Ultrapowers and the approximation property

Gabriel Goldberg

UC Berkeley

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Outline

1. 10 minute verbatim recitation of Woodin's stump speech on the HOD Conjecture.
2. Entire PhD thesis... in a single slide.
   - But first: what is **the Ultrapower Axiom?**
   - And more important still: what is it **not?**
3. A prediction: some of my thesis results can be proved from large cardinals alone.
4. Confirmation of prediction, consisting of...
   - Vintage 1970s style ultrafilter combinatorics.
   - An inconsistent large cardinal axiom.
   - A different inconsistent large cardinal axiom... and its relationship with vintage 1970s style ultrafilter combinatorics.
   - Also in between, an inner model called HCD.
5. A whole slide just to say "Thanks!"

Theorem (L Dichotomy, Jensen)

One of the following holds:

1. *Every uncountable set of ordinals can be covered by a constructible set of the same cardinality.*
2. *Every uncountable cardinal is inaccessible in L.*

Theorem (HOD Dichotomy, Woodin)

Suppose δ is extendible. Then one of the following holds.

1. *For every cardinal $\kappa \geq \delta$, every set of ordinals of cardinality less than κ can be covered by an ordinal definable set of cardinality less than κ.*
2. *Every regular cardinal $\kappa \geq \delta$ is measurable in* HOD.

Comparing L and HOD

The covering property in these dichotomies is called the *Jensen covering property* (past $\aleph_2$ and δ). Woodin's HOD *conjecture* asserts that ZFC proves that HOD has the Jensen covering property past the least extendible cardinal.

- Constructibility is absolute but ordinal definability is not. For this reason $V = L$ answers essentially all set-theoretic questions, while $V = \text{HOD}$ answers essentially none.
- Scott's theorem: large cardinals imply that L does not have the Jensen covering property.
- All known large cardinal axioms that are consistent with ZFC are consistent with the axiom $V = \text{HOD}$.

The inner model problem

- The inner model problem (IMP) for a large cardinal axiom A: define a generalization L_{A} of L in which A holds.
  - Tension: the axiom $V = L_{\mathrm{A}}$ must be absolute enough to answer all set-theoretic questions, yet permissive enough to be consistent with A.
- Amazingly, the IMP has been solved for measurable cardinals, strong cardinals, Woodin cardinals, and more.
  - Kunen, Silver, Solovay, Mitchell, Martin, Steel, Jensen, Neeman, Sargsyan, Trang, Woodin...
- The IMP for supercompact cardinals is open.

The inner model problem for supercompacts

Let S = "there is a supercompact cardinal."

- For every large cardinal axiom A such that L_{A} is known, all stronger large cardinal axioms refute $V = L_{\mathrm{A}}$.
- Woodin's work suggests that if the inner model problem for S can be solved, then every large cardinal axiom is consistent with $V = L_{\mathrm{S}}$.
  - The hypothetical model L_{S} is called Ultimate L.
  - Reason for ultimateness: one can encode all large cardinals into supercompactness measures that descend to Ultimate L, so *all* large cardinals descend to Ultimate L.
- The *Ultimate L conjecture* is a precise test question approximating the vague speculation above.
  - If the Ultimate L conjecture is provable, then the HOD conjecture is true.

On what the Ultrapower Axiom is not

Suitable Extender Models II: Beyond ω-Huge 409

Definition 183 (Ultrafilter Axiom at $\boldsymbol{\lambda}$). Suppose that there is a proper class of Woodin cardinals, λ is a limit of supercompact cardinals, and there is a proper elementary embedding,

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1}).$$

Then in $L_\lambda(H(\lambda^+))$ for each regular (infinite) cardinal $\gamma < \lambda$, the club filter on λ^+ is an ultrafilter restricted to $\{\eta < \lambda^+ \mid \operatorname{cof}(\eta) = \gamma\}$.

Figure: Not the Ultrapower Axiom

Ultrapowers

If U is an ultrafilter, M_U denotes the ultrapower of the universe of sets by U and $j_U : V \to M_U$ denotes the associated ultrapower embedding.

- If U is countably complete, then M_U is a wellfounded extensional structure, and we identify it with its Mostowski collapse, which is an inner model of ZFC.
- An inner model obtained in this way is called an *ultrapower*.
    - If M is a model of ZFC, an *internal ultrapower of M* is a definable inner model N of M such that M satisfies "N is an ultrapower."
- Similar convention for (internal) ultrapower embeddings.

The Ultrapower Axiom

- The Weak Ultrapower Axiom: any two ultrapowers of V have a common internal ultrapower.
- The Ultrapower Axiom (UA): the Weak Ultrapower Axiom plus the statement that for any ultrapower M, there is a unique ultrapower embedding from V to M.
- Equivalently: the **category** of ultrapowers and internal ultrapower embeddings is filtered.
  - It actually turns out to form a lattice!

Ubiquitous diamond diagram

$$\begin{tikzcd}
M_0 & & M_1 \\
& V \arrow[ul, "j_0"] \arrow[ur, "j_1"'] &
\end{tikzcd}$$

Ubiquitous diamond diagram

$$
\begin{tikzcd}
& N & \\
M_0 \arrow[ur, "i_0"] & & M_1 \arrow[ul, "i_1"'] \\
& V \arrow[ul, "j_0"] \arrow[ur, "j_1"'] &
\end{tikzcd}
$$

The Ultrapower Axiom, continued

- The Ultrapower Axiom holds in all canonical inner models .
  - So if the Ultimate L conjecture is true, then the Ultrapower Axiom should be consistent with all large cardinals.
- UA is vacuously true, and therefore useless, if there are no measurable cardinals. But UA becomes very powerful in the context of large cardinals:
  - UA answers lots of questions about the structure of the category of ultrapowers.
  - If there is a strongly compact cardinal κ, one can encode complicated set theoretic structures into this category.
  - Then using UA one can solve classical set theoretic questions.

The Ultrapower Axiom, continued, continued

Theorem (UA)

Suppose κ is strongly compact. Then:

- *The Generalized Continuum Hypothesis holds at all cardinals $\lambda \geq \kappa$.*
- *V is a generic extension of* HOD *for a forcing of size $\leq \kappa^{++}$.*
  - HOD *has the Jensen covering property beyond κ.*
- *For all cardinals $\lambda \geq \kappa$, $\diamondsuit(S^{\lambda^{++}}_{\lambda^+})$ holds definably over $H_{\lambda^{++}}$.*
  - *In particular, $H_{\lambda^{++}}$ is definably wellordered in ordertype λ^{++}.*
- *The Mitchell order is linear on the class of normal fine ultrafilters.*
- *Either κ is supercompact or κ is a limit of supercompacts.*

Traces of UA in V

- Assuming the Ultimate L conjecture, there should be an inner model of UA that inherits all large cardinals above the least extendible cardinal δ.
  - In particular, it sees all δ-complete ultrafilters.
  - And it sees their associated embeddings.
- As a consequence, the structure of very large cardinals in V should to some extent resemble the structure imposed by the Ultrapower Axiom.

Prediction

Certain consequences of UA are provable from large cardinals.

The rest of the talk consists of some results confirming this prediction.

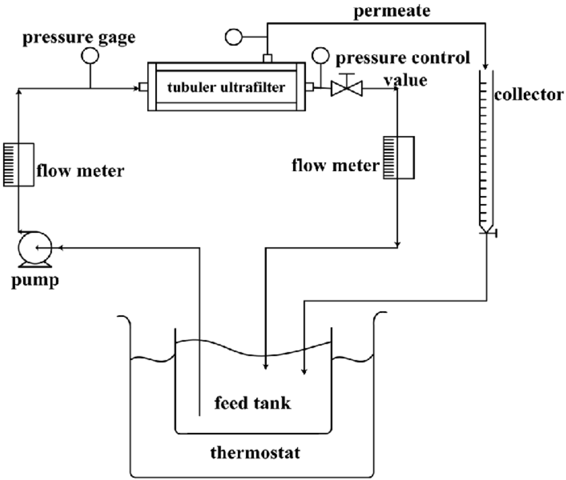


Figure: Consider a tubuler ultrafilter for your feed tank.

Indecomposable ultrafilters

Suppose U is an ultrafilter.

- U is *uniform* if every set in U has the same cardinality.
- A *U-cover* is a family of sets whose union belongs to U.
- If $\kappa \leq \delta$ are cardinals, then U is *(κ, δ)-indecomposable* if every U-cover of cardinality less than δ has a U-subcover of cardinality less than κ.

Definition

A uniform ultrafilter on a cardinal λ is said to be *indecomposable* if it is (ω_1, λ)-indecomposable.

Silver's question

Note that an ultrafilter is λ-complete if and only if it is (ω, λ)-indecomposable or equivalently $(2, \lambda)$-indecomposable.

Question (Silver)

If an inaccessible cardinal carries an indecomposable ultrafilter, must it be measurable?

- If λ is the supremum of countably many measurable cardinals, λ carries an indecomposable ultrafilter: fix U uniform on ω, $(\lambda_n)_{n<\omega}$ cofinal in λ, W_n's λ_n-complete on λ, and let

$$W = \int_{n\in\omega} W_n \, dU$$

- Every measurable cardinal carries indecomposable ultrafilters that are not countably complete.

The independence of Silver's question

- Silver, Jensen, Koppelberg, and finally Donder showed that the consistency strength of an indecomposable ultrafilter is exactly one measurable cardinal.
  - The proofs use variants of Jensen's principle $\square$, which is strange given our use of strongly compacts below.
  - Probably the proof generalizes to show that in the known canonical inner models, the answer to Silver's question is yes.
- By forcing, Sheard built a model where the answer is no.
  - In Sheard's model, there is an indecomposable ultrafilter on an inaccessible cardinal that is not even weakly compact.

Silver's question from large cardinals

UA resolves the countably complete version of Silver's question:

Theorem (UA)

Suppose κ is a cardinal and $\lambda > 2^\kappa$ carries a countably complete (κ, λ)-indecomposable ultrafilter. Then λ is the supremum of fewer than κ measurable cardinals.

Large cardinals resolve Silver's question in **exactly** the same way:

Theorem (An indecomposable ultrafilter theorem)

Suppose $\kappa \leq \lambda$ are cardinals with a strongly compact cardinal strictly between them and λ carries a (κ, λ)-indecomposable ultrafilter. Then λ is the supremum of fewer than κ measurable cardinals.

Proving an indecomposable ultrafilter theorem

Theorem (Approximation property for ultrapowers)

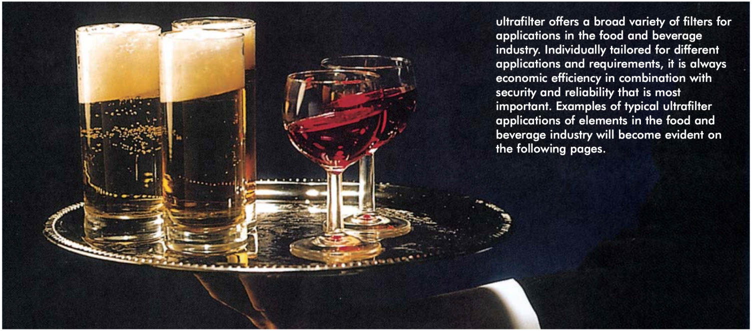
Suppose δ is strongly compact and U is an ultrafilter in V_δ. Then every M_U-δ-complete M_U-ultrafilter belongs to M_U.

Woodin and Usuba independently proved a weaker version of this theorem assuming U is countably complete, δ is supercompact, and the M_U-ultrafilter is truly δ-complete.

Theorem (Silver's factorization theorem)

Suppose κ is regular, $\lambda > 2^\kappa$, and W is a (κ, λ)-indecomposable ultrafilter. Then there is an ultrafilter U on a cardinal less than κ and an M_U-λ-complete M_U-ultrafilter W^* such that $j_W = j_{W^*} \circ j_U$.

ultrafilter in the Beverage Industry



Cardinal preserving embeddings

- An elementary embedding $j : V \to M$ is said to be *cardinal preserving* if M is an inner model that correctly computes the class of cardinals.
- Washed out version of Reinhardt's elementary embedding from V to V, the large cardinal axiom that was famously refuted by Kunen.

Question (Caicedo)

Can there be a nontrivial cardinal preserving embedding?

The question is still open.

The Kunen inconsistency

Suppose M and N are transitive sets. The *critical sequence* of an elementary embedding $j : M \to N$ is the maximum sequence $\langle \kappa_n(j) : n < m \rangle$ with $m \leq \omega + 1$ satisfying:

$$\kappa_n(j) = \begin{cases} \text{crit}(j) & \text{if } n = 0 \\ j(\kappa_{n-1}(j)) & \text{if } 0 < n < \omega \\ \sup_{n<\omega} \kappa_n(j) & \text{if } n = \omega \end{cases}$$

If $\lambda = \kappa_\omega(j)$ exists and is a continuity point of j, for instance if $\text{cf}^M(\lambda) = \omega$, λ is the least fixed point of j above its critical point.

Theorem (Kunen)

Suppose $j : V \to M$ and $\lambda = \kappa_\omega(j)$. Then $V_{\lambda+1} \not\subseteq M$.

Cardinal preserving embeddings and UA

The similarity between cardinal preserving embeddings and Reinhardt embeddings is more than just an analogy:

Theorem (UA)

Suppose δ is an ordinal and $j : V_\delta \to M$ is an elementary embedding such that $\text{Ord}^M = \delta$ and $\text{Card}^M = \text{Card} \cap \delta$. Let $\kappa = \text{crit}(j)$ and $\lambda = \kappa_\omega(j)$. Then $V_\lambda \subseteq M$ and $\delta < \lambda^{+\kappa}$.

This theorem is best possible assuming the consistency of a Σ_1-elementary embedding from $V_{\lambda+1}$ to $V_{\lambda+1}$ (i.e., Axiom I_2.)

Corollary (UA)

There are no cardinal preserving embeddings.

Cardinal preserving embeddings vs strongly compacts

Strongly compact cardinals simulate UA and prove the same result:

Theorem

Assume there is a proper class of strongly compact cardinals. Then there are no cardinal preserving embeddings.

Theorem (A cardinal incorrectness theorem)

Suppose $j : V \to M$ is an elementary embedding with critical point κ and λ is a limit of strongly compact cardinals that is fixed by j. Then $(\lambda^{+\kappa+2})^M$ is not a cardinal.

Necessarily M correctly computes all cardinals in the interval $[\lambda, \lambda^{+\kappa+1}]$. NB: if $j : V \to M$ is cardinal preserving, then $\mathrm{crit}(j)$ is λ-strongly compact where $\lambda = \kappa_\omega(j)$.

Indecomposability and cardinal preserving embeddings

An indecomposable ultrafilter theorem actually implies a cardinal incorrectness theorem.

Sketch of a cardinal incorrectness theorem.

Assume towards a contradiction that $(\lambda^{+\kappa+2})^M$ is a cardinal.

- M is cardinal correct in $[\lambda, \lambda^{+\kappa+1}]$, so $(\lambda^{+\kappa+2})^M = \lambda^{+\kappa+2}$. Therefore $j(\lambda^{+\kappa}) = (\lambda^{+j(\kappa)})^M > \lambda^{+\kappa+2}$.
- Let $Y = \bigcup\{j(A) : A \in [\lambda^{+\kappa}]^{<\lambda}\}$. So

$$|Y| = \lambda \cdot (\lambda^{+\kappa})^{<\lambda} = (\lambda^{+\kappa})^{<\lambda} = \lambda^{+\kappa+1}$$

using Solovay's theorem SCH holds above a strongly compact.

Sketch of a cardinal incorrectness theorem, continued.

- Fix $\xi \in j(\lambda^{+\kappa}) \setminus Y$, and let $U = \{A \subseteq \lambda^{+\kappa} : \xi \in j(A)\}$.
- U is a uniform ultrafilter: Otherwise there is an $A \in U$ with $|A| < \lambda^{+\kappa}$, but then $\xi \in j(A) \subseteq Y$, contradiction.
- U is $(\lambda, \lambda^{+\kappa})$-indecomposable: Assume not, towards a contradiction.
  - Let $\gamma < \kappa$ be the least cardinal such that there is a U-cover $(A_\alpha)_{\alpha<\lambda^{+\gamma}}$ that has no U-subcover of cardinality less than λ.
    - $(A_\alpha)_{\alpha<\lambda^{+\gamma}}$ has no U-subcover of cardinality less than $\lambda^{+\gamma}$.
  - Let $(B_\alpha)_{\alpha<\lambda^{+\gamma}} = j((A_\alpha)_{\alpha<\lambda^{+\gamma}})$.
  - $\bigcup_{\alpha<\lambda^{+\gamma}} A_\alpha \in U$, so $\xi \in j(\bigcup_{\lambda^{+\gamma}} A_\alpha) = \bigcup_{\alpha<\lambda^{+\gamma}} B_\alpha$.
  - Let $\nu = \min\{\alpha : \xi \in B_\alpha\}$. Then $\xi \in \bigcup_{\alpha\leq\nu} B_\alpha \subseteq j(\bigcup_{\alpha\leq\nu} A_\alpha)$. Hence $\bigcup_{\alpha\leq\nu} A_\alpha \in U$, contradiction.

Sketch of a cardinal incorrectness theorem, continued, continued.

- Using the fact that $\lambda^{\aleph_0} = \lambda^+$, one can show U must in fact be $(\rho, \lambda^{+\kappa})$-indecomposable for some $\rho < \lambda$. Details are a trade secret.
- There is a strongly compact cardinal between ρ and $\lambda^{+\kappa}$, and using the indecomposable ultrafilter theorem, it follows that $\lambda^{+\kappa}$ is measurable or a limit of measurables, contradiction.

$\square$

A wild ground appeared!

An inner model M of ZFC is a *ground* if there is a partial order $\mathbb{P} \in M$ and an M-generic filter $G \subseteq \mathbb{P}$ such that $V = M[G]$.

Theorem (UA)

Suppose κ is strongly compact. Then HOD *is a ground.*

A set is *κ-completely definable*, or $\mathrm{CD}(\kappa)$, if it is definable from a κ-complete ultrafilter on an ordinal. $\mathrm{HCD}(\kappa)$ = hereditarily $\mathrm{CD}(\kappa)$.

$$V = \mathrm{HCD}(\omega) \supseteq \mathrm{HCD}(\omega_1) \supseteq \cdots \supseteq \mathrm{HCD}(\infty) = \mathrm{HOD}$$

Theorem

Suppose κ is strongly compact. Then $\mathrm{HCD}(\kappa)$ is a ground. If κ is supercompact, then κ is supercompact in $\mathrm{HCD}(\kappa)$.

The Ketonen order

Suppose δ is an ordinal.

- A homomorphism $h : P(\delta) \to P(\delta)$ is *Lipschitz* if for all $A \in P(\delta)$, $\min(A) \leq \min(h(A))$.
- The *Ketonen order* (at δ) is defined on countably complete ultrafilters on δ by setting $U \leq_{\Bbbk} W$ if there is a countably complete Lipschitz $h : P(\delta) \to P(\delta)$ with $h^{-1}[W] = U$.

Theorem (ZF + DC)

The Ketonen order is a wellfounded partial order, and UA *holds if and only if it is linear at every ordinal.*

The choiceless cardinals

- Dropping AC, one can study *choiceless large cardinals* stronger than the ones that Kunen refuted in ZFC.
- If $j : V \to V$, $\kappa_\omega(j)$ is a *revised Reinhardt cardinal*.
- $\mathcal{U}(\delta, \xi) =$ set of ultrafilters on δ of $\leq_{\Bbbk}$-rank ξ.
  - UA holds iff $|\mathcal{U}(\delta, \xi)| \leq 1$ for all δ and ξ.

Theorem

Suppose λ is a revised Reinhardt cardinal and λ-DC holds. Then

- *For all ordinals δ and ξ, $|\mathcal{U}(\delta, \xi)| < \lambda$.*
- *Every λ^+-complete filter extends to a λ^+-complete ultrafilter.*
- *For any λ^+-complete ultrafilter U on an ordinal, $U \cap \mathrm{HOD} \in \mathrm{HOD}$.*

Thanks!

References I