Cofinalities of ultrafilters

Tom Benhamou

Department of Mathematics,
Rutgers University

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In general, (1) is more precise than (2).

The isomorphism class of a $\subseteq^*$-generating set for U

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1. *If U is a simple $P_{\mathbb{D}}$-point, then $\mathbb{D}$ is κ^+-directed.*
2. *If $2^\kappa = \kappa^+$, then U is simple P_{κ^+}-point.*

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If $a \in [\gamma]^{<\omega}$, $E_0(a) \subseteq U_a$, where U_a is the ultrafilter derived from j_U and $k(a)$. Let $f_a : \kappa \to [\kappa]^{|a|}$ witness $U_a \leq_{RK} U$.

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These condition ensure that $j_U \restriction K = k_F \circ j_F$ for some factor map k_F such that $crit(k_F) \geq (2^\kappa)^{+M_F} \geq \alpha$. Hence $F = E_0 \restriction \alpha$. □

- $\Rightarrow$ Since $E_0(a) \in K$, in V we have $|E_0(a)| < \kappa^{++}$. Since U is a $P_{\kappa^{++}}$-point, there is $B_a \in U$ which is a $\subseteq^*$-lower bound for $f_\alpha^{-1}[E_0(a)]$.
- $\Rightarrow$ Note that $f_a, B_a, P^K(\kappa) \in M_U$ from which we can compute $E_0(a) \in M_U$. Moreover, $E_0 \restriction \alpha \in M_U$ for each $\alpha < \gamma$. (indeed the critical point from $j_{E_0(\alpha)}$ to j_{E_0} is greater than α, and $j_{E_0(\alpha)} \restriction P^K(\kappa) \in M_U$).
- $\Rightarrow$ Using the $P_{\kappa^{++}}$-pointness again, there is $B \in U$ such that for every $a \in [\gamma]^{<\omega}$, $B \subseteq^* B_a$. In M_U, let U' be the filter on κ generated by B.

The contradiction is obtained by showing that E_0 is definable inside M_U. This is true since $E_0 \restriction \alpha$ is the unique $F \in M_U$ such that $M_U \models \varphi(F, f_\kappa, U', \alpha)$, where $\varphi(F, f_\kappa, U', \alpha)$ is the statement that F is a K-extender of length $\alpha \leq ((2^\kappa)^+)^{M_F}$, and $\exists \langle g_a \mid a \in [\alpha]^{<\omega} \rangle$ such that:

1. $F_a \subseteq g_{a*}(U')$.
2. For each $a \subseteq b$, $\pi_{a,b} \circ g_b = g_a \mod U'$.
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Question

What is the consistency strength of a $P_{\kappa^{++}}$-point?

Definition 9

Let $\mathbb{D}$ be a directed poset. We say that an ultrafilter U over $\kappa \geq \omega$ is a *simple $P_{\mathbb{D}}$-point* if there is a $\subseteq^*$-base $\mathcal{B} \subseteq U$, such that $(\mathcal{B}, \supseteq^*) \simeq \mathbb{D}$.

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Suppose that κ is a supercompact cardinal, then there is a forcing extension where κ is supercompact and for every κ^+-directed, well-founded poset $\mathbb{D}$ there is a $< \kappa$-directed κ^+-cc forcing in which there is a normal ultrafilter U which is a simple $P_{\mathbb{D}}$-point.

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Same result for κ is measurable and **the club filter** being a simple $P_{\mathbb{D}}$-point (i.e. has a generating sequence isomorphic to $\mathbb{D}$).

The Tukey order

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In particular, if U is a κ-complete ultrafilter over κ which is not $(\kappa, 2^\kappa)$-cohesive implies that U is maximal in the Tukey order in the class of κ-complete ultrafilters.

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Theorem 16 (B.-Gitik 22')

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2. *In L[U] every ultrafilter is (κ, κ^+)-cohesive.*

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$$\sum_W W_x = \{A \subseteq X \times Y \mid \{x \in X \mid (A)_x \in W_x\} \in W\}$$

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- $\Rightarrow$ Supercompact cardinals always yes (B. [2] '23).

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Do κ-complete non-$(\kappa, 2^\kappa)$-cohesive/κ-Tukey-top ultrafilters exist?

- $\Rightarrow$ Cons. no, $L[U]$.
- $\Rightarrow$ Cons. yes, forcing from a measurable cardinal (B.-Gitik [4] '22)
- $\Rightarrow$ Supercompact cardinals always yes (B. [2] '23).

Question

Can they exist in the canonical inner models?

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Consistently, there is a κ-complete (κ, κ^+)-cohesive ultrafilter which is not an iterated sum of p-points.

Theorem 21 (B.-Goldberg [6] '23)

In $L[\mathbb{E}]$, the following are equivalent for every κ-complete ultrafilter U over κ:

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If U is (κ, κ^+)-cohesive then $\diamondsuit^-_{thin}(U)$ fails.

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Assume U is λ-sound and for all $f : \kappa \to \kappa$, $j_U(f)(\kappa) < \lambda$, then $\diamondsuit^-_{thin}(U)$ holds.

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If U is a Dodd-sound ultrafilter which is not p-point, then $\diamondsuit^-_{thin}(U)$ holds (and in particular U is not (κ, κ^+)-cohesive).

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and $j_W = j^{M_n}_{D^*} \circ j_{0,n}$, where in M_k, $j_{k,k+1}$ is the ultrapower by a κ_k-complete p-point U_k over $\kappa_k = j_{0,k}(\kappa)$. Namely, that $U = \sum_{U_0} U_1$.

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- ⇒ In the complete proof, we need to consider several cases for the possible position of ρ^* and δ^*, with respect to κ_{m-1} and κ_m. Let us cover two of them:

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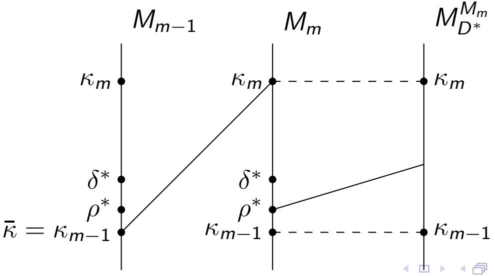
(1) $\kappa_{m-1} < \rho^* \leq \delta^* < \kappa_m$. In this case, the two-step iteration $j^{M_m}_{D^*} \circ j_{m-1,m}$ is the ultrapower by a κ_{m-1}-complete p-point on κ_{m-1}, contradicting the assumption about W:



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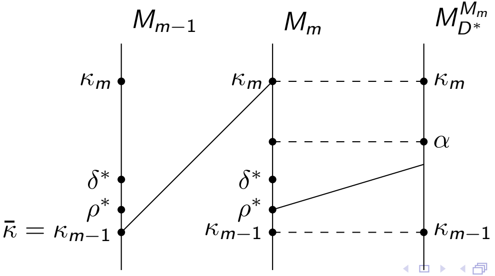
⇒ In the complete proof, we need to consider several cases for the possible position of ρ^* and δ^*, with respect to κ_{m-1} and κ_m. Let us cover two of them:

(1) $\kappa_{m-1} < \rho^* \leq \delta^* < \kappa_m$. In this case, the two-step iteration $j_{D^*}^{M_m} \circ j_{m-1,m}$ is the ultrapower by a κ_{m-1}-complete p-point on κ_{m-1}, contradicting the assumption about W:



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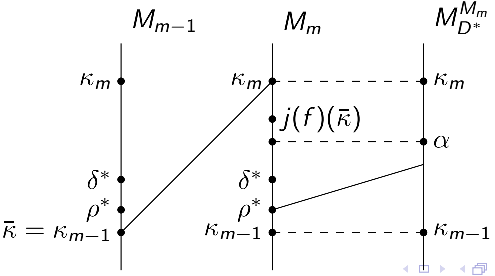

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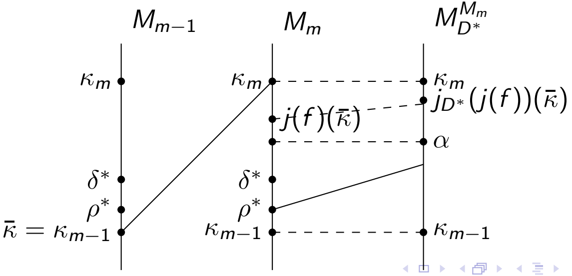

(2) Assume that $\rho^* = \delta^* = \kappa_m$. Working in M_m, D^* is a κ_m-complete ultrafilter over κ_m and by the maximality of U, D^* cannot be a p-point.

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If U is a normal ultrafilter on $P_\kappa(\lambda)$ then U is not (κ^+, λ^+)-cohesive

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Theorem 31 (B.-Weltsch 25+)

If U is a normal ultrafilter on $P_\kappa(\kappa^+)$ then U is $(\kappa, 2^{\kappa^+})$-cohesive.

Thank you for your attention!

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