta$ is defined by a certain game on $\mathbb{R}$ of length $\omega^{1+\beta}$.)

Using these models, Trang–Woodin obtained the same type of equivalence theorem for games of length $\omega^{1+\alpha}$ for any $\alpha < \omega_1$. Building on their work, we could show the following.

Theorem

For any $\alpha = \omega^\beta + \gamma < \omega_1$, where $\gamma < \omega^\beta$, the following are equivalent:

1. *For all $x \in \mathbb{R}$ coding α, $M^\sharp_\alpha(x)$ exists and is ω_1-iterable.*
2. *$G_{\omega \cdot (1+\alpha)}(A)$ are determined for all $<\omega^2$-$\mathbf{\Pi}^1_1$ sets $A \subseteq \mathbb{N}^{\omega \cdot (1+\alpha)} \simeq \mathbb{R}^{1+\alpha}$.*
3. *$M^\sharp_\gamma(\mathbb{R}, \mu_{<\beta})$ exists, is countably ω_1-iterable, and satisfies "AD $+ \forall \beta < \alpha \, (\mu_\beta$ is an ulrafilter on $\wp_{\omega_1}(\mathbb{R})^{\omega^\beta}$)."*

Remaining question

Conjecture

For all $1 \leq \alpha < \omega_1$, the following are equivalent:

1. *$G_{\omega \cdot \alpha}(A)$ are determined for all $\boldsymbol{\Pi}^1_1$ sets $A \subseteq \mathbb{R}^\alpha$.*
2. *$G_{\omega \cdot \alpha}(A)$ are determined for all $<\omega^2$-$\boldsymbol{\Pi}^1_1$ sets $A \subseteq \mathbb{R}^\alpha$.*

The conjecture is true for $\alpha < \omega$ and additively indecomposable ordinals α.

Thank you for your attention!

Remaining question

Conjecture

For all $1 \leq \alpha < \omega_1$, *the following are equivalent:*

1. $G_{\omega \cdot \alpha}(A)$ *are determined for all* $\mathbf{\Pi}^1_1$ *sets* $A \subseteq \mathbb{R}^\alpha$.
2. $G_{\omega \cdot \alpha}(A)$ *are determined for all* $<\omega^2$-$\mathbf{\Pi}^1_1$ *sets* $A \subseteq \mathbb{R}^\alpha$.

The conjecture is true for $\alpha < \omega$ and additively indecomposable ordinals α.

Thank you for your attention!