The uniqueness of elementary embeddings

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Outline

1. Conjecture: an elementary embedding of V is uniquely determined by its target model.
   - Proofs (from various hypotheses):
     - Anti-large cardinal assumptions.
     - Properties of HOD.
     - Cardinal arithmetic.
     - The Ultrapower Axiom.
2. Embeddings between levels of the cumulative hierarchy.
   - The HOD Conjecture as a uniqueness principle.
   - The failure of the HOD Conjecture as an embedding principle.
   - A new class of HOD-ultrafilters.

Embeddings of K

Theorem (Dodd-Jensen-Koepke-Mitchell)

Assume $0^{¶}$ does not exist. Then any elementary embedding from the core model K into an inner model is a normal iteration of K.

Since there is at most one normal iteration from K into a given inner model, the theorem has a fine structure-free consequence:

Corollary

Assume $0^{¶}$ does not exist. If $j_0, j_1 : V \to M$ are elementary embeddings, then $j_0 \restriction \mathrm{Ord} = j_1 \restriction \mathrm{Ord}$.

Embeddings of V

Conjecture

If $j_0, j_1 : V \to M$ are elementary embeddings, $j_0 \restriction \text{Ord} = j_1 \restriction \text{Ord}$.

- Equivalently, $j_0 \restriction \text{HOD} = j_1 \restriction \text{HOD}$, so one can think of uniqueness of embeddings as a K-like property of HOD.
- To what extent does the unconstrained large cardinal structure of V differ from the special case when K exists?
  - This sort of question turns out to hinge on the structure of HOD under large cardinal hypotheses, and in particular, on Woodin's HOD Conjecture.

Definable embeddings

Theorem (Woodin)

Suppose $j_0, j_1 : V \to M$ are elementary embeddings that are definable from parameters. Then $j_0 \restriction \mathrm{Ord} = j_1 \restriction \mathrm{Ord}$.

Proof.

- Assume the scheme fails for Σ_n-definable embeddings.
- Consider the least ordinal α such that there exist elementary embeddings $i_0, i_1 : V \to N$ that are Σ_n-definable from parameters with $i_0(\alpha) \neq i_1(\alpha)$.
- α is definable without parameters! Say by a formula φ.
- So $i_0(\alpha) = i_1(\alpha)$, since both are the unique ordinal satisfying φ in M. Contradiction. $\square$

Two embeddings into a single model

Proposition

It is consistent that there are distinct elementary embeddings from the universe of sets into a common inner model.

Sketch.

- Take U normal measure on κ, $j : V \to M$ the ultrapower.
- $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle_{\alpha \leq \kappa}$ Easton iteration adding a Cohen subset to every non-Mahlo inaccessible $\delta \leq \kappa$. Take V-generic $G \subseteq \mathbb{P}_\kappa$.
- For any $M[G]$-generic H for the tail forcing $(\mathbb{P}_{\kappa,j(\kappa)})^M$, j extends to $i : V[G] \to M[G * H]$ such that $i(G) = G * H$.
- Assume GCH in V, so $\left|P(\mathbb{P}_{\kappa,j(\kappa)}) \cap M[G]\right|^{V[G]} = \kappa^+$. Build $M[G]$-generics $H_0 \neq H_1 \subseteq \mathbb{P}_{\kappa,j(\kappa)}$ with $M[G * H_0] = M[G * H_1]$. Yields distinct $i_0, i_1 : V[G] \to N$. $\square$

The Ultrapower Axiom

An embedding $i : M \to N$ between transitive models is an *ultrapower* if i is the embedding associated to an ultrafilter of M.

Ultrapower Axiom (UA)

For any ultrapowers $j_0 : V \to M_0$ and $j_1 : V \to M_1$, there are ultrapowers $i_0 : M_0 \to N$ and $i_1 : M_1 \to N$ with $i_0 \circ j_0 = i_1 \circ j_1$.

$$\begin{tikzcd}
& N & \\
M_0 \arrow[ur, "i_0"] & & M_1 \arrow[ul, "i_1"'] \\
& V \arrow[ul, "j_0"] \arrow[ur, "j_1"'] &
\end{tikzcd}$$

The uniqueness of ultrapowers

Theorem (UA)

Suppose $j_0, j_1 : V \to M$ are ultrapower embeddings. Then $j_0 = j_1$.

Proof.

- Fix ultrapowers $i_0, i_1 : M \to N$ with $i_0 \circ j_0 = i_1 \circ j_1$.
- $M = H^M(j_n[V] \cup \text{Ord})$ so $i_n[M] = H^N(i_n \circ j_n[V] \cup i_n[\text{Ord}])$.
  - $i_0, i_1 : M \to N$ are definable over M, so $i_0 \restriction \text{Ord} = i_1 \restriction \text{Ord}$.
  - Thus $i_0 \circ j_0[V] \cup i_0[\text{Ord}] = i_1 \circ j_1[V] \cup i_1[\text{Ord}]$.
  - Therefore $i_0[M] = i_1[M]$, and the uncollapse map is $i_0 = i_1$.
- Since $i_0 \circ j_0 = i_1 \circ j_1$ and $i_0 = i_1$, $j_0 = j_1$. $\square$

HOD Hypotheses

- S^δ_ω denotes the set of countable cofinality ordinals below δ.
- A cardinal κ is "*not ω-strongly measurable in* HOD" if for all $\gamma < \kappa$, one of the following holds:
  - $(2^\gamma)^{\text{HOD}} \geq \kappa$.
  - There is an OD partition of S^κ_ω into γ-many stationary sets.

Definition (HOD Hypothesis)

There is a proper class of cardinals that are not ω-strongly measurable in HOD.

Definition (Strong HOD Hypothesis)

For every singular strong limit cardinal λ of uncountable cofinality, λ^+ is not ω-strongly measurable in HOD.

HOD Conjectures

The HOD Hypothesis is true if $V = \mathrm{HOD}$ (by the Solovay splitting lemma), so the hypothesis is consistent with all large cardinals. The real question is whether it is provable:

Definition (HOD Conjecture)

ZFC proves the HOD Hypothesis.

Perhaps a more plausible conjecture is that ZFC + some large cardinal axiom implies the HOD Conjecture.

Solovay's Lemma

Lemma (Solovay)

If $j : V \to M$ is elementary, δ is regular, $\langle S_\xi \rangle_{\xi < \gamma}$ is a stationary partition of S^δ_ω, $\langle T_\xi \rangle_{\xi < j(\gamma)} = j(\langle S_\xi \rangle_{\xi<\gamma})$, and $\delta_* = \sup j[\delta]$, then

$$j[\gamma] = \{\xi < j(\gamma) : T_\xi \cap \delta_* \textit{ is stationary}\}$$

Sketch.

- $j : \delta \to \delta_*$ induces a Rudin-Keisler equivalence between the ω-club filters $\mathcal{C}^\delta_\omega$ and $\mathcal{C}^{\delta_*}_\omega$ since j is one-to-one and $j[\delta] \in \mathcal{C}^{\delta_*}_\omega$.
  - Stationary subsets of S^κ_ω are positive sets for $\mathcal{C}^\kappa_\omega$.
  - So $T \subseteq \delta_* \cap S^{j(\delta)}_\omega$ is stationary iff $j^{-1}[T]$ is stationary.
- But $j^{-1}[T_\xi]$ is stationary iff $\xi = j(\bar{\xi})$ for some $\bar{\xi} < \gamma$:
  - If $\xi = j(\bar{\xi})$, $j^{-1}[T_\xi] = S_{\bar{\xi}}$. If $\xi \notin j[\gamma]$, $j^{-1}[T_\xi] = \emptyset$. □

Uniqueness of embeddings and the HOD Conjecture, I

Theorem (Woodin)

Assume that the Strong HOD *Hypothesis holds. Then for any* $j_0, j_1 : V \to M$, $j_0 \restriction \mathrm{Ord} = j_1 \restriction \mathrm{Ord}$.

Proof.

- Take strong limit singular λ fixed by j_0 and j_1, $\mathrm{cf}(\lambda) > \omega$.
- Fix $\gamma < \lambda$. Let $\langle S_\xi \rangle_{\xi<\gamma}$ be the $<_{\mathrm{OD}}$-least stationary partition of $S^{\lambda^+}_\omega$. Note that $j_0(\vec{S}) = j_1(\vec{S})$. Let $\vec{T} = j_n(\vec{S})$.
- $j_n[\gamma] = \{\xi < \gamma : T_\xi \text{ is stationary}\}$ by Solovay's Lemma. □

Corollary

If $V = \mathrm{HOD}$*, then for any* $j_0, j_1 : V \to M$, $j_0 = j_1$.

The uniqueness of embeddings under SCH

Theorem (SCH)

Suppose $j_0, j_1 : V \to M$ are elementary embeddings. Then $j_0 \restriction \mathrm{Ord} = j_1 \restriction \mathrm{Ord}$.

Idea.

- Consider the Skolem hull $X = H^M(j_0[V] \cup j_1[V])$.
  - Let $\pi : X \to N$ be the transitive collapse.
  - Let $i_0 = \pi \circ j_0$, $i_1 = \pi \circ j_1$.
  - Suffices to show that $i_0 \restriction \mathrm{Ord} = i_1 \restriction \mathrm{Ord}$.
- Using the SCH hypothesis, one can show that i_0 and i_1 are ultrapowers.
- So i_0 and i_1 are definable from parameters, and hence they agree on the ordinals by Woodin's theorem. $\square$

The uniqueness of embeddings under UA

Theorem

Assume the Ultrapower Axiom. Suppose $j_0, j_1 : V \to M$ are elementary embeddings. Then $j_0 = j_1$.

No restriction to the ordinals, no restriction to ultrapowers.

Proof assuming SCH.

By the proof of the previous theorem, the theorem reduces to the case that j_0 and j_1 are ultrapower embeddings, which we dispensed with earlier. $\square$

One eliminates the SCH hypothesis using heavier machinery from the analysis of strongly compact cardinals under UA.

Uniqueness of embeddings and the HOD Conjecture, II

Assuming large cardinal axioms, the HOD Hypothesis is equivalent to a different uniqueness property of elementary embeddings:

Theorem

Suppose κ is extendible. The following are equivalent:

- *The* HOD *Hypothesis.*
- *For all regular $\delta \geq \kappa$ and all sufficiently large $\lambda \geq \delta$, if $j_0, j_1 : V_\lambda \to V_{\lambda'}$ are elementary embeddings with $\sup j_0[\delta] = \sup j_1[\delta]$ and $j_0(\delta) = j_1(\delta)$, then $j_0 \restriction \delta = j_1 \restriction \delta$.*

In the forward direction, one uses Solovay's Lemma and the existence of ordinal definable stationary partitions. In the opposite direction, one uses uniqueness of embeddings to show roughly that HOD inherits too much of the large cardinal structure of V to be a small model.

Choiceless cardinals and HOD

Theorem (Woodin, ZF)

If there is a Reinhardt cardinal and an extendible cardinal, then the HOD *Conjecture is false.*

So large cardinals beyond choice imply the failure of the HOD Conjecture. The question roughly is whether this means:

1. The HOD Hypothesis is consistently false.
2. Berkeley cardinals are inconsistent, without assuming AC.

Rest of talk shows the HOD Hypothesis has consequences similar to those of the choiceless large cardinals.

Embeddings into HOD

- Let T_2 denote the Σ_2-theory of V with ordinal parameters.
- The structure (HOD, T_2) computes the theory of V.

Theorem

For any definable embeddings $j_0, j_1 : (\mathrm{HOD}, T_2) \to (M, S)$, $j_0 = j_1$.

A variant of the Kunen inconsistency states that if λ is inaccessible, there is no ω-continuous embedding $j : M \to V_\lambda$.

Theorem

Assume κ is extendible. Then exactly one of the following holds:

- *The* HOD *Hypothesis.*
- *For every inaccessible $\lambda > \kappa$, there is a transitive structure (M, S) of height λ, admitting a nontrivial ω-continuous elementary embedding $j : (M, S) \to (\mathrm{HOD} \cap V_\lambda, T_2 \cap V_\lambda)$.*

Equalizers

Suppose $V_\gamma \prec_{\Sigma_2} V$, $j_0, j_1 : V_\gamma \to V_{\gamma'}$ are elementary embeddings, δ is regular, $\sup j_0[\delta] = \sup j_1[\delta]$, and $j_0(\delta) = j_1(\delta)$.

Definition

The *equalizer of* j_0 *and* j_1 is $\text{Eq}(j_0, j_1) = \{x \in V_\gamma : j_0(x) = j_1(x)\}$.

- In V_γ:
  - $\text{Eq}(j_0, j_1)$ is definably closed.
  - $\text{Eq}(j_0, j_1)$ is closed under ω-sequences, and in fact, under $<\kappa$-sequences where $\kappa = \text{crit}(j)$.
- In particular, $\text{Eq}(j_0, j_1) \cap \text{HOD} \prec (\text{HOD} \cap V_\gamma, T_2 \cap V_\gamma)$.
  - Let $j : (M, S) \to (\text{HOD} \cap V_\gamma, T_2 \cap V_\gamma)$ be the inverse of the transitive collapse of $\text{Eq}(j_0, j_1) \cap \text{HOD}$.
  - Then $j(\delta) = \delta$ since $\sup j_0[\delta] = \sup j_1[\delta]$ and $j_0(\delta) = j_1(\delta)$.
  - If $j_0 \restriction \delta \neq j_1 \restriction \delta$, then j is a nontrivial elementary embedding.

The fixed point filter

Under choiceless large cardinal axioms, a filter called the *fixed point filter* is often more useful than the club filter.

Definition

Suppose $\kappa \leq \delta \leq \nu$ are ordinals below some cardinal γ. The *fixed point filter* associated to these ordinals is the filter generated by sets of the form

$$\mathsf{Fix}(j_\alpha)_{\alpha<\eta} = \{\xi < \delta : \forall \alpha < \eta \; j_\alpha(\xi) = \xi\}$$

where $\eta < \kappa$ and for all $\alpha < \eta$, $j_\alpha : V_\gamma \to V_\gamma$ is an elementary embedding fixing ν.

The fixed point filter, continued

A cardinal λ is a *revised Reinhardt cardinal* if for some elementary $j : V \to V$, λ is the least ordinal above $\text{crit}(j)$ with $j(\lambda) = \lambda$.

Theorem (NBG)

Suppose λ is a revised Reinhardt cardinal and λ-DC holds. Then every λ^+-complete ultrafilter on an ordinal is of the form $\mathcal{F} \restriction A$ for some fixed point filter $\mathcal{F}$.

This is used to prove the following theorem:

Theorem (NBG)

Suppose λ is a revised Reinhardt cardinal and λ-DC holds. Then every λ^+-complete filter on an ordinal extends to a λ^+-complete ultrafilter.

The equalizer filter

Assume κ is extendible, $\gamma \geq \kappa^+$, $j : V_\gamma \to V_{\gamma'}$ is elementary, p is a nonempty finite set of regular cardinals in the interval than (κ, γ).

- $\mathcal{K}_{j,p}$ denotes the set of elementary embeddings $k : V_\gamma \to V'_\gamma$ such that for all $\delta \in p$, $\sup k[\delta] = \sup j[\delta]$ and $k(\delta) = j(\delta)$.
- The *equalizer filter constrained by j on p*, denoted $\mathcal{EF}_{j,p}$, is the filter generated by sets of the form

$$\mathrm{Eq}(\sigma) = \{\xi < \max(p) : \forall k, \ell \in \sigma \;\; k(\xi) = \ell(\xi)\}$$

for all $\sigma \subseteq \mathcal{K}_{j,p}$ with $|\sigma| < \min p$.

Theorem (in this context)

Assume the HOD *Hypothesis is false. Then there is a ordinal definable partition $\mathcal{P}$ of* $\max(p)$ *of size less than* $\min(p)$ *such that for all $A \in \mathcal{P}$, $(\mathcal{EF}_{j,p} \cap \mathrm{HOD}) \restriction A$ is an ultrafilter in* HOD.

Thanks!