The Ultrapower Axiom from Determinacy

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Introduction

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This connection has been the subject of half a century of intense study in set theory. The history is barely covered in this talk.

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But UA does not refer to these specific models and can instead be studied abstractly in various contexts.

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So the measures supplied by AD carry the same rigid structure first observed in canonical models of ZFC.

Determinacy

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I wins a run of G_A if the sequence $x = \langle x_n : n < \omega \rangle$ belongs to A.

Determinacy and regularity properties

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Philosophy: AD is the theory of definable sets of reals.

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The proof of Martin's theorem involves the "large cardinal theory of small cardinals."

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Theorem (Solovay, 1967)

Assuming ZF + AD, $\aleph_1$ *and* $\aleph_2$ *are measurable.*

Solovay shows the closed unbounded filter induces a measure on $\aleph_1$, called the *club measure*:

$$\nu_{\text{club}}(A) = \begin{cases} 1 & \text{if } A \text{ contains a closed unbounded set} \\ 0 & \text{otherwise} \end{cases}$$

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Notation: The ultrapower of a structure $\mathcal{M}$ by a measure μ is denoted $\mathcal{M}_\mu = \mathcal{M}^\kappa/\mu$. The ultrapower embedding is denoted

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Theorem (Kunen)

Assume ZF + AD. *For* $n \geq 1$, $(\aleph_n)_\nu = \aleph_{n+1}$ *where* $\nu = \nu_{club}$.

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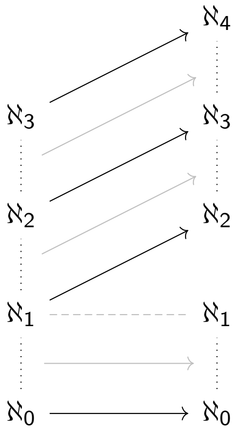

Figure: The ultrapower of the cardinals by the club measure.

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Theorem (Jackson)

Under ZF + AD, *the first eight infinite regular cardinals are*

$$\aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \aleph_{\omega+1} \quad \aleph_{\omega+2} \quad \aleph_{\omega\cdot 2+1} \quad \aleph_{\omega^\omega+1} \quad \aleph_{\omega^{\omega^\omega}+1}$$

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Theorem (Moschovakis, 1970)

Under ZF + AD*,* Θ *is a limit of weakly inaccessible cardinals.*

What is the structure of cardinals below Θ? To answer this, we'd need a *global classification of measures and their ultrapowers*.

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UA holds in all these models and provides an apparently complete picture of the behavior of measures within them.

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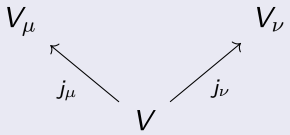

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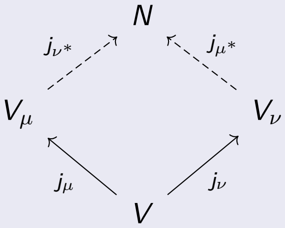

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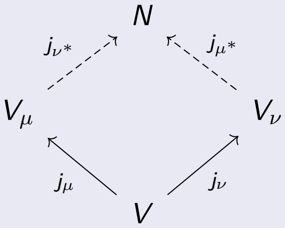

Informally: any two ultrapowers have a common ultrapower.

The complexity hierarchy of measures

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The following are equivalent:

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Conclusion: Under UA, the measures on a cardinal are classified by ordinal invariants.

The theorem

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Still we have our classification of measures:

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- A realizability lemma for Woodin's ultrapowers using the theory of precipitous ideals.
- The proof that large cardinals imply the existence of inner models with Woodin cardinals.

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In any case, we have a surprising connection between canonical models of ZFC and models of determinacy which will hopefully shed light on both subjects.

Thanks!