Adding elementary embeddings by small forcing

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Adding elementary embeddings

It's been around 40 years now that we know how to add by forcing, in the presence of large cardinals, an elementary embedding $j : V \longrightarrow M$ with small critical point:

Theorem

(Foreman-Magidor-Shelah) If κ is a supercompact cardinal, then there is a semiproper poset $\mathbb{P} \subseteq V_\kappa$ (thus preserving ω_1) forcing $\omega_2 = \kappa$ and Martin's Maximum, and hence forcing that NS_{ω_1} is saturated.

Theorem

(Foreman-Magidor-Shelah) If κ is a supercompact cardinal and $\mu < \kappa$ is regular, $\mathrm{Coll}(\mu, {<}\kappa)$ forces that NS_μ is precipitous.

Jech asked: Does any large cardinal notion imply that NS_{ω_1} is precipitous?

F-M-S showed that the answer is No:

Theorem

(Foreman-Magidor-Shelah) If μ is a regular cardinal, then there is a $<\mu$-closed poset $\mathbb{Q} \subseteq 2^{\mu}$ forcing that NS_{μ} is not precipitous.

A notable refinement of the first theorem:

Theorem

(Woodin) If δ is a Woodin cardinal, the stationary tower $\mathbb{P}_{<\delta} \subseteq V_\delta$ is such that, if G is $\mathbb{P}_{<\delta}$-generic, then there is, in $V[G]$, an elementary embedding $j : V \longrightarrow M$, ${}^{<\delta}M \cap V[G] \subseteq M$, with critical point ω_1^V (or critical point ω_2^V, ω_3^V, ...).

General question: Assume large cardinals. Is there a *small* forcing $\mathcal{P}$ such that if G is $\mathcal{P}$-generic, then there is an elementary embedding $j : V \longrightarrow M$, M transitive, with $\mathrm{crit}(j) = \omega_1^V$?

Collapsing functions and extender models

Definition

(Schimmerling-Veličković) Given a cardinal κ, $f : \omega_1 \longrightarrow \omega_1$ is a *collapsing function for* κ if there is a club $D \subseteq [H_\kappa]^{\aleph_0}$ such that $\mathrm{ot}(X \cap \kappa) < f(X \cap \omega_1)$ for every $X \in D$.

Note: By condensation, in L there is a collapsing function for every κ: just let $f(\alpha)$ be the least β such that $L_\beta \models |\alpha| = \aleph_0$.

Much more is true:

Theorem

(Schimmerling-Veličković) Suppose $L[\vec{E}]$ is a coherent extender model such that every countable model in $L[\vec{E}]$ embedding into some level of $L[\vec{E}]$ is $\omega_1 + 1$-iterable. Then
$L[\vec{E}] \models$ *For every $n \geq 2$ there is a collapsing function for $\aleph_n$.*

Theorem

(Schimmerling-Veličković) Suppose there is a forcing $\mathcal{P} \subseteq 2^{\aleph_1}$ *adding an elementary embedding* $j : V \longrightarrow M$ *with* M *transitive and* $\text{crit}(j) = \omega_1^V$. *Then*
$V \models$ *There is no collapsing function for* $(2^{\aleph_1})^+$.

Proof.

Let $\kappa = (2^{\aleph_1})^+$. Suppose $F : [H_\kappa]^{<\omega} \longrightarrow H_\kappa$ generates a club $D \subseteq [H_\kappa]^{\aleph_0}$ witnessing that $f : \omega_1 \longrightarrow \omega_1$ is a collapsing function for κ.
Let G be $\mathcal{P}$-generic and let $j : V \longrightarrow M$ be an elementary embedding in $V[G]$ with M transitive and $\text{crit}(j) = \omega_1^V$. Since $|\mathcal{P}| \leq 2^{\aleph_1}$, $j(\omega_1^V) \leq \kappa$.
Let $\alpha = j(f)(\omega_1^V) < j(\omega_1^V) \leq \kappa$ and $X = j``H_\kappa^V$. Then X is closed under $j(F)$, $X \cap j(\omega_1) = \omega_1^V$, and
$j(f)(X \cap j(\omega_1)) = \alpha < \kappa = \text{ot}(X)$.
By well-foundedness of M, in M there is a set $X \in [j(H_\kappa)]^{\aleph_0}$ closed under $j(F)$, and hence in $j(D)$, such that $X \cap j(\omega_1) = \omega_1^V$, and $j(f)(X \cap j(\omega_1)) = \alpha < \text{ot}(X)$. Contradiction. □

It is easy to force a collapsing function for ω_2 (Wu): forcing with finite symmetric systems $\mathcal{N} \subseteq H_{\omega_2}$ will do (Asperó-Schindler).

Question

Is there a poset forcing the existence of a collapsing function for $(2^{\aleph_1})^+$?

For some while, in joint work with Schindler, we thought the answer would be yes. However, the natural-looking forcings we were coming up with didn't quite work...

Main result

Theorem

Suppose κ is an inaccessible limit of $<\kappa$-supercompact cardinals. Then $\mathrm{Coll}(\omega, 2^{\aleph_1})$ forces the existence of an elementary embedding $j : V \longrightarrow M$ with M transitive and $\mathrm{crit}(j) = \omega_1^V$.

Corollary

Suppose κ is an inaccessible limit of $<\kappa$-supercompact cardinals. Then there is no collapsing function for $(2^{\aleph_1})^+$.

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Suppose κ is an inaccessible limit of $<\kappa$-supercompact cardinals. Then V is not a coherent extender model $L[\vec{E}]$ with the property that every countable model embedding into some level of $L[\vec{E}]$ is $\omega_1 + 1$-iterable.

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Main ideas

The proof relies very heavily on

- Lietz's proof that "NS_{ω_1} is $\aleph_1$-dense" can be forced from the same large cardinal assumption;
- the proof of the Asperó-Schindler theorem that MM^{++} implies $(*)$.

Theorem

(Woodin) Assume $\mathsf{AD}^{L(\mathbb{R})}$*. Then there is a* $\mathbb{P}_{\max}$*-variation* $\mathbb{Q}_{\max} \in L(\mathbb{R})$ *such that forcing with* $\mathbb{Q}_{\max}$ *produces a model of* ZFC *in which* NS_{ω_1} *is* $\aleph_1$*-dense; i.e.,* $\mathcal{P}(\omega_1)/\mathsf{NS}_{\omega_1}$ *has a dense subset of size* $\aleph_1$ *(equivalently,* $\mathcal{P}(\omega_1)/\mathsf{NS}_{\omega_1}$ *is forcing-equivalent to* $\mathsf{Coll}(\omega, \omega_1)$*).*

A longstanding question was whether the $\aleph_1$-density of NS_{ω_1} can be forced over a ZFC-model with enough large cardinals.

In 2023, Lietz proved that the answer is yes.

Theorem

(Lietz) Suppose κ *is an inaccessible limit of* $<\kappa$*-supercompact cardinals.Then there is a forcing* $\mathcal{P} \subseteq V_\kappa$ *preserving stationary subsets of* ω_1 *and forcing the* $\aleph_1$*-density of* NS_{ω_1}*.*

One crucial component in the definition of $\mathbb{Q}_{\max}$-condition, and the starting point of Lietz's proof, is the following.

Definition

(Woodin) Let $f : \omega_1 \longrightarrow H_{\omega_1}$ be such that $f(\alpha)$ is a filter of $\mathrm{Coll}(\omega, \omega_1)$ for every $\alpha < \omega_1$ (*f guesses filters of* $\mathrm{Coll}(\omega, \omega_1)$).

(0) Given $p \in \mathrm{Coll}(\omega, \omega_1)$, $S^f_p = \{\alpha < \omega_1 \,:\, p \in f(\alpha)\}$.

(1) $S \subseteq \omega_1$ is *f-stationary* if for every $\theta \geq \omega_2$,

$$\{X \in [H_\theta]^{\aleph_0} \,:\, f(\delta_X) \text{ is } \mathrm{Coll}(\omega, \delta_X)\text{-gen. over } X, \delta_X \in S\}$$

is stationary. [Notation: $\delta_X = X \cap \omega_1$.]

(2) f is a *witness for* $\diamondsuit(\omega_1^{<\omega})$ if for every $p \in \mathrm{Coll}(\omega, \omega_1)$, S^f_p is f-stationary.

In order to prove the theorem, let H be $\mathrm{Coll}(\omega_1, \omega_1)$-generic over V. Working in $V[H]$ construct a witness $f : \omega_1 \longrightarrow H_{\omega_1}$ for $\diamondsuit(\omega_1^{<\omega})$. Along with f we also construct a sequence $\vec{\lambda} = (\lambda_n)_{n<\omega}$ of functions $\lambda_n : \omega_1 \longrightarrow \omega_1$ such that

(0) for all n and $\alpha < \omega_1$, $\lambda_n^{-1}(\alpha)$ is f-stationary;

(1) for all n and $\beta < \omega_1$ there is a (unique) α such that $\lambda_{n+1}^{-1}(\beta) \subseteq \lambda_n^{-1}(\alpha)$;

(2) for all $n < \omega$ and $\alpha < \omega_1$ there is some $p \in \mathrm{Coll}(\omega, \omega_1)$ with $S^f_{p'} \cap \lambda_n^{-1}(\alpha)$ f-stationary for every $p' \in \mathrm{Coll}(\omega, \omega_1)$ extending p.

We make sure that

(3) for all $S \subseteq \omega_1$ in V, $p \in \mathrm{Coll}(\omega, \omega_1)$, $n < \omega$ and $\alpha < \omega_1$, if $S \cap S^f_p \cap \lambda_n^{-1}(\alpha)$ is f-stationary, then there is some $\beta < \omega_1$ such that $\lambda_{n+1}^{-1}(\beta) \subseteq S \cap S^f_p \cap \lambda_n^{-1}(\alpha)$.

If (0), (1) and (2) hold, we say that f *is a witness for* $\diamondsuit(\omega_1^{<\omega})$ *with homogeneous labelling sequence* $\vec{\lambda}$.

The sets $\lambda_n^{-1}(\alpha)$ (for $n < \omega$ and $\alpha < \omega_1$) are called *labels*.

We then force with $\mathcal{Q} = (\{\lambda_n^{-1}(\alpha) \,:\, n < \omega, \alpha < \omega_1\}, \subseteq)$ over $V[H]$. Let J be the generic and consider the generic ultrapower $\mathrm{Ult}(V, J)$ of V induced by J; i.e.,

$$\mathrm{Ult}(V, J) = (\{[f]_J \,:\, f : \omega_1^V \longrightarrow V, f \in V\}, \in_J),$$

where $f =_J g$ iff there is $S \in J$ with

$$S \subseteq \{\alpha < \omega_1^V \,:\, f(\alpha) = g(\alpha)\}$$

and similarly for $\in_J$.

By (3), Łoś's theorem holds and the canonical map $j : V \longrightarrow M$ given by $j(x) = [c_x]_J$ is an elementary embedding with critical point ω_1^V.

The main claim is that $\mathrm{Ult}(V, J)$ is well-founded. For this we will use:

Theorem

Suppose κ is an inaccessible cardinal which is a limit of $<\kappa$-supercompact cardinals. Suppose f is a witnesses of $\diamondsuit(\omega_1^{<\omega})$ with homogeneous labelling sequence $\vec{\lambda} = (\lambda_n)_{n<\omega}$. Then there is an ω_1-preserving forcing notion $\mathcal{P} \subseteq V_\kappa$ forcing that:

(1) *f is a witness of $\diamondsuit(\omega_1^{<\omega})$ with labelling sequence $\vec{\lambda}$;*

(2) *$\{[\lambda_n^{-1}(\alpha)]_{\mathrm{NS}_{\omega_1}} : n < \omega, \alpha < \omega_1\}$ is dense in $\mathcal{P}(\omega_1)/\,\mathrm{NS}_{\omega_1}$; in particular, NS_{ω_1} is $\aleph_1$-dense.*

We can thus find such a $\mathcal{P}$ in $V[H]$. Now suppose, towards a contradiction, that some $\lambda_{n_0}^{-1}(\alpha_0)$ forces over $V[H]$ that $\mathrm{Ult}(V, \dot{J})$ is ill-founded.

Let K be $\mathcal{P}$-generic over $V[H]$. By the Theorem, the map

$$i : \mathcal{Q} \longrightarrow (\mathcal{P}(\omega_1)/\,\mathrm{NS}_{\omega_1})^{V[H][K]}$$

sending $\lambda_n^{-1}(\alpha)$ to $[\lambda_n^{-1}(\alpha)]_{\mathrm{NS}_{\omega_1}^{V[H][K]}}$ is a dense embedding.

Let now J^* be $(\mathcal{P}(\omega_1)/\,\mathrm{NS}_{\omega_1})^{V[H][K]}$-generic over $V[H][K]$ with

$$[\lambda_{n_0}^{-1}(\alpha_0)]_{\mathrm{NS}_{\omega_1}^{V[H][K]}} \in J^*$$

and let $J = i^{-1}(J^*)$.

Then $\lambda_{n_0}^{-1}(\alpha_0) \in J$ and every $([f_n]_J)_{n<\omega}$ witnessing ill-foundedness of $\mathrm{Ult}(V, J)$ is such that $([f_n]_{J^*})_{n<\omega}$ witnesses ill-foundedness of $\mathrm{Ult}(V[H][K], J^*)$. But this contradicts the $\aleph_1$-density of $(\mathcal{P}(\omega_1)/\,\mathrm{NS}_{\omega_1})^{V[H][K]}$ in $V[H][K]$.

To prove the theorem we build on Lietz's proof. A variant of the A.-Schindler proof of $\mathsf{MM}^{++} \Rightarrow (*)$ will be a crucial ingredient.

Definition

Suppose f is a witness of $\diamondsuit(\omega_1^{<\omega})$ with homogeneous labelling sequence $\vec{\lambda} = (\lambda_n)_{n<\omega}$. A partial order $\mathbb{Q}$ *freezes* NS_{ω_1} *along* f *and* $\vec{\lambda}$ in case for every $\mathbb{Q}$-generic filter G we have that

(1) f is, in $V[G]$, a witness of $\diamondsuit(\omega_1^{<\omega})$ with homogeneous labelling sequence $\vec{\lambda}$ and

(2) for every $S \in \mathcal{P}(\omega_1) \cap V$, either $S \in \mathsf{NS}_{\omega_1}^{V[G]}$ or else there are n, α with $\lambda_n^{-1}(\alpha) \subseteq S$ mod. $\mathsf{NS}_{\omega_1}^{V[G]}$.

$\mathcal{P}$ will be $\mathcal{P}_\kappa$ for a certain nice-support iteration $(\mathcal{P}_\alpha : \alpha \leq \kappa)$ along which we keep freezing NS_{ω_1} along f and $\vec{\lambda}$.

We will need the following strengthening of $\diamondsuit(\omega_1^{<\omega})$.

Definition

(Woodin) $f : \omega_1 \longrightarrow H_{\omega_1}$ is a *witness for* $\diamondsuit^+(\omega_1^{<\omega})$ iff

(1) for every $\theta \geq \omega_2$,

$$\{X \in [H_\theta]^{\aleph_0} \ : \ f(\delta_X) \text{ is } \mathrm{Coll}(\omega, \delta_X)\text{-gen. over } X\}$$

contains a club of $[H_\theta]^{\aleph_0}$;

(2) for every $p \in \mathrm{Coll}(\omega, \omega_1)$, S_p^f is f-stationary.

Fact

If f is a witness for $\diamondsuit^+(\omega_1^{<\omega})$, then $(\{S_p^f \ : \ p \in \mathrm{Coll}(\omega, \omega_1)\}, \subseteq)$ *embeds completely into* $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$.

At a given stage $\alpha + 1$ of our iteration we first force:

($+$) that f is a witness for $\diamondsuit^+(\omega_1^{<\omega})$ with homogeneous labelling sequence $\vec{\lambda}$;

(🔨) the Strong Reflection Principle (SRP) below κ.

This can be easily done with a nice-support f-semiproper iteration of length the first available $<\kappa$-supercompact cardinal.

Now, using the fact that f is a witness for $\diamondsuit^+(\omega_1^{<\omega})$ with homogeneous labelling sequence $\vec{\lambda}$ and that NS_{ω_1} is saturated, we find a forcing $\mathbb{C}$ shooting a club freezing NS_{ω_1} along f and $\vec{\lambda}$.

$\mathbb{C}$ is a variant of the main forcing from the $\mathsf{MM}^{++} \Rightarrow (*)$ proof, adapted to a certain variant Λ-$\mathbb{Q}^-_{\max}$ of $\mathbb{Q}_{\max}$ instead of $\mathbb{P}_{\max}$.

Definition

Λ-$\mathbb{Q}_{\max}^{-}$ is the forcing notion consisting of tuples $p = (M, f, \vec{\lambda})$ such that:

(1) M is a countable transitive model of ZFC^*;

(2) $(M, \mathrm{NS}_{\omega_1}^M)$ is generically iterable;

(3) $f \in M$ is, in M, a function guessing filters of $\mathrm{Coll}(\omega, \omega_1^M)$, $\vec{\lambda} \in M$ is an ω-sequence of functions from ω_1^M into ω_1^M and, in M, f witnesses $\diamondsuit^+(\omega_1^{<\omega})$ with homogeneous labelling sequence $\vec{\lambda}$.

Given Λ-$\mathbb{Q}^-_{\max}$-conditions, $p = (M, f, \vec{\lambda})$ and $q = (N, h, \vec{\mu})$, q extends p in Λ-$\mathbb{Q}^-_{\max}$ if in N there is a generic iteration

$$\langle (M_\alpha, f_\alpha, \vec{\lambda}_\alpha), j_{\alpha,\beta} \,:\, \alpha \leq \beta \leq \omega_1^N \rangle$$

of p such that:

(i) $f_{\omega_1^N} = g$ (mod. $\mathrm{NS}^N_{\omega_1}$);

(ii) for every $n < \omega$, $(\vec{\lambda}_{\omega_1^N})_n = \mu_n$ (mod. $\mathrm{NS}^N_{\omega_1}$);

(iii) for every $S \in M_{\omega_1^N} \cap (\mathrm{NS}^N_{\omega_1})^+$ there is some $n < \omega$ and $\alpha < \omega_1^N$ such that $(\vec{\mu}_n)^{-1}(\alpha) \subseteq S$ (mod. $\mathrm{NS}^N_{\omega_1}$).

Lemma

Suppose $p_0 = (M, f, \vec{\lambda}) \in \Lambda\text{-}\mathbb{Q}^-_{\max}$, *h witnesses* $\diamondsuit(\omega_1^{<\omega})$, *and* δ *is a Woodin cardinal. Then there is a generic extension* $V[G]$ *of* V *by a partial order* $\mathcal{R} \subseteq V_\delta$ *such that* $\mathrm{Coll}(\omega, \delta)$ *forces over* $V[G]$ *that, for some sequence* $\vec{\mu}$,

$$(H_{\omega_2}^{V[G]}, h, \vec{\mu})$$

is a condition in $\Lambda\text{-}\mathbb{Q}^-_{\max}$ *extending* p_0.

Proof sketch: We first add a Cohen-generic $F : \omega_1 \longrightarrow \omega_1 \times \omega_1$. Using F we build a generic iteration

$$\mathcal{I} = \langle (M_\alpha, f_\alpha, \vec{\lambda}_\alpha), j_{\alpha,\beta} \ : \ \alpha \leq \beta \leq \omega_1 \rangle$$

of p_0, together with a sequence $\langle g_\alpha \ : \ \alpha < \omega_1 \rangle$, where each g_α is a generic over M_α for

$$\mathcal{Q}_\alpha := (\{S_p^{f_\alpha} \ : \ p \in \mathrm{Coll}(\omega, \omega_1^{M_\alpha})\}, \subseteq)$$

Let $x \in \mathbb{R}$ code p_0. We consider a generic $G \in V[F]$ over $L[x]$ for a certain variant $\mathrm{Coll}_{p_0}(\omega, {<}\omega_1)$ of $\mathrm{Coll}(\omega, {<}\omega_1)$.
$\mathrm{Coll}_{p_0}(\omega, {<}\omega_1)$ is a finite-support iteration $\langle \mathbb{A}_\alpha, \dot{\mathbb{B}}_\beta : \alpha \leq \omega_1, \beta < \omega_1 \rangle$. At every stage α, $\dot{\mathbb{B}}_\alpha$

- adds g_α and
- picks a certain label $\overline{S}_\alpha \in \{(\vec{\lambda}_\alpha)_n^{-1}(\nu) : n < \omega, \nu < \omega_1^{M_\alpha}\}$ such that there is a $(\mathcal{P}(\omega_1) \setminus \mathrm{NS}_{\omega_1})^{M_\alpha}$-generic filter U over M_α extending $g_\alpha \cup \{\overline{S}_\alpha\}$.

We then let U_α be the least such U in the canonical well-order of $L[x][G_\alpha]$.

The construction of G is guided by F. By genericity of F we may assume that:

(1) for each $p \in \mathrm{Coll}(\omega, \omega_1)$ there is an h-stationary set of stages $\alpha \in S_p^h$ such that $g_\alpha = h(\alpha)$;

(2) for each label $(\vec{\lambda}_{\omega_1})_n^{-1}(\nu)$ there is an h-stationary set of stages $\alpha < \omega_1$ such that $j_{\alpha,\omega_1}(\overline{S}_\alpha) = (\vec{\lambda}_{\omega_1})_n^{-1}(\nu)$.

This can be arranged, using the genericity of F, thanks to the homogeneity of $\vec{\lambda}$ in M. [Rec.: This is

(2) for all $n < \omega$ and $\alpha < \omega_1$ there is some $p \in \mathrm{Coll}(\omega, \omega_1)$ with $S^f_{p'} \cap \lambda_n^{-1}(\alpha)$ f-stationary for every $p' \in \mathrm{Coll}(\omega, \omega_1)$ extending p.

in the definition of f being a witness for $\diamondsuit(\omega_1^{<\omega})$ with homogeneous labelling sequence $\vec{\lambda}$.]

By a standard argument using Silver indiscernibles $\gamma < \omega_1$ for $L[x]$ and exploiting the canonical construction of our generic iteration from the generic G for $\mathrm{Coll}_p(\omega, \omega_1)$, it follows that for every $S \in \mathcal{P}(\omega_1)^{M_{\omega_1}}$ which is stationary in $V[H]$ there is some $p \in \mathrm{Coll}(\omega, \omega_1)$ and some $(\vec{\lambda}_{\omega_1})_n^{-1}(\nu)$ such that $S_p^{f_{\omega_1}} \cap (\vec{\lambda}_{\omega_1})_n^{-1}(\nu) \subseteq S$ (mod. NS_{ω_1}).

Let $\vec{\mu} = j_{0,\omega_1}(\vec{\lambda})$. Finally, using the Woodin cardinal δ there is an $j_{0,\omega_1}(f)$-semiproper $\mathcal{R}_0 \subseteq V_\delta$ forcing $\delta = \omega_2$, that $\diamondsuit^+(\omega_1^{<\omega})$ holds as witnessed by $j_{0,\omega_1}(f)$, and the saturation of NS_{ω_1}. This finishes the proof. $\square$

The forcing $\mathbb{C}$ for shooting a club freezing NS_{ω_1} along f and $\vec{\lambda}$ is a natural recursively constructed $\mathcal{L}$-forcing, much as the one in the $\mathsf{MM}^{++} \Rightarrow (*)$ proof, adding a 'certificate' of the following configuration:

There are Λ-$\mathbb{Q}^-_{\max}$-conditions $p = (M_0, f_0, \vec{\lambda}_0)$ and $q = (N_0, h_0, \vec{\mu}_0)$, such that the following statement $\star_{p,q}$ holds.
$\star_{p,q}$: q extends p in Λ-$\mathbb{Q}^-_{\max}$ as witnessed by a generic iteration

$$\mathcal{I}_0 = \langle (M_\alpha, f_\alpha, \vec{\lambda}_\alpha), i_{\alpha,\beta} \ : \ \alpha \leq \beta \leq \omega_1^{N_0} \rangle$$

of p in N_0 and there is a generic iteration

$$\mathcal{J} = \langle (N_\alpha, h_\alpha, \vec{\mu}_\alpha), j_{\alpha,\beta} \ : \ \alpha \leq \beta \leq \omega_1 \rangle$$

of q such that, letting

$$\mathcal{I} = j_{0,\omega_1}(\mathcal{I}_0) = \langle (M_\alpha, f_\alpha, \vec{\lambda}_\alpha), i_{\alpha,\beta} \ : \ \alpha \leq \beta \leq \omega_1 \rangle,$$

(1) $(M_{\omega_1}, f_{\omega_1}, \vec{\lambda}_{\omega_1}) = (H^V_{\omega_2}, f, \vec{\lambda})$ and
(2) every member of $(\mathcal{S}^{h_{\omega_1}}_{\vec{\mu}_{\omega_1}})^{N_{\omega_1}}$ is f-stationary.

As in the $\mathsf{MM}^{++} \Rightarrow (*)$ proof, this certificate incorporates a certain $\subseteq$-chain of models (not from V) as side conditions:

Let us fix a tree T on $\omega \times 2^{\aleph_2}$ such that

$$\Vdash_{\mathrm{Coll}(\omega,\omega_2)} \text{“}p[T] \text{ codes the members of } \Lambda\text{-}\mathbb{Q}^{-}_{\max}\text{”}$$

Let $\kappa = (2^{\aleph_2})^+$. Let d be $\mathrm{Add}(\kappa, 1)$-generic over V. In $V[d]$ there is a club $C \subseteq \kappa$ of ordinals above ω_2 and a “diamond sequence”

$$(\langle Q_\rho, B_\rho\rangle \,:\, \rho \in C)$$

such that $(Q_\rho \,:\, \rho \in C)$ is a strictly $\subseteq$-increasing and $\subseteq$-continuous sequence of transitive elementary submodels of $H_\kappa^{V[d]} = H_\kappa^V$ and for each $\rho \in C$, $B_\rho \subseteq Q_\rho$ and $|B_\rho| = |\rho|$.

$\mathbb{C}$ will be $\mathcal{P}_\kappa$, where

$$\langle \mathcal{P}_\rho \,:\, \rho \in C \cup \{\kappa\}\rangle$$

is the sequence of forcings defined by letting $\mathcal{P}_\rho$ be the set, ordered under $\supseteq$, of finite sets p of sentences, in a suitable fixed language $\mathcal{L}$, such that $\mathrm{Coll}(\omega, \rho)$ forces that there is a *ρ-certificate for p*. We will have that $\mathcal{P}_\rho \subseteq Q_\rho$ for all $\rho \in C$, and that $\mathcal{P}_\kappa \subseteq H_\kappa$.

A *ρ-pre-certificate (relative to $(H^V_{\omega_2}, f, \vec{\lambda})$ and T)* is a complete set Σ of sentences, in our fixed language $\mathcal{L}$, describing the following objects.

(1) Λ-$\mathbb{Q}^-_{\max}$-conditions $p_0 = (M_0, f_0, \vec{\lambda}_0)$ and $q = (N_0, h_0, \vec{\mu}_0)$;

(2) a real $x = \langle k_n : n < \omega \rangle$ coding p and a real $y = \langle k'_n : n < \omega \rangle$ coding q;

(3) branches $\langle (k_n, \alpha_n) : n < \omega \rangle$ and $\langle (k'_n, \alpha'_n) : n < \omega \rangle$ through T;

(4) $\langle p_i, \pi_{i,j} : i \leq j \leq \omega_1^{N_0} \rangle \in N_0$, a generic iteration of p witnessing $q \leq_{\Lambda\text{-}\mathbb{Q}^-_{\max}} p$;

(5) $\langle q_i, \sigma_{i,j} : i \leq j \leq \tau \rangle$, a generic iteration of q for some $\tau \leq \omega_1^V$;

(6) $\langle p_i, \pi_{i,j} : i \leq j \leq \tau \rangle = \sigma_{0,\tau}(\langle \mathcal{M}_i, \pi_{i,j} : i \leq j \leq \omega_1^{N_0} \rangle)$ and

$$(M_\tau, f_\tau, \vec{\lambda}_\tau) = (H^V_{\omega_2}, f, \vec{\lambda});$$

(7) generic $\mathrm{Coll}(\omega, \omega_1^{N_i})$-filters g_i^* over N_i, where $q_i = (N_i, h_i, \vec{\mu}_i)$, such that

$$N_{i+1} = \mathrm{Ult}(N_i, U_i^*)$$

for a $\mathcal{P}(\omega_1)^{N_i} \setminus \mathrm{NS}^{N_i}_{\omega_1}$-generic filter U_i^* over N_i with

$$\{S_a^{h_i} : a \in g_i^*\} \subseteq U_i^*;$$

(8) $K \subset \tau$, and for all $\delta \in K$,

(a) $\rho_\delta \in C \cap \rho$,

(b) $X_\delta \preccurlyeq (Q_{\rho_\delta}; \in, \mathcal{P}_{\rho_\delta}, B_{\rho_\delta})$ such that $X_\delta \cap \omega_1^V = \delta$,

(c) if $\gamma < \delta$ is in K, then $\rho_\gamma < \rho_\delta$ and $X_\gamma \cup \{\rho_\gamma\} \subset X_\delta$.

A ρ-pre-certificate Σ is a ρ-*certificate* if, in addition:

(9) For every $\delta \in K$,

(a) $[\Sigma]^{<\omega} \cap X_\delta \cap E \neq \emptyset$ for every dense $E \subseteq \mathcal{P}_{\rho_\delta}$ definable over the structure

$$(Q_{\rho_\delta}; \in, \mathcal{P}_{\rho_\delta}, B_{\rho_\delta})$$

from parameters in X_δ;

(b) if B_{ρ_δ} codes a pair $B^0_{\rho_\delta}$, $B^1_{\rho_\delta}$ of subsets of H_{ρ_δ} and $B^1_{\rho_\delta}$ codes a sequence $\langle \dot{D}_\xi : \xi < \omega_1^V \rangle$ of nice $\mathcal{P}_{\rho_\delta}$-names for dense subsets of $\mathrm{Coll}(\omega, \omega_1^V)$, then $g^*_\delta \cap (\dot{D}_\xi)_\Sigma \neq \emptyset$ for each $\xi < \delta$, where

$$(\dot{D}_\xi)_\Sigma = \{t \in \mathrm{Coll}(\omega, \omega_1^V) : s \Vdash_{\mathcal{P}_{\rho_\delta}} t \in \dot{D}_\xi \text{ for some } s \in \Sigma\}.$$

- The previous lemma is used to show $\mathbb{C} \neq \emptyset$.
- The (generically added) side conditions X_δ (for $\delta \in K$) are used to show that every member of $(\mathcal{S}^{h_{\omega_1}}_{\vec{\mu}_{\omega_1}})^{N_{\omega_1}}$ is f-stationary.

Back to our iteration $(\mathcal{P}_\alpha : \alpha \leq \kappa)$:

Recall that at every stage we first force

(🔨) the Strong Reflection Principle (SRP) below κ.

The point of doing this is that we want $\mathcal{P}_\kappa$ to preserve ω_1. In order to prove the relevant preservation theorem we need to guarantee, at every stage $\alpha < \kappa$, that arbitrarily long tail forcings $\mathcal{P}_\beta / G_\alpha$ are 'respectful' (in Lietz's terminology). This is guaranteed by SRP below κ.

Definition

(Lietz) Given a witness f of $\diamondsuit(\omega_1^{<\omega})$, a forcing $\mathbb{P}$ is *respectful* if $\mathbb{P}$ preserves ω_1 and whenever θ is sufficiently large and regular, $X \preccurlyeq H_\theta$ is countable and such that $f, \mathbb{P} \in X$, $p \in X \cap \mathbb{P}$, and $\dot{I} \in X$ is a $\mathbb{P}$-name for a normal and uniform ideal on ω_1, then exactly one of the following holds.

(1) There is some $(X, \mathbb{P})$-semigeneric $q \leq_{\mathbb{P}} p$ such that

$$q \Vdash_{\mathbb{P}} \text{“}X[\dot{G}] \text{ respects } \dot{I}\text{”}$$

(i.e., q forces that $\delta_X \notin A$ for every $A \in \dot{I} \cap X[\dot{G}]$);

(2) X does not respect

$$\dot{I}_p^{\mathbb{P}} = \{A \subseteq \omega_1 \ : \ p \Vdash_{\mathbb{P}} A \in \dot{I}\}$$

(i.e., there is some $A \in X$ such that $p \Vdash_{\mathbb{P}} A \in \dot{I}$ and $\delta_X \in A$).

The above is the reason we need κ to be a limit of $<\kappa$-supercompact cardinals and the main source of consistency strength in our hypothesis.

Question

Is there a version of the main result using (significantly) less than an inaccessible κ which is a limit of $<\kappa$-supercompact cardinals?

Happy birthday, Ralf!


Edzerdla
STRAFFE
HENDRIK