Basis for Uncountable Linear Orders

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Berkeley Inner Model Theory

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The Regular Choiceless Universe

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AC has the side effect of producing phenomenon that may be regarded as "exotic" and "non-regular". However when sets are definable or possess nice "regularity properties", there is often an intrinsic reason why some instance of choice hold.

Descriptive set theorist have come up with many regularity properties on subsets of $\mathbb{R}$ and sets which are images of $\mathbb{R}$.

Let $A \subseteq {}^\omega\omega$. The game G_A is a two-player game where each player takes turns picking an element of ω.

$$
G_A \quad \begin{array}{lccccccccc} \mathrm{I} & a_0 & & a_2 & & a_4 & & \cdots & \\ & & & & & & & & \vec{a} \\ \mathrm{II} & & a_1 & & a_3 & & a_5 & \cdots & \end{array}
$$

Player 1 wins if and only of $\vec{a} \in A$.

Definition
The axiom of determinacy, AD, is the assertion that for all $A \subseteq {}^\omega\omega$, one of the two players has a winning strategy in G_A.

AD^+ is Woodin's extension of the determinacy.

The Regular Choiceless Universe

Determinacy axioms are approximations of a regular choiceless universe just up to the sets which are surjective images of $\mathbb{R}$ which may be regarded as the descriptive set theoretic world. Θ, the supremum of the ordinals onto which $\mathbb{R}$ surjects, is the natural height of this descriptive set theoretic world.

Determinacy empirically seems very good at deciding basic combinatorial questions for sets which are images of $\mathbb{R}$. The direct method of forcing cannot be used to produce independence results over AD.

Fact (Chan-Jackson; Ikegami-Trang)
If $\mathbb{P}$ is a nontrivial forcing which is a image of $\mathbb{R}$, then $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \neg\mathrm{AD}$.

The Regular Choiceless Universe

From basic set construction principle and the definition of cardinal exponentiation, one can conclude that for all cardinals $\alpha, \beta, \gamma, \delta$, if $\alpha \leq \gamma$ and $\beta \leq \delta$, then $|{}^{\alpha}\beta| \leq |{}^{\gamma}\delta|$.

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Definition (ABCD Hypothesis)
For all cardinals $\omega \leq \alpha \leq \beta$ and $\omega \leq \gamma \leq \delta$, then $|^{\alpha}\beta| \leq |^{\gamma}\delta|$ if and only if $\alpha \leq \gamma$ and $\beta \leq \delta$.

The ABCD hypothesis is the aesthetically minimal behavior for infinite cardinal exponentiation which states this is all that can be inferred. It completely classifies the cardinality relation between any pair of infinite cardinal exponentiations.

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Theorem (Chan; The ABCD Conjecture)
Assume AD^+*. Let* $\omega \leq \alpha \leq \beta < \Theta$ *and* $\omega \leq \gamma \leq \delta < \Theta$ *be cardinals.* $|{}^{\alpha}\beta| \leq |{}^{\gamma}\delta|$ *if and only if* $\alpha \leq \gamma$ *and* $\beta \leq \delta$*.*

Linear Orderings

Definition
If $(X, \prec)$ and $(Y, \sqsubset)$ are two linear orderings, then a function $\Phi : (X, \prec) \to (Y, \sqsubset)$ is an order embedding if and only if for all $x_0, x_1 \in X$, $x_0 \prec x_1$ implies $\Phi(x_0) \sqsubset \Phi(x_1)$.

Let $\mathscr{B}$ and $\mathscr{C}$ be two classes of linear orderings. $\mathscr{B}$ is a basis for $\mathscr{C}$ if and only if $\mathscr{B} \subseteq \mathscr{C}$ and for all $\mathcal{L} \in \mathscr{C}$, there is a $\mathcal{J} \in \mathscr{B}$ so that $\mathcal{J}$ order embeds into $\mathcal{L}$.

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The following will be the main results of the talk.

Theorem (Chan)
Assume AD^+.

- *There is a four element basis for the linear orderings of cardinality greater than or equal to* $|\mathbb{R} \times \kappa|$ *when* κ *is a regular cardinal below* Θ.
- *There is a twelve element basis for the linear ordering of cardinality greater than or equal to* $|\mathbb{R} \times \kappa|$ *when* κ *is a singular cardinal of uncountable cofinality below* Θ.

Structure Under Determinacy

Fact

Assume $\mathsf{AC}^{\mathbb{R}}_{\omega}$ *and subsets of* $\mathbb{R}$ *have the Baire property, then wellordered unions of meager sets are meager. Thus* $\mathbb{R}$ *is not wellorderable.*

Fact

Assume all subsets of $\mathbb{R}$ *have the perfect set property. Then there is no injection of* ω_1 *into* $\mathbb{R}$ *(which is sometimes called the boldface* GCH *at* ω*).*

Structure under Determinacy

Fact (Woodin's Perfect Set Dichotomy)
Assume AD^+. *Let* X *be an image of* $\mathbb{R}$. *Exactly one of the following occurs.*

1. X *is wellorderable.*
2. $|\mathbb{R}| \leq |X|$ *(and so* X *is not wellorderable).*

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E_0 is the equivalence relation ${}^\omega 2$ defined by $x \; E_0 \; y$ if and only if there exists an $m \in \omega$ so that for all $m \leq n < \omega$, $x(n) = y(n)$.

Fact (Hjorth E_0-dichotomy)
Assume AD^+*. Let* X *be an image of* $\mathbb{R}$*. Exactly one of the following occurs.*

1. X *injects into* $\mathscr{P}(\kappa)$ *for some ordinal* κ *(and hence* X *is linearly orderable).*
2. $|\mathbb{R}/E_0| \leq |X|$ *(and hence* X *is not linearly orderable).*

Basis above $\mathbb{R}$

Definition
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Fact (Partition relation for $\mathbb{R}$)
Assume $\mathsf{AC}^{\mathbb{R}}_{\omega}$ and all subsets of $\mathbb{R}$ have the Baire property. Let $\Phi : [\mathbb{R}]^2 \to 2$. Then there is a perfect tree p on 2 and an $i \in 2$ so that for all $\{r, s\} \in [[p]]^2$, $P(\{r, s\}) = i$.

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Theorem
Assume $\mathsf{AC}^{\mathbb{R}}_\omega$ *and all subsets of* $\mathbb{R}$ *have the Baire property. Let* $(X, \prec)$ *be a linear ordering so that* $|\mathbb{R}| \leq |X|$. *Then there is an order embedding of* $(\mathbb{R}, <_{\mathrm{lex}})$ *into* $(X, \prec)$.

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Theorem
Assume $\mathsf{AC}^{\mathbb{R}}_{\omega}$ and all subsets of $\mathbb{R}$ have the Baire property. Let $(X, \prec)$ be a linear ordering so that $|\mathbb{R}| \leq |X|$. Then there is an order embedding of $(\mathbb{R}, <_{\mathrm{lex}})$ into $(X, \prec)$.

Proof.
Let $\Phi : \mathbb{R} \to X$ be an injection. Define $P : [\mathbb{R}]^2 \to 2$ by $P(\{r, s\}) = 0$ if and only if $\Phi(r) \prec \Phi(s)$ (where the convention is that writing $\{r, s\}$ means $r <_{\mathrm{lex}} s$). There is a perfect tree p so that $[p]$ which is homogeneous for P. Either homogeneous value gives an ordering embedding of $(\mathbb{R}, <_{\mathrm{lex}})$ into $(X, \prec)$. □

Partition relations

Definition (Ordinary Partition Relation)
Let κ be a cardinal, $\epsilon \leq \kappa$, and $\gamma < \kappa$. The ordinary partition relation $\kappa \to (\kappa)^{\epsilon}_{\gamma}$ asserts that for all $P : [\kappa]^{\epsilon} \to \gamma$, there is an $A \subseteq \kappa$ with $|A| = \kappa$ and $\beta < \gamma$ so that for all $f \in [A]^{\epsilon}$, $P(f) = \beta$.

Partition relations

Definition
A function $f : \epsilon \to \mathrm{ON}$ has the correct type if and only if

1. f is discontinuous everywhere: For all $\alpha < \epsilon$, $\sup(f \restriction \alpha) < f(\alpha)$.
2. f has uniform cofinality ω: There is a function $F : \epsilon \times \omega \to \mathrm{ON}$ so that for all $\alpha < \epsilon$, $F(\alpha, n) < F(\alpha, n+1)$ and $f(\alpha) = \sup\{F(\alpha, n) : n \in \omega\}$.

If $A \subseteq \kappa$, $[A]^{\epsilon}_{*}$ denote all the function $f : \epsilon \to A$ of the correct type.

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If $A \subseteq \kappa$, $[A]^{\epsilon}_{*}$ denote all the function $f : \epsilon \to A$ of the correct type.

Definition (Correct Type Partition Relation)
Let κ be an uncountable cardinal, $\epsilon \leq \kappa$, and $\gamma < \kappa$. The correct type partition relation $\kappa \to_{*} (\kappa)^{\epsilon}_{\gamma}$ asserts that for all $P : [\kappa]^{\epsilon}_{*} \to \gamma$, there is a (unique) $\beta < \gamma$ and club $C \subseteq \kappa$ so that for all $f \in [C]^{\epsilon}_{*}$, $P(f) = \beta$.

Definition
For $\epsilon \leq \kappa$, let μ_κ^ϵ be the filter on $[\kappa]_*^\epsilon$ defined by $A \in \mu_\kappa^\epsilon$ if and only if there is a club $C \subseteq \kappa$ so that $[C]_*^\epsilon \subseteq A$.

$\kappa \to_* (\kappa)_\gamma^\epsilon$ implies that μ_κ^ϵ is a γ^+-complete ultrafilter.

$\kappa \to_* (\kappa)_2^2$ implies the ω-club filter μ_κ^1 is normal ultrafilter and ${}^\omega\kappa$ is not wellorderable.

Basis above Regular Cardinals

Definition
Let κ be an ordinal and $<$ be its usual ordering. Let $\kappa = (\kappa, <)$. Let $<^*$ be the reverse of the usual ordering on κ. Let $\kappa^* = (\kappa, <^*)$.

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Fact
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Proof.
Let $\Phi : \kappa \to X$ be an injection. Define $P : [\kappa]^2 \to 2$ by $P(\alpha, \beta) = 0$ if and only if $\Phi(\alpha) \prec \Phi(\beta)$ (where the convention is that $\alpha < \beta$). There is an $A \subseteq \kappa$ with $|A| = |\kappa|$ which is homogeneous for P. Homogeneous taking value 0 induces an order embedding of κ into $(X, \prec)$ and homogeneous taking value 1 induces an order embedding of κ^* into $(X, \prec)$. $\square$

Basis above Regular Cardinals

Fact (Steel, Woodin)
Assume AD^+. *If* κ *is regular, then* κ *is measurable. In fact,* μ^1_κ *is a normal* κ*-complete ultrafilter which implies* $\kappa \to_* (\kappa)^2_2$.

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Theorem
Assume AD^+*. Let* $\kappa < \Theta$ *be regular. There is a two-element basis for linear orderings whose cardinality is greater than or equal to* $|\kappa|$*.*

Cantor showed $\mathbb{R}$ is the only separable complete linear ordering without endpoints. Suslin asked whether $\mathbb{R}$ is the only complete linear ordering without endpoints with the countable chain condition. A counterexample to the Suslin problem is called a Suslin line.

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Assume AD^+*. There are no Suslin line on a set which is an image of* $\mathbb{R}$*.*

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Theorem (Chan-Jackson)
Assume AD^+*. There are no Suslin line on a set which is an image of* $\mathbb{R}$*.*

Definition
An Aronszajn line is a linear ordering such that $\mathbb{R}$, ω_1 and ${\omega_1}^*$ does not order embed into it.

Basis for Uncountable Linear Orderings

Fact (Moore)
Assume ZFC *and* PFA. *There is a five-element basis for the uncountable linear orderings.*

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Theorem (Weinert)
Assume AD^+. $\mathbb{R}$, ω_1 *and* ${\omega_1}^*$ *order embeds into any uncountable linear ordering which is an image of* $\mathbb{R}$. *Hence there is a three element basis for the uncountable linear ordering which is an image of* $\mathbb{R}$ *and there are no Aronszajn lines which are images of* $\mathbb{R}$.

Proof.
By Woodin's perfect set dichotomy, every uncountable set X has an injective copy of ω_1 or $\mathbb{R}$. The result follows from the previous two basis results. □

Basis above Cartesian Product $\mathbb{R}$ and a Regular Cardinal

Definition
Let $\mathcal{X} = (X, \prec)$ and $\mathcal{Y} = (Y, \sqsubset)$. Let $(\mathcal{X} \otimes \mathcal{Y}, \ll)$ be the linear ordering on $X \times Y$ defined by $(a, b) \ll (x, y)$ if and only if the disjunction of the following holds:

1. $a \prec x$.
2. $a = x$ and $b \sqsubset y$.

Theorem
Let κ be an uncountable cardinal. Any two distinct linear orderings from $\{\mathbb{R} \otimes \kappa, \kappa \otimes \mathbb{R}, \mathbb{R} \otimes \kappa^*, \kappa^* \otimes \mathbb{R}\}$ do not order embed into each other.

Basis above Cartesian Product $\mathbb{R}$ and a Regular Cardinal

Theorem
Assume $\mathsf{AC}^{\mathbb{R}}_{\omega}$, *all subsets of* $\mathbb{R}$ *have the Baire property, and boldface* GCH *at* ω *holds. Let* κ *satisfy* $\kappa \rightarrow_{*} (\kappa)^2_2$. *Let* $(X, \prec)$ *be a linear ordering such that* $|\mathbb{R} \times \kappa| \leq |X|$. *Then at least one linear ordering from* $\{\mathbb{R} \otimes \kappa, \kappa \otimes \mathbb{R}, \mathbb{R} \otimes \kappa^*, \kappa^* \otimes \mathbb{R}\}$ *order embeds into* $(X, \prec)$.

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Suppose $(\mathbb{R} \times \kappa, \prec)$ is a linear ordering.

Fix $r \in \mathbb{R}$. Let $Q_r : [\kappa]^2_* \to 2$ by $Q_r(\alpha, \beta) = 0$ if and only if $(r, \alpha) \prec (r, \beta)$. By $\kappa \to_* (\kappa)^2_2$, there is a club $C \subseteq \kappa$ so that Q_r takes constant value $j_r \in 2$ on $[C]^2_*$. In other words, there is an $A \in \mu^1_\kappa$ which is homogeneous for Q_r.

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Fix $r <_{\mathrm{lex}} s$. Let $P_{r,s} : [\kappa]^1_* \rightarrow 2$ by defined by $P_{r,s}(\alpha)$ if and only if $(r, \alpha) \prec (s, \alpha)$. By $\kappa \rightarrow_* (\kappa)^1_2$, there is an $i_{r,s} \in 2$ so that $P_{r,s}$ takes constant value $i_{r,s}$ almost everywhere with respect to μ^1_κ.

Basis above Cartesian Product $\mathbb{R}$ and a Regular Cardinal

Now to stabilize the relation between the real and ordinal coordinate.

- Define $T^{0,0}_{r,s} : [\kappa]^2_* \to 2$ by $T^{0,0}_{r,s}(\alpha, \beta) = 0$ if and only if $(r, \beta) \prec (s, \alpha)$. Let $u^{0,0}_{r,s}$ be its homogeneous value.
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Using the partition relation on $\mathbb{R}$, one can find a perfect tree p_0, $\bar{i}, \bar{j}, u^{\bar{i},\bar{j}} \in 2$ so that for all $r \in [p_0]$ and $\{r, s\} \in [[p_0]]^2$, $i_{r,s} = \bar{i}$, $j_r = \bar{j}$, and $u^{\bar{i},\bar{j}}_{r,s} = u^{\bar{i},\bar{j}}$.

Basis above Cartesian Product $\mathbb{R}$ and a Regular Cardinal

We need to pick μ^1_κ-large homogeneous set for $P_{r,s}$, Q_r and $T^{\bar{i},\bar{j}}_{r,s}$ for all $r, s \in [p_1]$ in order to build the desired order embedding.

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Fact (Chan-Jackson-Trang)
Assume all subsets of reals have the Baire property and boldface GCH *at* ω *holds. Let* $\Phi : \mathbb{R} \to \mathscr{P}(\mathrm{ON})$*. Then there is a countable* $\mathcal{E} \subseteq \mathscr{P}(\mathrm{ON})$ *consisting of pairwise disjoint sets and a comeager* $K \subseteq \mathbb{R}$ *so that for all* $x \in K$*, there is an* $\mathcal{F} \subseteq \mathcal{E}$ *so that* $\Phi(x) = \bigcup \mathcal{F}$*.*

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Fact
Assume all subsets of $\mathbb{R}$ have the Baire property and boldface GCH *at ω holds. Let $\Phi : \mathbb{R} \times \mathbb{R} \to \mathscr{P}(\mathrm{ON})$ be E_0-invariant. Then there is a comeager $K \subseteq \mathbb{R} \times \mathbb{R}$ so that Φ is constant on K.*

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Fact
Assume all subsets of $\mathbb{R}$ *have the Baire property and boldface* GCH *at* ω *holds. Let* μ *be a countably complete filter on* κ*. Let* $\Phi : \mathbb{R} \to \mathscr{P}(\kappa)$ *be such that* $\Phi(x) \in \mu$ *for all* $x \in \mathbb{R}$*. Then uniformly from* Φ*, there is a* $Z \in \mu$ *and a comeager* $K \subseteq \mathbb{R}$ *so that for all* $x \in K$*,* $Z \subseteq \Phi(x)$*.*

Proof.
Let $\Psi(x) = \bigcap\{\Phi(y) : y \in [x]_{E_0}\} \in \mu$. $\Psi(x)$ is E_0-invariant. $\square$

Fact

Assume all subsets of $\mathbb{R}$ have the Baire property and boldface GCH at ω holds. Let μ be a countably complete ultrafilter on κ. Let $\Phi : \mathbb{R} \to \mathscr{P}(\kappa)$ so that $\Phi(x) \in \mu$ for all $x \in \mathbb{R}$. Then uniformly from Φ, there is a $Z \in \mu$ and a comeager $K \subseteq \mathbb{R}$ so that for all $x \in K$, $Z \subseteq \Phi(x)$.

Let $\Phi_0 : [[p_0]]^2 \to \mu^1_\kappa$ by $\Phi_0(r, s) = P^{-1}_{r,s}[\{\bar{i}\}]$. Let $\Phi_1 : [p_0] \to \mu^2_\kappa$ be defined by $\Phi_1(r) = Q^{-1}_r[\{\bar{j}\}]$. Let $\Phi_2 : [[p_0]]^2 \to \mu^2_\kappa$ be $\Phi_2(r, s) = (T^{\bar{i},\bar{j}}_{r,s})^{-1}[\{u^{\bar{i},\bar{j}}\}]$. By applying the fact and then using the Mycielski theorem, one gets a perfect tree $p_1 \subseteq p_0$ and $D \in \mu^1_\kappa$ which is homogeneous for all relevant partitions simultaneously.

Basis above Cartesian Product $\mathbb{R}$ and a Regular Cardinal

- If $\bar{i} = 0$, $\bar{j} = 0$, and $\bar{u}^{\bar{i},\bar{j}} = 0$, then $[p_1] \times D$ is order isomorphic to $\mathbb{R} \otimes \kappa$.
- If $\bar{i} = 0$, $\bar{j} = 0$, and $\bar{u}^{\bar{i},\bar{j}} = 1$, then $[p_1] \times D$ is order isomorphic to $\kappa \otimes \mathbb{R}$.
- If $\bar{i} = 0$, $\bar{j} = 1$, and $\bar{u}^{\bar{i},\bar{j}} = 0$, then $[p_1] \times D$ is order isomorphic to $\mathbb{R} \otimes \kappa^*$.
- If $\bar{i} = 0$, $\bar{j} = 1$, and $\bar{u}^{\bar{i},\bar{j}} = 1$, then $[p_1] \times D$ is order isomorphic to $\kappa^* \otimes \mathbb{R}$.
- If $\bar{i} = 1$, $\bar{j} = 0$, and $\bar{u}^{\bar{i},\bar{j}} = 0$, then $[p_1] \times D$ is order isomorphic to $\mathbb{R} \otimes \kappa$
- If $\bar{i} = 1$, $\bar{j} = 0$, and $\bar{u}^{\bar{i},\bar{j}} = 1$, then $[p_1] \times D$ is order isomorphic to $\kappa \otimes \mathbb{R}$.
- If $\bar{i} = 1$, $\bar{j} = 1$, and $\bar{u}^{\bar{i},\bar{j}} = 0$, then $[p_1] \times D$ is order isomorphic to $\mathbb{R} \otimes \kappa^*$.
- If $\bar{i} = 1$, $\bar{j} = 1$, and $\bar{u}^{\bar{i},\bar{j}} = 1$, then $[p_1] \times D$ is order isomorphic to $\kappa^* \times \mathbb{R}$.

This shows that linear ordering above $|\mathbb{R} \times \kappa|$ has a four element basis when $\kappa \rightarrow_* (\kappa)^2_2$.

Basis above Cartesian Product $\mathbb{R}$ and a Regular Cardinal

- If $\bar{i} = 0$, $\bar{j} = 0$, and $\bar{u}^{\bar{i},\bar{j}} = 0$, then $[p_1] \times D$ is order isomorphic to $\mathbb{R} \otimes \kappa$.
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- If $\bar{i} = 0$, $\bar{j} = 1$, and $\bar{u}^{\bar{i},\bar{j}} = 0$, then $[p_1] \times D$ is order isomorphic to $\mathbb{R} \otimes \kappa^*$.
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This shows that linear ordering above $|\mathbb{R} \times \kappa|$ has a four element basis when $\kappa \rightarrow_* (\kappa)^2_2$.

By the HOD analysis, regular cardinals satisfy the exponent two correct type partition.

Theorem
Assume AD^+. If $\kappa < \Theta$ is a regular cardinal, then there is a four element basis for the linear orderings whose cardinality is greater than or equal to $|\mathbb{R} \times \kappa|$.

Definition
Let κ be a singular cardinal, $\delta = \mathrm{cof}(\kappa)$, and $\rho : \delta \to \kappa$ be an increasing cofinal function. Let $L^{\kappa}_{\rho} = \{(\alpha, \beta) : \alpha < \delta \wedge \beta < \rho(\alpha)\}$ (which has cardinality κ). Let $<^{+}$ denote usual ordinal ordering and $<^{-}$ denote the reverse ordinal ordering. Let $\iota, \ell \in \{+, -\}$. Define $\mathcal{L}^{\kappa,\rho}_{\iota\ell} = (L^{\kappa}_{\rho}, \prec^{\kappa,\rho}_{\iota\ell})$ be defined by $(\alpha_0, \beta_0) \prec^{\kappa,\rho}_{\iota\ell} (\alpha_1, \beta_1)$ if and only if the disjunction of the following holds:

- $\alpha_0 <^{\iota} \alpha_1$.
- $\alpha_0 = \alpha_1$ and $\beta_0 <^{\ell} \beta_1$.

Note that $\mathcal{L}^{\kappa,\rho}_{++}$ is order isomorphic to usual κ and $\mathcal{L}^{\kappa,\rho}_{--}$ is order isomorphic to κ^*.

Definition
Let κ be a singular cardinal, $\delta = \mathrm{cof}(\kappa)$, and $\rho : \delta \to \kappa$ be an increasing cofinal function. Let $L^\kappa_\rho = \{(\alpha, \beta) : \alpha < \delta \wedge \beta < \rho(\alpha)\}$ (which has cardinality κ). Let $<^+$ denote usual ordinal ordering and $<^-$ denote the reverse ordinal ordering. Let $\iota, \ell \in \{+, -\}$. Define $\mathcal{L}^{\kappa,\rho}_{\iota\ell} = (L^\kappa_\rho, \prec^{\kappa,\rho}_{\iota\ell})$ be defined by $(\alpha_0, \beta_0) \prec^{\kappa,\rho}_{\iota\ell} (\alpha_1, \beta_1)$ if and only if the disjunction of the following holds:

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- $\alpha_0 = \alpha_1$ and $\beta_0 <^\ell \beta_1$.

Note that $\mathcal{L}^{\kappa,\rho}_{++}$ is order isomorphic to usual κ and $\mathcal{L}^{\kappa,\rho}_{--}$ is order isomorphic to κ^*.

Theorem
Any two distinct linear orderings from the following four linear orderings $\{\mathcal{L}^{\kappa,\rho}_{++}, \mathcal{L}^{\kappa,\rho}_{--}, \mathcal{L}^{\kappa,\rho}_{+-}, \mathcal{L}^{\kappa,\rho}_{-+}\}$ do not order embed into each other.

Basis above Singular Cardinals

Theorem
Let κ be a singular cardinal which is a limit of exponent 2 correct-type partition cardinals. Then $\{\mathcal{L}^{\kappa,\rho}_{++}, \mathcal{L}^{\kappa,\rho}_{--}, \mathcal{L}^{\kappa,\rho}_{+-}, \mathcal{L}^{\kappa,\rho}_{-+}\}$ forms a four element basis for the linear orderings whose cardinality is greater than or equal to κ.

Theorem
Assume AD^+. Let $\kappa < \Theta$ be a singular cardinal which is a limit of regular cardinals. Then $\{\mathcal{L}^{\kappa,\rho}_{++}, \mathcal{L}^{\kappa,\rho}_{--}, \mathcal{L}^{\kappa,\rho}_{+-}, \mathcal{L}^{\kappa,\rho}_{-+}\}$ forms a four element basis for the linear orderings whose cardinality is greater than or equal to κ.

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Let $\delta = \mathrm{cof}(\kappa)$ and $\rho : \delta \to \kappa$ be a cofinal sequence of exponent 2 correct-type partition cardinals. One needs to be able to choose homogeneous subsets of $\rho(\alpha)$ for various partitions for all $\alpha < \delta$. If $\lambda \to_* (\lambda)^2_2$ holds, then μ^1_λ is normal. For any $A \in \mu^2_\lambda$, one can uniformly obtain a $\mathfrak{C}_{\mu^1_\lambda}(A) \in \mu^1_\lambda$ so that $[\mathfrak{C}_{\mu^1_\lambda}(A)]^2_* \subseteq A$ by using a diagonal intersection (as in the proof of Rowbottom lemma).

Basis above Singular Cardinals

Under AD, ω_1 and ω_2 are regular and even partition cardinals. For $3 \leq n < \omega$, ω_n is a singular cardinal of cofinality ω_2. Thus ω_ω is singular cardinal which is not a limit of regular cardinals. ω_3 is singular cardinal which is not even a limit of cardinals.

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Fact (Jackson-Ketchersid-Schlutzenberg-Woodin)
Assume AD *and* $V = L(\mathbb{R})$. *If* $\omega_1 < \kappa < \Theta$, *then for any* $x \in \mathbb{R}$, $\mathrm{HOD}_{\{x\}}$ *has* κ*-many measurable cardinals below* κ.

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Fact (Jackson-Ketchersid-Schlutzenberg-Woodin)
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In $V = L(\mathbb{R})$ every set is $\mathrm{OD}_{\{z\}}$ for some $z \in \mathbb{R}$. A linear ordering on κ then belongs to $\mathrm{HOD}_{\{z\}}$ for some $z \in \mathbb{R}$. Use the measurable cardinals (or its weak compactness) to prove the existence of homogeneous sets and select them uniformly.

Theorem
Assume AD *and* $V = L(\mathbb{R})$. *(Also under* AD^+*) Let* $\kappa < \Theta$ *be a singular cardinal. Then* $\{\mathcal{L}^{\kappa,\rho}_{++}, \mathcal{L}^{\kappa,\rho}_{--}, \mathcal{L}^{\kappa,\rho}_{+-}, \mathcal{L}^{\kappa,\rho}_{-+}\}$ *forms a four element basis for the linear ordering whose cardinality are greater than or equal to* κ.

Definition
Let κ be a singular cardinal, $\delta = \mathrm{cof}(\kappa)$, and $\rho : \delta \to \kappa$ be a cofinal map.

Define $\sqsubset^{\kappa,\rho}_{\mathbb{R}\iota\ell}$ on $\mathbb{R} \times L^{\kappa}_{\rho}$ by $(r_0, \alpha_0, \beta_0) \sqsubset^{\kappa,\rho}_{\mathbb{R}\iota\ell} (r_1, \alpha_1, \beta_1)$ if and only if the disjunction of the following holds.

- $r_0 <_{\mathrm{lex}} r_1$.
- $r_0 = r_1$ and $\alpha_0 <^{\iota} \alpha_1$.
- $r_0 = r_1$, $\alpha_0 = \alpha_1$, and $\beta_0 <^{\ell} \beta_1$.

Let $\mathcal{H}^{\kappa,\rho}_{\mathbb{R}\iota\ell} = (\mathbb{R} \times L^{\kappa}_{\rho}, \sqsubset^{\kappa,\rho}_{\mathbb{R}\iota\ell})$. Note that $\mathcal{H}^{\kappa,\rho}_{\mathbb{R}\iota\ell}$ is order isomorphic to $\mathbb{R} \otimes \mathcal{L}^{\kappa,\rho}_{\iota\ell}$.

Basis above Cartesian Product $\mathbb{R}$ and a Singular Cardinal

Definition
Let κ be a singular cardinal, $\delta = \mathrm{cof}(\kappa)$, and $\rho : \delta \to \kappa$ be a cofinal map.

Define $\sqsubset_{\mathbb{R}\iota\ell}^{\kappa,\rho}$ on $\mathbb{R} \times L_\rho^\kappa$ by $(r_0, \alpha_0, \beta_0) \sqsubset_{\mathbb{R}\iota\ell}^{\kappa,\rho} (r_1, \alpha_1, \beta_1)$ if and only if the disjunction of the following holds.

- $r_0 <_{\mathrm{lex}} r_1$.
- $r_0 = r_1$ and $\alpha_0 <^\iota \alpha_1$.
- $r_0 = r_1$, $\alpha_0 = \alpha_1$, and $\beta_0 <^\ell \beta_1$.

Let $\mathcal{H}_{\mathbb{R}\iota\ell}^{\kappa,\rho} = (\mathbb{R} \times L_\rho^\kappa, \sqsubset_{\mathbb{R}\iota\ell}^{\kappa,\rho})$. Note that $\mathcal{H}_{\mathbb{R}\iota\ell}^{\kappa,\rho}$ is order isomorphic to $\mathbb{R} \otimes \mathcal{L}_{\iota\ell}^{\kappa,\rho}$.

Define $\sqsubset_{\iota\ell\mathbb{R}}^{\kappa,\rho}$ on $\mathbb{R} \times L_\rho^\kappa$ by $(r_0, \alpha_0, \beta_0) \sqsubset_{\iota\ell\mathbb{R}}^{\kappa,\rho} (r_1, \alpha_1, \beta_1)$ if and only if the disjunction of the following holds.

- $\alpha_0 <^\iota \alpha_1$.
- $\alpha_0 = \alpha_1$ and $\beta_0 <^\ell \beta_1$.
- $\alpha_0 = \alpha_1$, $\beta_0 = \beta_1$, and $r_0 <_{\mathrm{lex}} r_1$.

Let $\mathcal{H}_{\iota\ell\mathbb{R}}^{\kappa,\rho} = (\mathbb{R} \times L_\rho^\kappa, \sqsubset_{\iota\ell\mathbb{R}}^{\kappa,\rho})$. Note that $\mathcal{H}_{\iota\ell\mathbb{R}}^{\kappa,\rho}$ is order isomorphic to $\mathcal{L}_{\iota\ell}^{\kappa,\rho} \otimes \mathbb{R}$.

Definition
Define $\sqsubset^{\kappa,\rho}_{\iota\mathbb{R}\ell}$ on $\mathbb{R} \times L^{\kappa}_{\rho}$ by $(r_0, \alpha_0, \beta_0) \sqsubset^{\kappa,\rho}_{\iota\mathbb{R}\ell} (r_1, \alpha_1, \beta_1)$ if and only if the disjunction of the following holds.

- $\alpha_0 <^{\iota} \alpha_1$.
- $\alpha_0 = \alpha_1$ and $r_0 <_{\mathrm{lex}} r_1$.
- $\alpha_0 = \alpha_1$, $r_0 = r_1$, and $\beta_0 <^{\ell} \beta_1$.

Let $\mathcal{H}^{\kappa,\rho}_{\iota\mathbb{R}\ell} = (\mathbb{R} \times L^{\kappa}_{\rho}, \sqsubset^{\kappa,\rho}_{\iota\mathbb{R}\ell})$.

Basis above Cartesian Product $\mathbb{R}$ and a Singular Cardinal

Definition
Define $\sqsubset^{\kappa,\rho}_{\iota\mathbb{R}\ell}$ on $\mathbb{R} \times L^{\kappa}_{\rho}$ by $(r_0, \alpha_0, \beta_0) \sqsubset^{\kappa,\rho}_{\iota\mathbb{R}\ell} (r_1, \alpha_1, \beta_1)$ if and only if the disjunction of the following holds.

- $\alpha_0 <^{\iota} \alpha_1$.
- $\alpha_0 = \alpha_1$ and $r_0 <_{\mathrm{lex}} r_1$.
- $\alpha_0 = \alpha_1$, $r_0 = r_1$, and $\beta_0 <^{\ell} \beta_1$.

Let $\mathcal{H}^{\kappa,\rho}_{\iota\mathbb{R}\ell} = (\mathbb{R} \times L^{\kappa}_{\rho}, \sqsubset^{\kappa,\rho}_{\iota\mathbb{R}\ell})$.

Theorem
Let κ be a singular cardinal of uncountable cofinality. Any two distinct linear orderings from the following twelve linear orderings $\{\mathcal{H}^{\kappa,\rho}_{\mathbb{R}\iota\ell}, \mathcal{H}^{\kappa,\rho}_{\iota\mathbb{R}\ell}, \mathcal{H}^{\kappa,\rho}_{\iota\ell\mathbb{R}} : \iota, \ell \in \{+,-\}\}$ do not order embed into each other.

When $\mathrm{cof}(\kappa) = \omega$, $\mathcal{H}^{\kappa,\rho}_{+\mathbb{R}+}$ order embeds into $\mathcal{H}^{\kappa,\rho}_{\mathbb{R}++}$, for example. (Tentatively, if one replaces the four $\mathcal{H}^{\kappa,\rho}_{\mathbb{R}\iota\ell}$ with four new orderings $\mathcal{S}^{\kappa,\rho}_{\mathbb{R}\iota\ell}$, you may get a 12 element basis for $\mathbb{R} \times \kappa$ when $\mathrm{cof}(\kappa) = \omega$.)

Basis above Cartesian Product $\mathbb{R}$ and a Singular Cardinal

Theorem
Assume $\mathsf{AC}_\omega^{\mathbb{R}}$*, the Baire property, and boldface* GCH *at* ω*. Let* κ *be a singular cardinal of* <u>*uncountable cofinality*</u>*,* $\delta = \mathrm{cof}(\kappa)$*,* $\delta \to_* (\delta)^2_2$*, and there is a cofinal function* $\rho : \delta \to \kappa$ *such that* $\rho(\alpha) \to_* (\rho(\alpha))^2_2$ *for all* $\alpha < \delta$*. Then* $\{\mathcal{H}^{\kappa,\rho}_{\mathbb{R}\iota\ell}, \mathcal{H}^{\kappa,\rho}_{\iota\mathbb{R}\ell}, \mathcal{H}^{\kappa,\rho}_{\iota\ell\mathbb{R}} : \iota, \ell \in \{+,-\}\}$ *forms a twelve element basis for linear orderings whose cardinality is above* $|\mathbb{R} \times \kappa|$*.*

By the HOD-analysis, regular cardinals satisfy the exponent two correct type partition relation,

Theorem
Assume AD^+*. Let* κ *be a singular cardinal of* <u>*uncountable cofinality*</u> *which is a limit of regular cardinal. Then* $\{\mathcal{H}^{\kappa,\rho}_{\mathbb{R}\iota\ell}, \mathcal{H}^{\kappa,\rho}_{\iota\mathbb{R}\ell}, \mathcal{H}^{\kappa,\rho}_{\iota\ell\mathbb{R}} : \iota, \ell \in \{+,-\}\}$ *forms a twelve element basis for linear orderings whose cardinality is above* $|\mathbb{R} \times \kappa|$*.*

Basis above Cartesian Product $\mathbb{R}$ and a Singular Cardinal

Theorem
Assume $\mathsf{AC}^{\mathbb{R}}_{\omega}$*, the Baire property, and boldface* GCH *at* ω*. Let* κ *be a singular cardinal of* <u>*uncountable cofinality*</u>*,* $\delta = \mathrm{cof}(\kappa)$*,* $\delta \to_* (\delta)^2_2$*, and there is a cofinal function* $\rho : \delta \to \kappa$ *such that* $\rho(\alpha) \to_* (\rho(\alpha))^2_2$ *for all* $\alpha < \delta$*. Then* $\{\mathcal{H}^{\kappa,\rho}_{\mathbb{R}\iota\ell}, \mathcal{H}^{\kappa,\rho}_{\iota\mathbb{R}\ell}, \mathcal{H}^{\kappa,\rho}_{\iota\ell\mathbb{R}} : \iota, \ell \in \{+,-\}\}$ *forms a twelve element basis for linear orderings whose cardinality is above* $|\mathbb{R} \times \kappa|$*.*

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These two theorems are easier since one can use the ω-club filter which are measures in the real world. However there are singular cardinals which are not even limits of cardinals. We will use the measures inside various $\mathrm{HOD}_{\{x,y,z\}}$ given by the Jackson-Ketchersid-Schlutzenberg-Woodin result; however, they are not measures in the real world.

Basis above Cartesian Product $\mathbb{R}$ and a Singular Cardinal

Theorem
Assume AD *and* $V = L(\mathbb{R})$. *(Also under* AD^+*.) Let* $\kappa < \Theta$ *be a singular cardinal of uncountable cofinality. Then* $\{\mathcal{H}^{\kappa,\rho}_{\mathbb{R}\iota\ell}, \mathcal{H}^{\kappa,\rho}_{\iota\mathbb{R}\ell}, \mathcal{H}^{\kappa,\rho}_{\iota\ell\mathbb{R}} : \iota, \ell \in \{+,-\}\}$ *forms a twelve element basis for linear orderings whose cardinality is greater than or equal to* $|\mathbb{R} \times \kappa|$.

Theorem
Assume AD *and* $V = L(\mathbb{R})$. *(Also under* AD^+.*) Let* $\kappa < \Theta$ *be a singular cardinal of uncountable cofinality. Then* $\{\mathcal{H}^{\kappa,\rho}_{\mathbb{R}\iota\ell}, \mathcal{H}^{\kappa,\rho}_{\iota\mathbb{R}\ell}, \mathcal{H}^{\kappa,\rho}_{\iota\ell\mathbb{R}} : \iota, \ell \in \{+,-\}\}$ *forms a twelve element basis for linear orderings whose cardinality is greater than or equal to* $|\mathbb{R} \times \kappa|$.

Let $(\mathbb{R} \times \kappa, \prec)$ be a linear ordering. Let $\delta = \mathrm{cof}(\kappa)$. Assume $\prec$ is OD and HOD believes δ is the cofinality of κ.

Fact (Jackson-Ketchersid-Schlutzenberg-Woodin)

Assume AD *and* $V = L(\mathbb{R})$. *If* $\omega_1 < \kappa < \Theta$, *then for any* $x \in \mathbb{R}$, $\mathrm{HOD}_{\{x\}}$ *has* κ*-many measurable cardinals below* κ.

If κ is not a limit of regular cardinal, some measurable cardinals in some $\mathrm{HOD}_{\{x\}}$ will not longer be measurable in $\mathrm{HOD}_{\{y\}}$ for sufficiently strong y. We need a comeager set of x so that the set of measurable cardinals of $\mathrm{HOD}_{\{x\}}$ are stabilized.

Fact

Assume all subsets of $\mathbb{R}$ have the Baire property and boldface GCH *at ω holds. Let $\Phi : \mathbb{R} \times \mathbb{R} \to \mathscr{P}(\mathrm{ON})$ be E_0-invariant. Then there is a comeager $K \subseteq \mathbb{R} \times \mathbb{R}$ so that Φ is constant on K.*

Fact

There is a comeager $K_1 \subseteq \mathbb{R}$ and comeager $K_2 \subseteq \mathbb{R} \times \mathbb{R}$ and $Z \subseteq \mathrm{ON}$ so that for all $r \in K_1$ and $(r, s) \in K_2$, Z is the set of measurable cardinals of $\mathrm{HOD}_{\{r\}}$ and $\mathrm{HOD}_{\{r,s\}}$.

Proof.

Let $\Phi(r)$ be the class of measurable cardinals of $\mathrm{HOD}_{\{r\}}$ and $\Psi(r, s)$ be the class of measurable cardinals of $\mathrm{HOD}_{\{r,s\}}$. Apply the previous fact to these two E_0-invariant functions. $\square$

Fact

Assume all subsets of $\mathbb{R}$ have the Baire property and boldface GCH *at ω holds. Let $\Phi : \mathbb{R} \times \mathbb{R} \to \mathscr{P}(\mathrm{ON})$ be E_0-invariant. Then there is a comeager $K \subseteq \mathbb{R} \times \mathbb{R}$ so that Φ is constant on K.*

Fact

There is a comeager $K_1 \subseteq \mathbb{R}$ and comeager $K_2 \subseteq \mathbb{R} \times \mathbb{R}$ and $Z \subseteq \mathrm{ON}$ so that for all $r \in K_1$ and $(r, s) \in K_2$, Z is the set of measurable cardinals of $\mathrm{HOD}_{\{r\}}$ and $\mathrm{HOD}_{\{r,s\}}$.

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Let $\Phi(r)$ be the class of measurable cardinals of $\mathrm{HOD}_{\{r\}}$ and $\Psi(r, s)$ be the class of measurable cardinals of $\mathrm{HOD}_{\{r,s\}}$. Apply the previous fact to these two E_0-invariant functions. $\square$

Note that K_1 and K_2 are OD. Pick $\rho \in \mathrm{HOD}$ so that $\rho : \delta \to Z$ is cofinal in κ. For each $(r, s) \in K_2$, let $\nu^{r,s}$ be the unique Mitchell order zero normal measure on δ and $\mu^{r,s}_\alpha$ be the unique Mitchell order zero normal measure on $\rho(\alpha)$ in $\mathrm{HOD}_{\{r,s\}}$. (The uniqueness follows from the fact that $\mathrm{HOD}_{\{r,s\}}$ is core model.) The assignment $(r, s) \mapsto (\nu^{r,s}, \mu^{r,s}_\alpha : \alpha < \delta)$ is E_0-invariant.

Basis above Cartesian Product $\mathbb{R}$ and a Singular Cardinal

For $r \in K_1$, consider $(\kappa, \ll^r)$ be r^{th}-section of $(\mathbb{R} \times \kappa, \prec)$. Uniformly, one has a order embedding of $\mathcal{L}^{\kappa,\rho}_{\iota_r \ell_r}$ into $(\kappa, \ll^r)$ for some $\iota_r, \ell_r \in \{+, -\}$ by the basis result for singular cardinals. One can piece them together into a new OD linear ordering $(\mathbb{R} \times L^{\kappa}_{\rho}, \sqsubset)$ so that each section is ordered by $\prec^{\kappa,\rho}_{\iota_r \ell_r}$. (r, s) is a good pair if $(\iota_r, \ell_r) = (\iota_s, \ell_s)$. If (r, s) is a good pair, let $I(r, s) = \iota_r = \iota_s$ and $L(r, s) = \ell_r = \ell_s$.

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Define $F : K_1 \to \{+,-\} \times \{+,-\}$ by $F(r) = (\iota_r, \ell_r)$. It is tempting to find a perfect tree homogeneous for F that stabilizes (ι_r, ℓ_r). However the perfect tree will destroy the ordinal definability and we need to use the measures in $\mathrm{HOD}_{\{r,s\}}$ to find the homogeneous sets on the ordinals. The key insight is to delay picking homogeneous perfect trees for the partitions on $\mathbb{R}$ until one has picks all the homogeneous sets for all the potentially relevant partitions on the ordinals.

Basis above Cartesian Product $\mathbb{R}$ and a Singular Cardinal

Let $\diamondsuit(r_0, r_1)$ be the disjunction of the following statement:

1. There exists $\alpha_0 < \delta$ so that for all $\alpha_1 < \delta$, $(r_1, \alpha_1, 0) \sqsubset^{I(r_0, r_1)} (r_0, \alpha_0, 0)$.
2. There exists $\alpha_1 < \delta$ so that for all $\alpha_0 < \delta$, $(r_0, \alpha_0, 0) \sqsubset^{I(r_0, r_1)} (r_1, \alpha_1, 0)$.

Define $P_0 : K_2 \to 3$ by

$$P_0(r_0, r_1) = \begin{cases} 0 & (r_0, r_1) \text{ is a good pair and } \diamondsuit(r_0, r_1) \\ 1 & (r_0, r_1) \text{ is a good pair and } \neg\diamondsuit(r_0, r_1) \\ 2 & (r_0, r_1) \text{ is not a good pair} \end{cases}.$$

Define $P_1 : K_2 \to 3$ by

$$P_1(r_0, r_1) = \begin{cases} 0 & P_0(r_0, r_1) = 0 \text{ and (1) of } \diamondsuit(r_0, r_1) \text{ holds} \\ 1 & P_0(r_0, r_1) = 0 \text{ and (2) of } \diamondsuit(r_0, r_1) \text{ holds} \\ 2 & P_0(r_0, r_1) \neq 1 \end{cases}.$$

Define $\Sigma : K_2 \to \delta$ as follows: If $P_1(r_0, r_1) = 0$, then let $\Sigma(r_0, r_1)$ be the least $\alpha_0 < \delta$ so that for all $\alpha_1 < \delta$, $(r_1, \alpha_1, 0) \sqsubset^{I(r_0, r_1)} (r_0, \alpha_0, 0)$. If $P_1(r_0, r_1) = 1$, then let $\Sigma(r_0, r_1)$ be the least $\alpha_1 < \delta$ so that for all $\alpha_0 < \delta$, $(r_0, \alpha_0, 0) \sqsubset^{I(r_0, r_1)} (r_1, \alpha_1, 0)$. If $P_1(r_0, r_1) = 2$, then let $\Sigma(r_0, r_1) = 0$.

Fact

Assume $\mathsf{AC}_\omega^\mathbb{R}$ *and all sets of reals have the Baire property. If* $\Sigma : \mathbb{R} \to \delta$ *and* $\mathrm{cof}(\delta) > \omega$*, then there is a comeager* $K \subseteq \mathbb{R}$ *so that* $\Sigma[K]$ *is bounded below* δ*.*

Applying this fact to Σ, one gets a comeager K_3 and a $\bar{\eta} < \delta$ so that $\Sigma[K_3] \subseteq \bar{\eta}$. This ordinal will important for defining the order embeddings. For this reason, the theorem fails when $\mathrm{cof}(\kappa) = \omega$.

Basis above Cartesian Product $\mathbb{R}$ and a Singular Cardinal

To continue this argument, many other partitions will be defined: For each good pair $(r_0, r_1) \in K_3$, we have $Q_0^{r_0,r_1}, \dots, Q_m^{r_0,r_1} : \delta \to 2$ homogeneous taking value $i_0^{r_0,r_1}, \dots, i_m^{r_0,r_1}$ and $T_0^{r_0,r_1}, \dots, T_n^{r_0,r_1} : [\delta]^2 \to 2$ homogeneous taking value $j_0^{r_0,r_1}, \dots, j_n^{r_0,r_1}$.

If $B \in \nu^{r_0,r_1} \times \nu^{r_0,r_1}$, then the proof of the Rowbottom lemma via a diagonal intersection gives uniformly $\mathfrak{C}_{\nu^{r_0,r_1}}(B) \in \nu^{r_0,r_1}$ so that $[\mathfrak{C}_{\nu^{r_0,r_1}}(B)]^2 \subseteq B$.

Define $\bar{\bar{\Xi}} : K_3 \to \mathscr{P}(\delta)$ by

$$\bar{\bar{\Xi}}(r_0, r_1) = \bigcap_{k \leq m} (Q_k^{r_0,r_1})^{-1}[\{i_k^{r_0,r_1}\}] \cap \bigcap_{k \leq n} \mathfrak{C}_{\nu^{r_0,r_1}}((T_k^{r_0,r_1})^{-1}[\{j_k^{r_0,r_1}\}]).$$

Note $\bar{\bar{\Xi}}(r_0, r_1) \in \nu^{r_0,r_1}$, but $\bar{\bar{\Xi}}$ is not E_0-invariant.

Let $N_{r_0,r_1} = \{\bar{\Xi}(s_0, s_1) : (s_0, s_1) \in K_3 \wedge s_0 \in [r_0]_{E_0} \wedge s_1 \in [r_1]_{E_0}\}$. Note $(r_0, r_1) \mapsto N_{r_0,r_1}$ is E_0-invariant. N_{r_0,r_1} is $\mathrm{OD}_{\{r_0,r_1\}}$ and hence belongs to $\mathrm{HOD}_{\{r_0,r_1\}}$. Since $[r_0]_{E_0}, [r_1]_{E_0} \in \mathrm{HOD}_{\{r_0,r_1\}}$ and is countable in $\mathrm{HOD}_{\{r_0,r_1\}}$, N_{r_0,r_1} is countable inside of $\mathrm{HOD}_{\{r_0,r_1\}}$. Since $\bar{\Xi}(s_0, s_1) \in \nu^{s_0,s_1} = \nu^{r_0,r_1}$ by E_0-invariance, $N_{r_0,r_1} \subseteq \nu^{r_0,r_1}$. Since ν^{r_0,r_1} is countably complete in $\mathrm{HOD}_{\{r_0,r_1\}}$ and N_{r_0,r_1} is countable in $\mathrm{HOD}_{\{r_0,r_1\}}$, $\Xi(r_0, r_1) = \bigcap N_{r_0,r_1} \in \nu^{r_0,r_1}$. The map $\Xi : K_3 \to \mathscr{P}(\delta)$ is E_0-invariant.

Fact

Assume all subsets of $\mathbb{R}$ have the Baire property and boldface GCH *at ω holds. Let $\Phi : \mathbb{R} \times \mathbb{R} \to \mathscr{P}(\mathrm{ON})$ is E_0-invariant. Then there is a comeager $K \subseteq \mathbb{R} \times \mathbb{R}$ so that Φ is constant on K.*

The previous fact gives a single $D \subseteq \delta$ and a comeager K_4 so that D is simulteneously homogeneous for all potentially relevant partitions on δ index by any good pair in K_4.

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Now there are many more partition defined on $\rho(\alpha)$ for each $\alpha < \delta$ and $(r_0, r_1) \in K_4$: $Q_0^{\alpha,r_0,r_1}, \ldots, Q_m^{\alpha,r_0,r_1} : \rho(\alpha) \to 2$ and $T_0^{\alpha,r_0,r_1}, \ldots, T_n^{\alpha,r_0,r_1} : [\rho(\alpha)]^2 \to 2$. By uniformly doing a similar argument as above, one gets a sequence $\langle \bar{\eta}^\alpha : \alpha < \delta \rangle$ and $\langle E_\alpha : \alpha < \delta \rangle$ and a comeager K_5 so that for all $(r_0, r_1) \in K_5$, $E_\alpha \in \mu_\alpha^{r_0,r_1}$ and homogeneous for all potentially relevant partitions on $\rho(\alpha)$. (Note that we can repeat the previous argument because $\rho(\alpha)$ being measurable in $\mathrm{HOD}_{\{r_0,r_1\}}$ means $\mathrm{cof}(\rho(\alpha)) > \omega$ in the real world by a result of Steel.)

Several more partitions $P_2, P_3, P_4 : [\mathbb{R}]^2 \to 2$ are also defined.

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With the homogeneous sets D and $\langle E_\alpha : \alpha < \delta \rangle$ found, we can now pick the perfect trees homogeneous for F, P_0, P_1, P_2, P_3, and P_4. The homogeneous values on these partition will determine which of the twelve linear orderings will embed into $(\mathbb{R} \times \kappa, \prec)$. The ordinals $\bar{\eta}$ and $\langle \bar{\eta}_\alpha : \alpha < \delta \rangle$ and the sets D and $\langle E_\alpha : \alpha < \delta \rangle$ are used to define and verify the order embedding.

Definition
If κ is an ordinal, let $<^+$ and $<^-$ denote the usual or reverse ordering on κ, respectively. Let $\iota, \ell \in \{+, -\}$. Define $\mathcal{W}^{\kappa,\omega}_{\iota,\ell} = ([\kappa]^\omega, \prec^{\kappa,\omega}_{\iota\ell})$ by $f \prec^{\kappa,\omega}_{\iota,\ell} g$ if and only if the conjunction of the following holds.

- $\sup(f) <^\iota \sup(g)$.
- $\sup(f) = \sup(g)$ and if $n \in \omega$ is the least m so that $f(m) \neq g(m)$, then $f(n) <^\ell g(n)$.

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Theorem
Assume $\kappa \to_* (\kappa)^{\omega+\omega}_2$. Then any two distinct linear orderings from $\{\mathcal{W}^{\kappa,\omega}_{++}, \mathcal{W}^{\kappa,\omega}_{--}, \mathcal{W}^{\kappa,\omega}_{+-}, \mathcal{W}^{\kappa,\omega}_{-+}\}$ do not order embed into each other.

Theorem
Assume $\kappa \to_* (\kappa)^{\omega+\omega}_2$. Then $\{\mathcal{W}^{\kappa,\omega}_{++}, \mathcal{W}^{\kappa,\omega}_{--}, \mathcal{W}^{\kappa,\omega}_{+-}, \mathcal{W}^{\kappa,\omega}_{-+}\}$ forms a four element basis for the linear orderings with cardinality above $|{}^\omega\kappa|$.

A more complete analysis for ω-sequence through cardinals can be obtained using diagonal Prikry forcing over HOD-type models.