Ultrafilters and definability

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Outline

Main question: what is the relationship between V and HOD under large cardinal hypotheses?

1. The Ultrapower Axiom and ordinal definability
2. Definability from ultrafilters
3. Uniqueness of elementary embeddings:
   - Consequence of $V = \mathrm{HOD}$: *elementary embeddings of V are uniquely determined by their target models.*
   - Assuming large cardinals, we show: *elementary embeddings of V with sufficiently large critical points are uniquely determined by their target models.*

The Ultrapower Axiom

An elementary embedding $i : M \to N$ between transitive models is an *ultrapower embedding* if i is given by an ultrafilter of M.

Ultrapower Axiom (UA)

For any ultrapower embeddings $j_0 : V \to M_0$ and $j_1 : V \to M_1$, there are ultrapower embeddings $i_0 : M_0 \to N$ and $i_1 : M_1 \to N$ with $i_0 \circ j_0 = i_1 \circ j_1$.

$$
\begin{tikzcd}
 & N & \\
M_0 \arrow[ur, "i_0"] & & M_1 \arrow[ul, "i_1"'] \\
 & V \arrow[ul, "j_0"] \arrow[ur, "j_1"'] &
\end{tikzcd}
$$

Wellordering ultrafilters

Definition (UA)

If U_0 and U_1 are countably complete ultrafilters on ordinals, set $U_0 \leq_{\Bbbk} U_1$ if there exist ultrapower embeddings $i_0 : M_{U_0} \to N$ and $i_1 : M_{U_1} \to N$ such that $i_0 \circ j_{U_0} = i_1 \circ j_{U_1}$ and $i_0([\mathrm{id}]_{U_0}) \leq i_1([\mathrm{id}]_{U_1})$.

A special case of this order was first discovered by Ketonen.

Theorem (UA)

For any ordinal δ, $\leq_{\Bbbk}$ wellorders the set of countably complete ultrafilters on δ.

The ordinal definability of ultrafilters

Theorem (UA)

Every countably complete ultrafilter on an ordinal is ordinal definable.

- Unexplained coincidence: also a consequence of AD + DC (for ultrafilters on ordinals less than Θ).
- Assuming AC, if *every* ultrafilter on an ordinal is ordinal definable, then $V = \mathrm{HOD}$.

With large cardinals

Question

Is UA consistent with a strongly compact cardinal?

Recall that an inner model M of ZFC is a *ground of V* if there is a partial order $\mathbb{P} \in M$ and an M-generic filter $G \subseteq \mathbb{P}$ such that $V = M[G]$.

Theorem (UA)

Assume there is a strongly compact cardinal. Then HOD *is a ground of V.*

We first give a (very easy) proof assuming a supercompact.

UA + supercompact $\implies$ HOD is a ground

Theorem (Vopenka)

For any set of ordinals A, HOD is a ground of HOD_A.

This reduces the theorem to the following proposition.

Proposition (UA)

Suppose κ is supercompact and A is a set of ordinals such that $V_\kappa \subseteq \mathrm{HOD}_A$. Then $V = \mathrm{HOD}_A$.

Proof.

Fix $\lambda \geq \kappa$. Take an ultrafilter U such that $V_\lambda \subseteq \mathrm{Ult}(V_\kappa, U)$. Then $V_\lambda \subseteq j_U(\mathrm{HOD}_A) \subseteq \mathrm{HOD}_A$ since U is OD. $\square$

The HOD Conjecture and the UA Conjecture

Conjecture (HOD Conjecture)

If κ is extendible, then HOD *correctly computes all successors of singular cardinals greater than κ.*

Conjecture (UA Conjecture)

If κ is strongly compact, there is an inner model M of ZFC + UA *such that M satisfies "U is a κ-complete ultrafilter on X" if and only if $U = W \cap M$ for some κ-complete ultrafilter W on X.*

Theorem

The UA *Conjecture implies the* HOD *Conjecture.*

The Weak Ultrapower Axiom

The Ultrapower Axiom can be decomposed into two principles:

- **Weak UA:** Any two ultrapowers of V have a common ultrapower.
- **Uniqueness of Ultrapower Embeddings:** An inner model M admits at most one ultrapower embedding $j : V \to M$.
  - Follows from $V = \text{HOD}$ and drives the proof that HOD is a ground of V under UA + a supercompact.
  - In proof that UA holds in canonical inner models, one uses $V = \text{HOD}$ to derive uniqueness.
  - Are the largeness properties of HOD under UA circular?

Question

Assume Weak UA plus a supercompact cardinal. Must V be a generic extension of HOD?

Completely definable sets

Definition

A set x is *κ-completely definable* if it is definable from a κ-complete ultrafilter on an ordinal.

- $\mathrm{CD}(\kappa)$ denotes the class of κ-completely definable sets.
- $\mathrm{HCD}(\kappa)$ denotes the union of all transitive sets $M \subseteq \mathrm{CD}(\kappa)$.

Equivalently, $\mathrm{CD}(\kappa)$ is the class of sets ordinal definable from ultrapower embeddings with critical point at least κ.

$$V = \mathrm{HCD}(\omega) \supseteq \mathrm{HCD}(\omega_1) \supseteq \cdots \mathrm{HCD}(\infty) = \mathrm{HOD}$$

Hereditarily completely definable sets

Theorem

If κ is strongly compact then $\mathrm{HCD}(\kappa)$ is a ground of V.

- If $g \subseteq \omega$ is Cohen generic over V, then $(\mathrm{HCD}(\kappa))^{V[g]} \subseteq V$ for any uncountable κ, so $\mathrm{HCD}(\kappa)$ can be a *nontrivial* ground.
- Logic showing that HOD is a ground under UA plus a supercompact κ almost shows $\mathrm{HCD}(\kappa)$ is a ground without the assumption of UA.
- **Issue:** does $\mathrm{HCD}(\kappa)$ satisfy the Axiom of Choice?

The Axiom of Choice in HCD(κ)

Theorem

Suppose κ is strongly compact. Then HCD(κ) *satisfies the Axiom of Choice.*

Key fact: If $\mathcal{W}$ on $P_\kappa(P(\lambda))$ is fine, then every κ-complete ultrafilter on λ is of the form $\{A \subseteq \lambda : \alpha \in j_{\mathcal{W}}(A)\}$.

Proof.

- Suffices to show that for any set S of κ-complete ultrafilters on ordinals, there is a κ-completely definable wellorder of S.
- Let λ be such every ultrafilter in S lies on an ordinal less than λ. and let $\mathcal{W}$ be a κ-complete fine ultrafilter on $P_\kappa(P(\lambda))$.
  - For every $U \in S$, let α_U be least such that $U = \{A \subseteq \lambda : \alpha_U \in j_{\mathcal{W}}(A)\}$.
  - The order defined by $U < W$ if $\alpha_U < \alpha_W$ is in CD(κ). □

Uniqueness of elementary embeddings

Suppose M is an inner model. How many elementary embeddings from V to M can exist?

Proposition

Letting κ be the least measurable cardinal, it is consistent that there is an inner model M such that $\mathrm{Ult}(V, U) = M$ for 2^{2^κ}-many distinct normal ultrafilters U on κ.

- The model is obtained by Kunen-Paris forcing.
- The embeddings all lift a single elementary embedding of the ground model.
- In particular, they agree on the ordinals and hence they agree on HOD.

Uniqueness of elementary embeddings, continued

Theorem (Woodin)

If $j_0, j_1 : V \to M$ are definable embeddings, then $j_0 \restriction \mathrm{HOD} = j_1 \restriction \mathrm{HOD}$.

Theorem (Eventual SCH)

If $j_0, j_1 : V \to M$ are elementary embeddings, then $j_0 \restriction \mathrm{HOD} = j_1 \restriction \mathrm{HOD}$.

Theorem (UA or $V = \mathrm{HOD}$)

For any inner model M, there is at most one elementary embedding from V to M.

Uniqueness of embeddings measures the closeness of V and HOD.

Uniqueness of elementary embeddings above an extendible

Our main theorem is that uniqueness of embeddings holds for all elementary embeddings with large enough critical points:

Theorem

Suppose there is a proper class of strongly compact cardinals, κ is extendible, and $j_0, j_1 : V \to M$ are elementary embeddings with $\text{crit}(j_0) > \kappa$. Then $j_0 = j_1$.

In other words: elementary embeddings with sufficiently large critical point are uniquely determined by their target models.

Elementary embeddings and set theoretic geology

Theorem

Suppose $j_0, j_1 : V \to M$ are the ultrapowers associated to ultrafilters in V_κ where κ is strongly compact. Then there is a ground $N \subseteq V$ such that $j_0 \restriction N = j_1 \restriction N$.

In fact, the following theorem implies one can take $N = \mathrm{HCD}(\kappa)$:

Theorem

Suppose $j_0, j_1 : V \to M$ are the ultrapowers associated to ultrafilters in V_κ. Then $j_0 \restriction \mathrm{CD}(\kappa) = j_1 \restriction \mathrm{CD}(\kappa)$.

Proof.

It suffices to show that if $i : V \to P$ is a an ultrapower embedding with $\mathrm{crit}(i) \geq \kappa$, then $j_0(i) = j_1(i)$. But by Kunen's Commuting Ultrapowers Lemma, $j_0(i) = i \restriction M = j_1(i)$. □

Usuba's Theorem

For any cardinal κ, a *κ-ground* is an inner model M of ZFC such that $V = M[G]$ for an M-generic filter G on a partial order $\mathbb{P} \in M$ with $|\mathbb{P}| < \kappa$.

Theorem (Usuba)

Suppose κ is an extendible cardinal. Then every ground is a $\beth_\omega(\kappa)$-ground.

- Combined with another result of Usuba, this implies there is a *minimum* ground. In fact, this ground is the intersection of all κ-grounds.
- $\beth_\omega(\kappa)$ is not optimal, but it open whether every ground must be a κ-ground if κ is extendible.

Reduction from arbitrary embeddings to ultrapowers

Theorem (Eventual SCH)

If $j_0, j_1 : V \to M$ are elementary embeddings, then one can find an elementary embedding $k : P \to M$ and ultrapower embeddings $i_0, i_1 : V \to P$ such that $j_0 = k \circ i_0$ and $j_1 = k \circ i_1$.

Since strongly compact cardinals yield the eventual SCH, to prove the uniqueness of elementary embeddings with critical point above the least extendible cardinal κ, it suffices to prove the uniqueness of *ultrapower embeddings* with critical point above κ.

Uniqueness of ultrapower embeddings above an extendible

Proposition

If there is a proper class of strongly compacts, κ is extendible, $j_0, j_1 : V \to M$ are ultrapowers, and $\mathrm{crit}(j_0) > \kappa$, then $j_0 = j_1$.

Proof.

- Fix a strongly compact δ past the underlying sets of ultrafilters inducing j_0 and j_1.
- j_0 and j_1 agree on $\mathrm{HCD}(\delta)$.
- By Usuba's Theorem, $\mathrm{HCD}(\delta)$ is a $\beth_\omega(\kappa)$-ground of V.
- Since $\mathrm{crit}(j_0) > (2^\kappa)^+$, the Lévy-Solovay theorem implies that $j_0 \restriction \mathrm{HCD}(\delta)$ lifts uniquely to an elementary embedding of V.
- Since j_0 and j_1 lift $j_0 \restriction \mathrm{HCD}(\delta)$, $j_0 = j_1$. □

HOD under Weak UA

Theorem (Weak UA)

If there is a proper class of supercompact cardinals and κ is extendible, then HOD *is a ground of V.*

- **Issue:** given ultrapowers $j_0 : V \to M_0$ and $j_1 : V \to M_1$ with critical point above κ, Weak UA yields ultrapowers $i_0 : M_0 \to N$ and $i_1 : M_1 \to N$, but it is not clear their critical points must lie above κ, so one cannot get $i_0 \circ j_0 = i_1 \circ j_1$.
- **Idea:** prove that the Mitchell order is linear on κ^+-complete supercompactness measures using argument for linearity of the Mitchell order under UA combined with uniqueness of elementary embeddings with critical point above κ. This implies κ^+-complete supercompactness measures are OD, which is enough to conclude $\mathrm{HCD}(\kappa^+) = \mathrm{HOD}$.

Conclusion

- Many tools to study the relationship between HOD and V assuming large cardinals:
  - Countably complete ultrafilter combinatorics
    - UA phenomena
    - Kunen's Commuting Ultrapowers Lemma
    - Mitchell order
    - Solovay's Theorem
  - Set theoretic geology
    - Lévy-Solovay phenomena
    - Vopenka's Theorem
    - Usuba's Theorems
    - Approximation and cover properties
- No reason to think the HOD Conjecture itself could not be established using simple (fine structure free) arguments like the ones from this talk.

Thanks

Thanks!