On the HOD conjecture and its failure

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July 3, 2024

Abstract

The subject of this tutorial is Woodin's HOD conjecture, one of the most prominent open problems in pure set theory. We begin with a proof of his HOD dichotomy theorem along with an improvement of the speaker's reducing the large cardinal hypothesis from an extendible to a strongly compact cardinal. Following this, we mostly discuss the implications of the failure of the HOD conjecture, especially ω-strongly measurable cardinals and a condition under which such a cardinal is locally supercompact in HOD.

1 Introduction

2 The HOD dichotomy theorem

For any ordinal δ and any regular cardinal $\gamma < \delta$, $S^\delta_\gamma = \{\alpha < \delta : \mathrm{cf}(\alpha) = \gamma\}$. If $\mathrm{cf}(\delta) > \gamma$, then S^δ_γ is stationary in δ.

If δ is an ordinal of uncountable cofinality, we the club filter on δ by $\mathcal{C}_\delta$. An ordinal definable set $S \subseteq \delta$ is said to be an OD-*atom of the club filter* if S cannot be partitioned into two disjoint ordinal definable stationary subsets of δ; in other words $(\mathcal{C}_\delta \restriction S) \cap \mathrm{HOD}$ is a HOD-ultrafilter.

A regular cardinal δ is *ω-strongly measurable in* HOD if there is a partition of S^δ_ω into fewer than δ OD-atoms of the club filter.

Exercise 1. If δ is ω-strongly measurable in HOD, then there is an ordinal definable partition of S^δ_ω into OD-atoms of the club filter.

The following lemma is proved in [5]. (Note however that Woodin takes 2 as the definition of an ω-strongly measurable cardinal.)

Lemma 2.1 (Woodin)**.** *The following are equivalent:*

1. *δ is ω-strongly measurable in* HOD*.*
2. *For some λ such that $(2^\lambda)^{\mathrm{HOD}} < \delta$, there is no ordinal definable partition of δ into λ disjoint stationary sets.* □

An inner model M has the *κ-cover property at an ordinal* λ if $P_\kappa(\lambda) \cap M$ is cofinal in $(P_\kappa(\lambda), \subseteq)$; M has the *κ-cover property* if it has the κ-cover property at every ordinal.

Theorem 2.2. *If κ is strongly compact, exactly one of the following holds:*

(1) HOD *has the κ-cover property.*

(2) All sufficiently large regular cardinals are ω-strongly measurable in HOD.

Proof. Note that if $\delta > \kappa$ is ω-strongly measurable in HOD, then HOD does not have the κ-cover property at δ. To see this, fix $S \subseteq S^\delta_\omega$ such that $(\mathcal{C}_\delta \restriction S) \cap \text{HOD}$ is a HOD-ultrafilter. Let $U = (\mathcal{C}_\delta \restriction S) \cap \text{HOD}$. Since HOD satisfies that U is a normal ultrafilter, the set of HOD-regular cardinals less than δ is in U. Since $S \in U$, HOD satisfies that there are arbitrarily large regular cardinals in S. But every ordinal in S has countable cofinality in V, which implies that the κ-cover property fails at δ.

Claim 1. *Suppose λ is an ordinal, $\delta \geq \lambda$ is a regular cardinal, and S^δ_ω admits an ordinal definable partition $\vec{S} = \langle S_\alpha \rangle_{\alpha<\lambda}$ into stationary sets. Then* HOD *has the κ-cover property at λ.*

Proof. To see this, we appeal to a version of Solovay's lemma [3] which was observed by Usuba [4]:

Theorem 2.3 (Usuba)**.** *Suppose $j : V \to M$ is an elementary embedding, δ is a regular cardinal, and $\vec{S} = \langle S_\alpha \rangle_{\alpha<\lambda}$ is a partition of S^δ_ω into stationary sets. Let $\delta_* = \sup j[\delta]$ and let*

$$R = \{\alpha < j(\lambda) : M \vDash j(\vec{S})_\alpha \textit{ is stationary in } \delta_*\}$$

Then $j[\lambda] \subseteq R$ and $|R|^M < \text{cf}^M(\delta_*)$. □

By the strong compactness of κ, there is an elementary $j : V \to M$ with critical point κ such that $\text{cf}^M(\delta_*) < j(\kappa)$, where $\delta_* = \sup j[\delta]$. Let

$$R = \{\alpha < j(\lambda) : M \vDash j(\vec{S})_\alpha \text{ is stationary in } \delta_*\}$$

and note that $R \in j(P_\kappa(\lambda)\cap\text{HOD})$ since $R \in \text{HOD}^M$ and $|R|^M < \text{cf}^M(\delta_*)$. If $\sigma \in P_\kappa(\lambda)$, then $j(\sigma) = j[\sigma] \subseteq j[\lambda] \subseteq R$, and hence M satisfies that $j(\sigma)$ is covered by a set in $j(P_\kappa(\lambda) \cap \text{HOD})$. By elementarity, σ is covered by a set in $P_\kappa(\lambda) \cap \text{HOD}$, which establishes the κ-cover property at λ. □

To finish the proof, note that trivially, either HOD has the κ-cover property or there is some λ such that HOD does not have the κ-cover property at λ. If the latter holds and $\delta > (2^\lambda)^{\text{HOD}}$ is regular, then by our observations above, S^δ_ω cannot be ordinal definably partitioned into λ disjoint stationary sets, and so by Woodin's Lemma 2.1, δ is ω-strongly measurable in HOD. □

Note that the proof shows that if $\delta > \kappa$ is ω-strongly measurable in HOD, then so is every regular cardinal above δ (but see Question 2.5). In fact, the proof establishes something slightly stronger that we will need later. If γ is a regular cardinal, $\lambda \leq \nu$ are ordinals, and $\text{cf}(\delta) > \gamma$, then δ is *(γ, λ)-strongly measurable in* HOD if there is a partition of S^δ_γ into fewer than λ OD-atoms of the club filter.

Theorem 2.4. *Suppose κ is a strongly compact cardinal and $\gamma > \kappa$ is ω-strongly measurable in* HOD*. Then for all ordinals ν with $\text{cf}(\nu) \geq \delta$ and all regular cardinals $\gamma < \kappa$, ν is (γ, δ)-strongly measurable in* HOD*.*

Sketch. Following the proof of Theorem 2.2, one shows that for all ordinals $\nu \geq \delta$ and all regular $\gamma < \kappa$, there is no ordinal definable partition of S^ν_γ into δ stationary sets. Then one appeals to a generalization of Woodin's lemma. □

Question 2.5. Is the previous theorem true with $\delta = \kappa$?

We now turn to the covering properties of HOD that follow in case HOD has the κ-cover property. Let us start with Woodin's HOD dichotomy theorem. A cardinal κ is HOD-*supercompact* if for all $\lambda \geq \kappa$, there is an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$, $M^\lambda \subseteq M$ and $\text{HOD}^M \cap P(\lambda) = \text{HOD} \cap P(\lambda)$.

Theorem 2.6 (Woodin)**.** *Suppose κ is* HOD*-supercompact. Either all sufficiently large regular cardinals are ω-strongly measurable in* HOD *or* HOD *has the κ-cover and approximation properties.*

Proof. The structure of the proof is identical to that of Theorem 2.2, but one proves the following stronger claim using HOD-supercompactness in place of strong compactness:

Claim 2. *Suppose λ is an ordinal, $\delta \geq \lambda$ is a regular cardinal, and S^δ_ω admits an ordinal definable partition $\vec{S} = \langle S_\alpha \rangle_{\alpha<\lambda}$ into stationary sets. Then* HOD *has the κ-cover and approximation properties at λ.*

For this, let $j : V \to M$ witness that κ is HOD-supercompact at δ. Instead of Usuba's theorem, we use Solovay's original lemma [3]:

Theorem 2.7 (Solovay)**.** *Suppose $j : V \to M$ is an elementary embedding, δ is a regular cardinal, and $\vec{S} = \langle S_\alpha \rangle_{\alpha<\lambda}$ is a partition of S^δ_ω into stationary sets. If $j[\delta] \in M$, then $j[\lambda] = \{\alpha < j(\lambda) : j(\vec{S})_\alpha$ is stationary in $\delta_*\}$.* □

Thus the assumption of the claim yields that $j[\lambda] \in \text{HOD}^M$. Fix a set $A \subseteq \lambda$ that is κ-approximated by HOD, and let us show that $A \in \text{HOD}$. Note that $j(A) \cap j[\lambda] \in \text{HOD}^M$ since $j(A)$ is $j(\kappa)$-approximated by HOD^M. Since $j \restriction \lambda \in \text{HOD}^M$, it follows that $A \in \text{HOD}^M$. But since $\text{HOD}^M \cap P(\lambda) = \text{HOD} \cap P(\lambda)$, we have $A \in \text{HOD}$. □

We now establish some stronger covering properties of HOD under the assumption that there is a strongly compact cardinal κ such that HOD has the κ-cover property.

Theorem 2.8. *Suppose* HOD *has the κ-cover property and κ is strongly compact. Then for any* HOD*-regular ordinal $\delta \geq \kappa$, $\text{cf}(\delta) = |\delta|$. As a consequence, for all singular cardinals $\lambda \geq \kappa$, λ is singular in* HOD *and $\lambda^{+\text{HOD}} = \lambda^+$.*

Theorem 2.8 is the author's main contribution; the rest of the proof is a reogranization of Woodin's techniques, but here one needs to do a little work because the proof of [5, Lemma 3.9] does not seem to generalize to the current situation.

This uses the following lemma which will be useful later:

Theorem 2.9. *Suppose δ is a* HOD*-regular ordinal and for some ordinal $\kappa < \delta$, $S \subseteq (S^\delta_{<\kappa})^{\text{HOD}}$ is stationary in V. Then there is an ordinal definable family $\langle S_\alpha \rangle_{\alpha<\delta}$ of stationary subsets of S such that for any $\sigma \in [\delta]^\kappa$, $\bigcap_{\alpha\in\sigma} S_\alpha = \emptyset$.*

Proof. Let $\langle c_\xi : \xi \in S \rangle$ be an ordinal definable ladder sequence, so $c_\xi \subseteq \xi$ is a cofinal set of ordertype $<\kappa$. For $\nu < \delta$, let ν' be the least ordinal such that $\{\xi \in S : c_\xi \cap [\nu, \nu')\}$ is stationary. Note that $\nu' < \delta$ by a regressive function argument. Also the function $\nu \mapsto \nu'$ is ordinal definable.

In HOD, define a sequence $\langle \nu_\alpha \rangle_{\alpha<\delta}$ by transfinite recursion, setting $\nu_0 = 0$, $\nu_{\alpha+1} = \nu'_\alpha$, and $\nu_\lambda = \sup_{\alpha<\lambda} \nu_\alpha$ when λ is a limit ordinal. The HOD-regularity of δ ensures that this construction does not break down at limit steps below δ.

Let $S_\alpha = \{\xi \in S : c_\xi \cap [\nu_\alpha, \nu_{\alpha+1}) \neq \emptyset\}$. Then S_α is stationary by construction, and for any $\sigma \in [\delta]^\kappa$, $\bigcap_{\alpha\in\sigma} S_\alpha = \emptyset$: if $\xi \in \bigcap_{\alpha\in\sigma} S_\alpha$, then $c_\xi \cap [\nu_\alpha, \nu_{\alpha+1}) \neq \emptyset$ for each $\alpha \in \sigma$, contradicting that $\text{ot}(c_\xi) < \kappa$. □

Proof of Theorem 2.8. Since HOD has the κ-cover property, $S = (S^\delta_{<\kappa})^{\text{HOD}}$ is stationary, so by Theorem 2.9, let $\langle S_\alpha \rangle_{\alpha<\delta}$ be a family of stationary subsets of S such that for any $\sigma \in [\delta]^\kappa$, $\bigcap_{\alpha\in\sigma} S_\alpha = \emptyset$. For each $\xi < \delta$, let $\sigma_\xi = \{\alpha < \delta : \xi \in S_\alpha\}$. Let $C \subseteq \delta$ be a closed unbounded set of ordertype $\text{cf}(\delta)$. Then $\delta = \bigcup_{\xi\in C} \sigma_\xi$ since for any $\alpha < \delta$, $S_\alpha \cap C \neq \emptyset$, and therefore for some $\xi \in C$, $\alpha \in \sigma_\xi$.

It follows that $|\delta| = |\bigcup_{\xi\in C} \sigma_\xi| \leq \text{cf}(\delta) \cdot \kappa = \text{cf}(\delta)$. □

3 Weak covering and HOD

A filter F on X is *λ-weakly saturated* if there is no partition of X into λ disjoint F-positive sets. For example, if F is the closed unbounded filter on an ordinal ν, then F is $\mathrm{cf}(\nu)^+$-weakly saturated. If δ is an ordinal, then F is *δ-descendingly complete* if for any F-positive set S and function $f : S \to \delta$, there is an F-positive set $T \subseteq S$ such that $f[T]$ is bounded below δ. If F is the closed unbounded filter on an ordinal of cofinality different from $\mathrm{cf}(\delta)$, then F is δ-descendingly complete. The filter F is *strongly δ-descendingly complete* if for any function $f : X \to \delta$, there is a set $A \in F$ such that $f[A]$ is bounded. Equivalently, every ultrafilter extending F is δ-descendingly complete.

Lemma 3.1. *If δ is a regular cardinal and F is δ-descendingly complete and δ-weakly saturated, then F is strongly δ-descendingly complete.*

Proof. Suppose $f : X \to \delta$ is a function and assume towards a contradiction that there is no $A \in F$ such that $f[A]$ is bounded below δ. For $\nu < \delta$, let ν' be least such that $\{x \in X : f(x) \in [\nu, \nu')\}$ is F-positive. Our assumption implies $\{x \in X : f(x) > \nu\}$ is F-positive, so ν' exists, and the descending completeness of F implies that $\nu' < \delta$.

By transfinite recursion, define a sequence $\langle \nu_\alpha \rangle_{\alpha<\delta}$ by setting $\nu_0 = 0$, $\nu_{\alpha+1} = \nu'_\alpha$, and $\nu_\lambda = \sup_{\alpha<\lambda} \nu_\alpha$ for λ a limit ordinal. Setting $S_\alpha = \{x \in X : f(x) \in [\nu_\alpha, \nu_{\alpha+1})\}$ contradicts that F is δ-weakly saturated. □

Exercise 2. Strong descending completeness is equivalent to the conjunction of descending completeness and weak saturation.

Theorem 3.2 (Prikry–Silver, [1, Theorem 2.9])**.** *If $\delta < \rho$ are regular cardinals and there is a uniform strongly δ-descendingly complete filter on ρ, then every subset of S^ρ_δ reflects.*

Theorem 3.3 (Goldberg–Casey)**.** *If δ is* HOD*-regular,* $\mathrm{cf}(\delta^{+\mathrm{HOD}}) \in \{\omega, \mathrm{cf}(\delta), |\delta|, \delta^+\}$.

Proof. Let $\rho = \delta^{+\mathrm{HOD}}$. Suppose $\omega < \mathrm{cf}(\rho) < |\delta|$, and we will show that $\mathrm{cf}(\rho) = \mathrm{cf}(\delta)$. Let F denote the club filter $\mathcal{C}$ on ρ intersected with HOD. Since $\mathrm{cf}(\rho) < |\delta|$, $\mathcal{C}$ is $|\delta|$-weakly saturated, and hence in HOD, F is $|\delta|$-weakly saturated.

Assume towards a contradiction that $\mathrm{cf}(\rho) \neq \mathrm{cf}(\delta)$. Then $\mathcal{C}$ is δ-descendingly complete, and hence F is δ-descendingly complete in HOD. Working in HOD, the fact that F is δ-descendingly complete and $|\delta|$-weakly saturated implies that F is strongly δ-descendingly complete. But this is a contradiction, since it implies that (in HOD), S^ρ_δ reflects. □

Corollary 3.4. *If δ is regular and* $\mathrm{cf}(\delta^{+\mathrm{HOD}}) > \omega$*, then* $\mathrm{cf}(\delta^{+\mathrm{HOD}}) \geq \delta$.

Exercise 3 (Casey)**.** If every subset of δ^+ has a sharp, then the set of ordinals $\{\delta^{+L[A]} : A \subseteq \delta^+\}$ contains a closed unbounded set. In particular, Corollary 3.4 does not apply to arbitrary inner models.

Theorem 3.3 does apply to a broad class of inner models; namely, all those inner models M that are *club amenable* at δ^{+M} in the sense that $\mathcal{C}_{\delta^{+M}} \cap M \in M$. In fact, it suffices that $F \cap M \in M$ for some filter F extending $\mathcal{C}_{\delta^{+M}} \cap M$.

A filter F on an ordinal ρ is *weakly normal* if every regressive function from a set in F to ρ is bounded on a set in F. Note that the club filter on any ordinal of uncountable cofinality is weakly normal in the weaker sense that every regressive function on a positive set is bounded on a positive set; this is equivalent to

The relationship between these two concepts is analogous to that between descending completeness and strong descending completeness:

Exercise 4. A filter on δ is weakly normal if and only if it is weakly normal in the weaker sense and δ-weakly saturated.

Theorem 3.5 (Ketonen)**.** *Suppose U is a weakly normal ultrafilter on a regular cardinal δ and $S^\delta_{<\kappa} \in U$ for some cardinal $\kappa < \delta$. Then U is γ-descendingly incomplete for every regular ordinal $\gamma \in [\kappa, \delta]$.*

Proof. Let $\delta_* = [\mathrm{id}]_U$. Since U is weakly normal, $\delta_* = \sup j_U[\delta]$. Since $S^\delta_{<\kappa} \in U$, M_U satisfies that $\mathrm{cf}(\delta_*) < j_U(\kappa)$. By Usuba's lemma (Theorem 2.3), there is a set $R \in M_U$ with $|R|^{M_U} < j_U(\kappa)$ such that $j_U[\delta] \subseteq R$. But then if $\gamma \in [\kappa, \delta]$ is regular, $R \cap j(\gamma)$ is cofinal in $\sup j[\gamma]$, so $\mathrm{cf}^M(\sup j[\gamma]) < j(\kappa)$, and hence $j(\gamma) > \sup j[\gamma]$. This means U is γ-descendingly incomplete. □

Theorem 3.6 (Goldberg–Casey)**.** *If ρ is* HOD*-regular, then one of the following holds:*

(1) $\mathrm{cf}(\rho) = \omega$.

(2) $\mathrm{cf}(\rho) = |\rho|$.

(3) There is a closed unbounded set of HOD*-regular ordinals less than ρ.*

(4) ρ is weakly inaccessible in HOD *and for all sufficiently large* HOD*-regular ordinals $\gamma < \rho$,* $\mathrm{cf}(\gamma) = \mathrm{cf}(\rho)$.

(5) $\rho = \lambda^{+\mathrm{HOD}}$ *where* $\mathrm{cf}(\lambda) = \mathrm{cf}(\rho)$.

(6) $\rho = \lambda^{+\mathrm{HOD}}$ *where λ is singular in* HOD *and for all sufficiently large* HOD*-regular ordinals $\gamma < \lambda$,* $\mathrm{cf}(\gamma) = \mathrm{cf}(\rho)$.

Proof. Assume $\omega < \mathrm{cf}(\rho) < |\rho|$, so (1) and (2) fail. Let F denote the club filter on ρ restricted to HOD. Assume that the set of HOD-singular ordinals is F-positive, so (3) fails as well. We must show that (4) or (5) holds.

By Exercise 4, F is weakly normal in HOD. By weak normality and the fact that F concentrates on singular ordinals, there is some $\kappa < \rho$ such that $(S^\rho_{<\kappa})^{\mathrm{HOD}} \in F$. If U is a HOD-ultrafilter extending F, U is weakly normal and hence is γ-descendingly incomplete for all HOD-regular $\gamma \in [\kappa, \rho]$. In particular, F cannot be strongly γ-descendingly complete.

Now fix $\gamma \geq \max\{\mathrm{cf}(\rho)^+, \kappa\}$. We have that F is γ-weakly saturated and not strongly γ-descendingly complete, and hence F is γ-descendingly incomplete by the contrapositive of Lemma 3.1. It follows that $\mathrm{cf}(\gamma) = \mathrm{cf}(\rho)$ since the club filter on ρ is descendingly incomplete only at ordinals with the same cofinality as ρ. Thus for all sufficiently large HOD-regular ordinals $\gamma < \rho$, $\mathrm{cf}(\gamma) = \mathrm{cf}(\rho)$.

In particular, if ρ is weakly inaccessible, we have (4) and if ρ is the successor of a HOD-regular ordinal, we have (5). Finally, if ρ is the successor of a HOD-singular λ, either F is λ-descendingly complete or F is γ-descendingly complete for all sufficiently large regular cardinals below λ, which yields that either (5) or (5) holds. □

Vaguely speaking, (3) states that the closed unbounded filter almost witnesses that ρ is measurable, while (4) asserts that it almost witnesses that κ is ρ-strongly compact where κ is least such that for all HOD-regular $\gamma \in [\kappa, \rho)$, $\mathrm{cf}(\gamma) = \mathrm{cf}(\rho)$. In particular, (3) implies that ρ is strongly Mahlo in HOD, and (4) implies that $\mathrm{HOD} \vDash \neg\square(\gamma)$ for all sufficiently large HOD-regular ordinals $\gamma < \rho$.

We highlight one particular mystery around Theorem 3.6.

Corollary 3.7. *If λ is a cardinal,* $\mathrm{cf}(\lambda) = \omega$*, and* $\mathrm{cf}(\lambda^{+\mathrm{HOD}}) > \omega$*, then* $\lambda^{+\mathrm{HOD}} = \lambda^+$.

The open problem around Theorem 3.6 is exemplified by the following open question:

Question 3.8. If λ is a singular cardinal and $\mathrm{cf}(\lambda^{+\mathrm{HOD}}) > \omega$, must $\lambda^{+\mathrm{HOD}} = \lambda^+$?

So it is the case $\mathrm{cf}(\lambda) > \omega$ that is unclear. The more general problem is whether one can rule out Theorem 3.6 (5) in the case that λ is singular in HOD.

Note that Theorem 3.6 implies this when $\mathrm{cf}(\lambda) = \omega$. Let us mention one result on weak covering in the successor of singular case, an analog of Silver's theorem:

Theorem 3.9 (Goldberg–Poveda)**.** *If λ is a strong limit singular cardinal of uncountable cofinality and $\{\nu < \lambda : \nu^{+\mathrm{HOD}} = \nu^+\}$ is stationary, then $\lambda^{+\mathrm{HOD}} = \lambda^+$.* $\square$

4 Supercompact cardinals

Theorem 4.1. *Suppose κ is strongly compact and $\delta > \kappa$ is ω-strongly measurable in* HOD*. If $\mathrm{cf}(\delta^{+\mathrm{HOD}}) > \omega$, then* HOD *satisfies that δ is $\delta^{+\mathrm{HOD}}$-supercompact.*

Proof. Let $\nu = \delta^{+\mathrm{HOD}}$. By Theorem 3.3, since δ is regular and $\mathrm{cf}(\nu) > \omega$, we have $\mathrm{cf}(\delta^{+\mathrm{HOD}}) \geq \delta$.

The main idea of the proof is to show that there is an ordinal definable stationary set $S \subseteq \nu$ such that if F_S is the club filter restricted to S, then $U_S = F_S \cap \mathrm{HOD}$ witnesses that δ is ν-supercompact in HOD. More precisely, U_S is a HOD-ultrafilter and $j_{U_S}[\nu] \in \mathrm{Ult}(\mathrm{HOD}, U_S)$.

By Theorem 2.4, if $\gamma < \kappa$ is regular, there exists a stationary set $S \subseteq S^\nu_\gamma$ such that U_S is a HOD-ultrafilter. Perhaps surprisingly, we are only able to show U_S witnesses the theorem — meaning $j_{U_S}[\nu] \in \mathrm{Ult}(\mathrm{HOD}, U_S)$ — when γ is uncountable. So fix any regular $\gamma \in (\omega, \kappa)$, and fix a set $S \subseteq S^\nu_\gamma$ such that U_S is a HOD-ultrafilter.

Let $T \subseteq S^\nu_\omega$ be such that U_T is a HOD-ultrafilter. By Theorem 2.9, there is an ordinal definable family $\langle T_\alpha \rangle_{\alpha<\nu}$ of stationary subsets of T such that for any $\sigma \in [\nu]^\delta$, $\bigcap_{\alpha\in\sigma} T_\alpha = \emptyset$. For each $\xi < \nu$, let

$$R_\xi = \{\alpha < \nu : T_\alpha \cap \xi \text{ is stationary in } \xi\}$$

We will prove that the function $\xi \mapsto R_\xi$ represents $j_{U_S}[\nu]$ in $\mathrm{Ult}(\mathrm{HOD}, U_S)$.

We first show that for each $\alpha < \nu$, $\{\xi < \nu : \alpha \in R_\xi\} \in U_S$. To see this, note that for $A \subseteq \nu$ in HOD, $A \in U_T$ if and only if for U_S-almost all $\xi < \nu$, $A \cap \xi \in U^\xi_T$. The reason is that

$$\{A \in P^{\mathrm{HOD}}(\nu) : \{\xi < \nu : A \cap \xi \in U^\xi_T\} \in U_S\}$$

is a filter in HOD extending the restriction of the club filter to HOD and containing T, and U_T is the unique such filter. Therefore since each T_α belongs to U_T, we have $\{\xi < \nu : T_\alpha \cap \xi \in U^\xi_T\} \in U_S$, which implies that $\{\xi < \nu : \alpha \in R_\xi\} \in U_S$.

Next we show that if $f : \nu \to \nu$ is ordinal definable and $f(\xi) \in R_\xi$ for U_S-almost all $\xi < \nu$, then f is constant on a set in U_S. Let $p : \delta \to \nu$ be a continuous cofinal map, which exists since $\mathrm{cf}(\nu) = \delta$. If $\beta < \delta$ has uncountable cofinality, let

$$h(\beta) = \min\{\gamma < \beta : p(\gamma) \in T_{f(p(\beta))}\}$$

Note that $h(\beta)$ exists because $p[\beta]$ is closed unbounded in $p(\beta)$ and $T_{f(p(\beta))}$ is stationary in $p(\beta)$ since $f(p(\beta)) \in R_{p(\beta)}$.

The function h is regressive and defined on the stationary set $p^{-1}[S]$. Therefore by Fodor's lemma, there is an ordinal $\gamma < \delta$ such that

$$E = \{\beta \in p^{-1}[S] : h(\beta) = \gamma\}$$

is stationary. It follows that $p[E]$ is a stationary subset of S. Note that if $\xi \in p[E]$, then $\xi = p(\beta)$ for some β such that $h(\beta) = \gamma$, and so by the definition of h, $p(\gamma) \in T_{f(\xi)}$. In other words $p[E]$ is contained in the set $A = \{\xi \in S : p(\gamma) \in T_{f(\xi)}\}$, which means that this set is an ordinal definable stationary subset of S. Since S is an atom of the club filter restricted to HOD and A is ordinal definable, it follows that $A \in U_S$.

On the other hand, f takes fewer than δ-many values on A. To see this, note that we have that $\bigcap_{\alpha\in f[A]} T_\alpha \neq \emptyset$ since for each $\alpha \in f[A]$, we have $p(\gamma) \in T_\alpha$. By our choice of the sequence $\langle T_\alpha \rangle_{\alpha<\nu}$, this means $|f[A]| < \delta$. Since $A \in U_S$, $|f[A]| < \delta$, and U_S is δ-complete, f is constant on a set in U_S, as desired. $\square$

The same proof yields:

Theorem 4.2. *Suppose κ is strongly compact and $\delta > \kappa$ is regular in V and ω-strongly measurable in* HOD*. If $\lambda \geq \delta$ is regular in* HOD *and $\{\xi < \lambda : \text{cf}^{\text{HOD}}(\xi) < \delta\}$ is stationary in V, then* HOD *satisfies that δ is λ-supercompact.*

5 Partition cardinals above Θ

A long-standing question in determinacy theory is whether there can exist partition cardinals above Θ. Here we show that if such cardinals exist far beyond Θ, then the HOD conjecture is false.

Theorem 5.1 (Goldberg–Blue)**.** *Suppose λ is an inaccessible limit of Lowenheim-Skolem cardinals and $\delta > \lambda$ satisfies $\delta \to (\delta)^\gamma$ for all $\gamma < \lambda$. Then there is a model of* ZFC *in which all regular cardinals are ω-strongly measurable in* HOD*.*

We prefer to prove the following theorem whose hypothesis is arguably better motivated:

Theorem 5.2 (Goldberg–Blue)**.** *Assume $\text{I}_0(\lambda)$ and that in $L(V_{\lambda+1})$, Dependent Choice holds and for all $\gamma < \lambda$, $\lambda^+ \to (\lambda^+)^\gamma$. Then for any limit of Lowenheim-Skolem cardinals γ of V_λ, either γ or γ^+ is measurable.*

Corollary 5.3. *Under the hypothesis of the previous theorem, the* HOD *conjecture is false.* □

The axiom I_0 is typically studied in the context of the Axiom of Choice. It is a conjecture of Woodin that ZFC plus I_0 implies that $L(V_{\lambda+1})$ satisfies $\lambda^+ \to (\lambda^+)^\alpha$ for all $\alpha < \omega_1$. On the other hand, assuming AC, $L(V_{\lambda+1})$ does not satisfy $\lambda^+ \to (\lambda^+)^{\omega_1}$, since this partition property implies $\mathbb{R}$ cannot be wellordered, whereas any wellorder of $\mathbb{R}$ in V is an element of $L(V_{\lambda+1})$.[1]

Could some choiceless extension of ZF + I_0-theory imply a structure theory even more closely analogous to that of $L(\mathbb{R})$? The theorem and corollary are a first step towards understanding this possibility.

We will use the following result of the author which is a consequence of Cramer's technique of inverse limit reflection in $L(V_{\lambda+1})$:

Theorem 5.4 (Goldberg, [2])**.** *Assume $\text{I}_0(\lambda)$. Suppose $L(V_{\lambda+1})$ satisfies* DC *and for some $\gamma < \lambda$, V_λ satisfies DC_γ. Then $L(V_{\lambda+1})$ satisfies DC_γ.* □

Sketch of Theorem 5.2. Suppose γ is a limit of Lowenheim-Skolem cardinals in V_λ. We will show that γ^+ is measurable. The key property of γ we will use is that one can force DC_γ over V_λ by a countably closed forcing $\mathbb{P} \in V_\lambda$ that preserves γ^+. Let $G \subseteq \mathbb{P}$ be a V-generic filter. Then $L(V_{\lambda+1})[G] = L(V[G]_{\lambda+1})$, $V[G]$ satisfies I_0, $L(V[G]_{\lambda+1})$ satisfies DC, and $V[G]_\lambda$ satisfies DC_γ. Therefore we can apply Theorem 5.4 to conclude that $L(V_{\lambda+1})[G]$ satisfies DC_γ.

We first show that there is no wellordered sequence $\langle A_\alpha \rangle_{\alpha<\gamma^+}$ of distinct subsets of γ. For this, we consider Moschovakis's generalized perfect set game. This is the ordinal game of length ω in which Players I and II alternate moves, with Player I playing ordinals less than γ and II playing either 0 or 1. At the end of a run, Player I has constructed $s \in \gamma^\omega$ and Player II has constructed $x \in 2^\omega$. Player I wins if there is some α such that x is the restriction of the characteristic function of A to ordinals in the range of s; more precisely, for all $n < \omega$, $x(n) = A_\alpha(s(n))$.

[1] A similar situation arises with the Ultrapower Axiom. In $L(V_{\lambda+1})$, the Ketonen order is semilinear in the sense that each rank of the order has size less than λ. It is natural to wonder whether $L(V_{\lambda+1})$ could satisfy UA itself, but this again is impossible assuming the Axiom of Choice.

If Player I has a winning strategy τ in this game, then there is an injection from 2^ω to γ^+ defined by sending $x \in 2^\omega$ to the least α such that for all $n < \omega$, $x(n) = A_\alpha((\tau * x)(n))$. Since $\lambda^+ \to (\lambda^+)^{\omega_1}$ implies $\mathbb{R}$ cannot be wellordered, Player I does not win this game.

This game is γ^+-Suslin. To see this, let $c_\alpha : \omega \to \{\alpha\}$ denote the constant function, and note that $B = \{(s, x, c_\alpha) : \forall n < \omega\, x(n) = A_\alpha(s(n))\}$ is a closed subset of $\gamma^\omega \times 2^\omega \times (\gamma^+)^\omega$ and $p(B)$ is the payoff set for Player I in the game of interest.

We now run the proof the determinacy of this game using that $\lambda^+ \to (\lambda^+)^{\gamma^+}$, which is based on Martin's proof of Π^1_1-determinacy. Consider the open auxiliary game in which Player I plays ordinals $s(n)$ less than γ while II responds with pairs $(x(n), f_n)$ where $x(n) \in \{0, 1\}$, $f_n : B_{s\restriction n, x\restriction n} \to \lambda^+$ is order-preserving in the Kleene–Brouwer order, and $f_n \supseteq f_{n-1}$ if $n > 0$.

If Player I has a winning strategy in the auxiliary game, then by using partition measures to integrate out the auxiliary moves as in Martin's proof, one shows that Player I wins the original game.

Therefore Player I does not win the auxiliary game. In this case, one would like to appeal to the Gale-Stewart theorem to assert that Player II wins the game. But since Player II's moves are drawn from a set that is not well-orderable, one can only conclude that Player II has a winning *quasi-strategy* in the auxiliary game.

For this reason, we move to $L(V_{\lambda+1})[G]$, where DC_γ holds and γ^+ is preserved. Since Player I plays ordinals less than γ, in the resulting extension, one can use DC_γ to thin out Player II's winning quasi-strategy to a winning strategy. It follows that in $L(V_{\lambda+1})[G]$, Player II has a winning strategy in the original game. (Note that since we have added no ω-sequences, we do not need to reinterpret the payoff set.)

By the usual argument from the perfect set theorem, this implies that in $L(V_{\lambda+1})[G]$, $|\{A_\alpha : \alpha < \gamma^+\}| \leq \gamma$. This contradicts that γ^+ is not collapsed in $L(V_{\lambda+1})[G]$.

Now assume γ is the least limit of supercompact cardinals in V_λ and suppose $\delta \in (\gamma, \lambda)$ is a regular cardinal. We will show that S^δ_ω is γ^+-unsplittable in V. Otherwise, suppose $\vec{A} = \langle A_\alpha \rangle_{\alpha < \gamma^+}$ partitions S^δ_ω into stationary sets. Then if κ is the least strongly compact cardinal of V_λ, $P_\kappa(\gamma^+) \cap \text{HOD}_{\vec{A}}$ is cofinal in $P_\kappa(\gamma^+)$ by the proof of Theorem 2.2. In particular, $P_\kappa(\gamma) \cap \text{HOD}_{\vec{A}}$ is cofinal in $P_\kappa(\gamma)$.

Since $P_\kappa(\gamma) \cap \text{HOD}_{\vec{A}}$ is well-orderable, we have $|P_\kappa(\gamma) \cap \text{HOD}_{\vec{A}}| = \gamma$. But this contradicts König's Theorem. To see this, let $Y \subseteq \gamma \times \gamma$ be such that for all $\tau \in P_\kappa(\gamma) \cap \text{HOD}_{\vec{A}}$, there is some $\alpha < \gamma$ such that $\tau = \{\beta < \gamma : (\alpha, \beta) \in Y\}$. Working in $L[Y]$, there is a cofinal subset of $P_\kappa(\gamma)$ of size γ, and this implies

$$L[Y] \vDash \gamma^+ \leq \gamma^{<\kappa} = |P_\kappa(\gamma)| = 2^{<\kappa} \cdot \text{cf}(P_\kappa(\gamma)) = \gamma$$

a contradiction.

Finally, suppose $\delta \in (\gamma, \lambda)$ is a limit of Lowenheim-Skolem cardinals. Then the closed unbounded filter on η is η-complete and one can run the argument of Lemma 2.1 (after forcing DC_{γ^+}) to obtain atoms for the ω-club filter. This yields the theorem. ☐

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