On the influence of inner model theory on forcing and ultrafilters

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Two Theorems of Ralf (1/2)

Theorem (The existence of K without hand grenades)

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the puzzle of constructing the core model for many strong cardinals. Specifically, we improve Jensen's notes [8] by establishing that K can be shown to exist in the theory

ZFC + "there is no sharp for an inner model with
a proper class of strong cardinals",

or

ZFC + "$0^{\P}$ (zero hand grenade) does not exist",

Figure: From: *The core model for almost linear iterations* (APAL)

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the puzzle of constructing the core model for many strong cardinals. Specifically, we improve Jensen's notes [8] by establishing that K can be shown to exist in the theory

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LOLLIPOP

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Figure: From: *The core model for almost linear iterations* (APAL)

Two Theorems of Ralf (2/2)

Theorem (Schindler)

The core model of $Ult(V, U)$ of V by a measure U is an iterate of K.

The theorem and its restricted version by Mitchell have been a central part of the forcing theory related to structure of measures/extenders.

Typically, one starts from an extender model $K = L[E]$, devise a special forcing extension $V = K[G]$ (depending on the application in mind) and use Ralf's theorem and the iteration associated with $j_U \upharpoonright K$ to derive consequences for $U \in V$

Some applications

- Showing the consistency of a model where every well-founded pre-ordering $(S, <_S)$ is isomorphic to the Mitchell order of normal measures on some measurable cardinal κ (BN)
- Determining the exact consistency strength of a model with a measurable cardinal and every ω-sequence of κ-complete measures are $\leq_{RK}$ an extender (Hayut-Poveda)
- Characterizing, for various κ-distributive forcings $\mathbb{Q}$, the consistency strength of $\mathbb{Q}$ being absorbed by a Tree Prikry forcing via κ-complete measures (Benhamou-Gitik-Hayut)
- Constructing a model of UA + $2^\kappa > \kappa^+$ for a measurable cardinal κ (BN-Kaplan)

Gitik and Kaplan found alternative methods for deriving similar conclusion for $j_U \restriction V$ for **normal** measures $U \in V[G]$, without the core model assumptions.

- Assuming $GCH_{\leq\kappa}$ in the ground model V and $\mathbb{P}_\kappa$ is an Easton support iteration of Prikry forcings (Gitik-Kaplan)
- Assuming $GCH_{\leq\kappa}$ in the ground model V and $\mathbb{P}_\kappa$ is a non-stationary support iteration of Prikry forcings (Gitik-Kaplan)
- Assuming $GCH_{\leq\kappa}$ in the ground model V and $\mathbb{P}_\kappa$ is a full-support iteration of Prikry forcings (Kaplan)

Questions:

1. Do these results extend to non-normal measures?
2. Can they extend to supercompact-type measures?

More applications of IMT to forcing theory

Theorem (BN-Kaplan)

It is consistent, relative to a 2-strong cardinal, that there is a model $L[A, U]$ with a single measurable cardinal κ such that

1. *There is a single normal measure U on κ*
2. *Every σ-complete measure is isomorphic to a finite power U^n of U*
3. $2^\kappa = \kappa^{++}$.

Ralf theorem plays a critical role in the proof. In addition, the proof makes an important use of a fine-structure-based iteration theory, following work of Zeman.

Coding with stationary sets of low cofinality ordinals 1/3

- Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha \leq \kappa \rangle$ be a poset that adds new subsets to κ and allows an ultrapower embedding $j : V \to M$ to extend to $j^* : V[G] \to M[G^*]$, from which we can derive $V[G]$-ultrafilters $U^* = \{A \subseteq \kappa \mid \gamma \in j^*(A)\}$.

- We would like to avoid having alternative generics to G^* for $j(\mathbb{P})$ over M, as this will result in alternatives to U^* that are not powers of U^*.

- One ingredient is using non-stationary supports (following Friedman–Magidor) to ensure uniqueness of G^*.

- Another key ingredient is to incorporate self coding posets to each step $\mathbb{Q}_\alpha$, $\alpha \leq \kappa$.

Coding with stationary sets of low cofinality ordinals 2/3

- For $\alpha \leq \kappa$ inaccessible, fix a sequence $\vec{S}^\alpha = \langle S_i^\alpha \mid i < \alpha^+ \rangle$ of pairwise disjoint nonreflecting stationary sets $S_i^\alpha \subseteq \alpha^+$ (or α^{++})

- A coding component of $\mathbb{Q}_\alpha$ codes the generic information by adds club(s) disjoint from specific sets S_i^α.

- If $S^\alpha \subseteq cof(\alpha)$ (consists of α-cofinality ordinals) then $\mathbb{Q}_\alpha$ can be α-closed.

- The need to control non-normal measures led us to work with S_α consisting of small cofinality ordinals (e.g., $S^\alpha \subseteq cof(\omega_1)$).

- The problem is that $\mathbb{Q}_\alpha$ is α-distributive but no longer α-closed

Safely iterating distributive posets over infinitely many cardinals requires large-cardinal strength.

Definition

A potential distributive iteration is a sequence $\vec{q} = \langle \dot{\mathbb{Q}}_n \mid n < \omega \rangle$ of names of posets, such that each $\mathbb{Q}_n$ is (forced) to be $\aleph_n$-distributive.

Theorem (Adolf-BN-Schindler-Zeman)

If for every potential distributive iteration $\vec{\mathbb{Q}} \in V$ there is a cardinal preserving generic extension $V[G]$ that contains generics to all finite initial iterations of $\vec{\mathbb{Q}}$ then Projective Determinacy (PD) holds.

Coding with stationary sets of low cofinality ordinals 3/3

- The concept that a suitable choice of a sequence $\vec{S} = \langle S^\alpha \mid \alpha \leq \kappa \text{ inaccessible}\rangle$ of nonreflecting stationary sets could lead to a cardinal preserving Easton support iteration was discovered by Foreman-Magidor-Zeman in their work on Welch games.

- The choice of the sets S^α and the proof relies on fine structure assumption of the ground.

- In developing an analogous result for nonstationary supports. Eyal and I found the emerging iteration theory achieves new results that address some known problems.

Given a singular cardinal κ, PCF theory gives examples of cardinal preserving "short" iterations $\mathbb{P} = \langle \mathbb{Q}_{\kappa_i} \mid i < \rho \rangle$ along cofinal sequence $\langle \kappa_i \rangle_{i<\rho} \subseteq \kappa$, consisting of $|\mathbb{Q}_{\kappa_i}| \leq \kappa_i^+$, and such that $\mathbb{P}$ changes the structure at κ^+.

Question: Can these constructions have analogs below an inaccessible cardinal?

Theorem (BN-Kaplan)

There is an iterated forcing $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha < \kappa \rangle$ of a Mahlo length κ of posets $\mathbb{Q}_\alpha$ of size $|\mathbb{Q}_\alpha| \leq \alpha^+$, which preserves cardinals and destroys the stationarity of some $S_\kappa \subseteq \kappa^+ \cap cof(\omega_1)$.

Theorem (BN-Kaplan)

There is an iterated forcing $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha < \kappa \rangle$ of a Mahlo length κ of posets $\mathbb{Q}_\alpha$ of size $|\mathbb{Q}_\alpha| \leq \alpha^+$, which preserves cardinals and adds a branch to a κ^+-Suslin tree T.

Structures-based support 1/3

We consider a family $W \subseteq \mathcal{P}(H_{\kappa^+})$ consisting of admissible structures $X \subseteq H_{\kappa^+}$ such that each X has an ordinal definable Skolem function $h^X : \kappa \to X$.

Definition

- For $X \in W$, let

$$I(X, h^X) = \{\alpha < \kappa \mid \alpha \in h^X[\alpha] \text{ and } h^X \text{ is definable from } p \in [\alpha]^{<\omega}\}.$$

- For each $\alpha \in I(X, h^X)$, define the localization of X at α is

$$\bar{X}^\alpha = tc(h^X[\alpha]).$$

If W is cofinal in H_{κ^+} then the collection $\{I(X, h^X) \mid X \in W\}$ generates a κ-complete ideal, $\mathcal{I}_W$ contained in the nonstationary ideal on κ.

Structures-based support 2/3

Definition (Suitable sequence $\vec{W}$)

$\vec{W} = \langle W_\alpha \mid \alpha \leq \kappa \text{ inaccessible}\rangle$ is suitable if for every $\beta < \alpha$ and $X \in W_\alpha$, if $\beta \in I(X, h^X)$ then $\bar{X}^\beta \in W_\beta$.

Definition ($\vec{W}$-support iteration)

A $\vec{W}$-support iteration $\mathbb{P}_\kappa = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha < \kappa\rangle$ is supported in sets $\sigma \subseteq \kappa$ satisfy that for all regular $\alpha \leq \kappa$, $\sigma \cap \alpha \in \mathcal{I}_{W_\alpha}$.

For many standard choices of posets $\mathbb{Q}_\alpha$, their $\vec{W}$-support iteration is equivalent to a nonstationary support iteration.

Structures-based support 3/3

The $\vec{W}$-support formulation is helpful in forming cardinal preserving iterations of various distributive posets. The following two results are key.

Proposition

Suppose that $X \in W$ and $Z \subseteq \kappa$ is such that $X \models$ "Z is nowhere stationary". Then $Z \subseteq^* I(X, h^X)$.

Proposition

Suppose that $\vec{X} = \langle X_i \mid i < \eta \rangle$, $\eta < \kappa$, is an internally approachable sequence of structure $X_i \in W_\kappa$, then for every sufficiently large $\alpha \in I(X_0, h^{X_0})$, $\vec{X}^\alpha = \langle \bar{X}_i^\alpha \mid i < \eta \rangle \subseteq W_\alpha$ is also internally approachable.

Example with Fine Structure

Example

In a fine structural extender model $L[E]$, given a sequence $\langle S^\alpha \mid \alpha < \kappa \rangle$ of stationary sets, let $W_\alpha = \{N_\eta \mid \text{ a collapsing structure of some } \eta \in \alpha^+ \setminus S_\alpha\}$.

Choosing the sets $\langle S^\alpha \mid \alpha < \kappa \rangle$ to be definable from similar fine structural properties will result in comparability of the sets $\langle W_\alpha \mid \alpha < \kappa \rangle$. For example, set for $\alpha < \kappa$

$$S^\alpha = \{\eta < \alpha^+ \mid n^{N_\eta} = 3, p_{n^{N_\eta}}^{N_\eta} = \emptyset, otp(c_\eta) = \omega_1\}.$$

With these assumptions we force with a $\vec{W}$-supported iteration $\mathbb{P}_\kappa = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha < \kappa \rangle$ such that $\mathbb{Q}_\alpha$ is nontrivial only when α is regular, and then $\mathbb{Q}_\alpha = \mathbb{Q}(S_\alpha)$ consists of closed bounded subsets $d \subseteq \alpha^+ \setminus S^\alpha$.

Assuming κ is Mahlo, the properties of the $\vec{W}$-based iteration shows that caridnals are not collapsed in a generic extension but S^κ is no longer stationary.

Question

Can we find examples with good families of structures $\vec{W} = \langle W_\alpha \mid \alpha < \kappa \rangle$ *outside of fine structural inner models?*

More specifically,

Question

Can this iteration theory be applied to ground model V where $2^\kappa = \kappa^{++}$ *and result in an iteration* $\mathbb{P}_\kappa$ *that preserve cardinals and destroys a stationary subset of* κ^{++}*?*

Question

Can these constructions be applied in models with κ *supercompact?*

Thank You!