UA from AD

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Introduction

UA and AD

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The Ultrapower Axiom (UA):

- A regularity principle for large cardinals
- Motivated by the methodology of inner model theory
- Extends inner model theory abstractly to all large cardinals
- Holds in all known canonical inner models of ZFC

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The Axiom of Determinacy (AD):

- A regularity principle for sets of reals
- Motivated by game-theoretic methods in descriptive set theory
- Extends descriptive set theory abstractly to all sets of reals
- Holds for all canonical sets of reals but contradicts AC

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- AD implies the existence of many measures below Θ, the least ordinal that is not the surjective image of $\mathbb{R}$
- Their structure is central to the set theory of determinacy models; e.g., cofinalities, partition properties, uniformization
- The connection between AD and large cardinals is mediated by the existence of measures (or systems of measures)

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Theorem (G.)

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Corollary (G.)

Assume $\mathsf{ZF} + \mathsf{AD}^{L(\mathbb{R})}$*. Then* $L(\mathbb{R})$ *satisfies* UA*.*

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- By work of Solovay, Martin–Paris, and Kunen from the 1970s, AD implies UA for measures on ω_2
- In fall 2024, G.–Jackson proved UA below $\aleph_\omega$ from AD

Outline

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- The Ultrapower Axiom and the Ketonen order
- Measures under AD
- Sketch of proof of UA from determinacy
- Applications and future directions

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Our background theory is ZF + DC.

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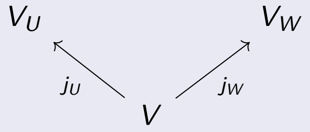

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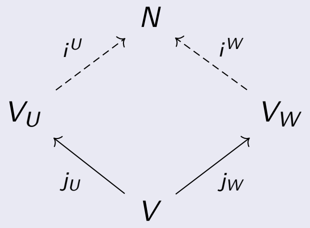

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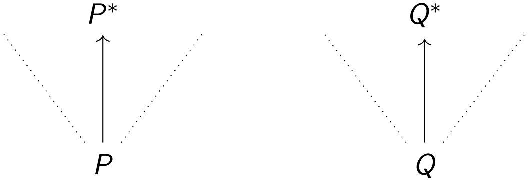

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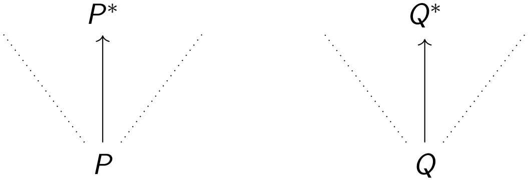

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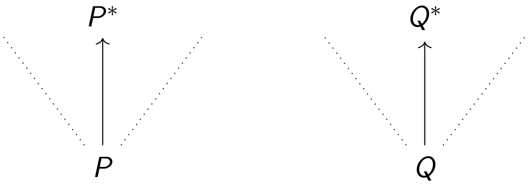

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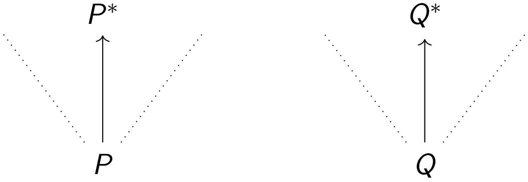

Outcome: Either $P^* \in Q^*$, $Q^* \in P^*$, or $P^* = Q^*$.

- If P and Q are ultrapowers of M, comparison simplifies to UA
- UA is a first-order statement that can be analyzed in ZFC

Results from grad school

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Theorem (G., ZFC + UA, 2018)

A cardinal κ is strongly compact if and only if it is supercompact or a measurable limit of supercompact cardinals.

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Theorem (G., UA)

For all κ, the Ketonen order on $\upsilon(\kappa)$ is linear.

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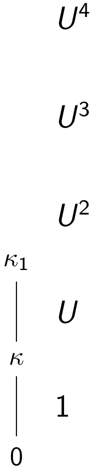

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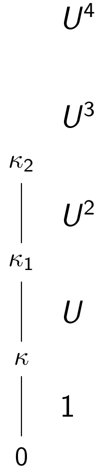

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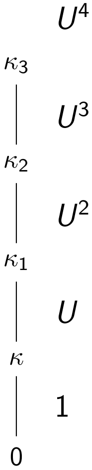

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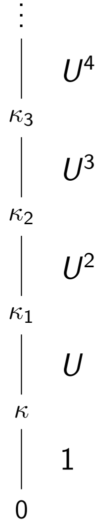

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The self-dual subsets of $P(\omega)$ are pre-well-ordered by L-reducibility.

UA itself implies a form of AD:

Proposition (G., UA)

The restriction of L-reducibility to $\upsilon(\kappa)$ is the Ketonen order. Thus $\upsilon(\kappa)$ is well-ordered by L-reducibility.

Measures and determinacy

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For $n \geq 3$, ω_n is singular of cofinality ω_2.

- ω_n's carry increasingly complex measures classified by Kunen
- Measures below least weakly inaccessible classified by Jackson

Definability of measures

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Theorem (Kunen, AD)

Every countably complete filter below Θ extends to a measure.

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Every measure is ordinal definable.

This is because they are well-ordered by the Ketonen order.

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Theorem (G.)

For any ordinal κ, the following are equivalent:

- UA *holds for measures on κ.*
- *The Ketonen order on $\upsilon(\kappa)$ is linear.*

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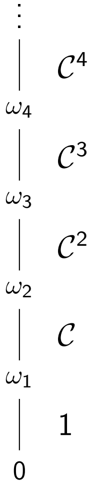

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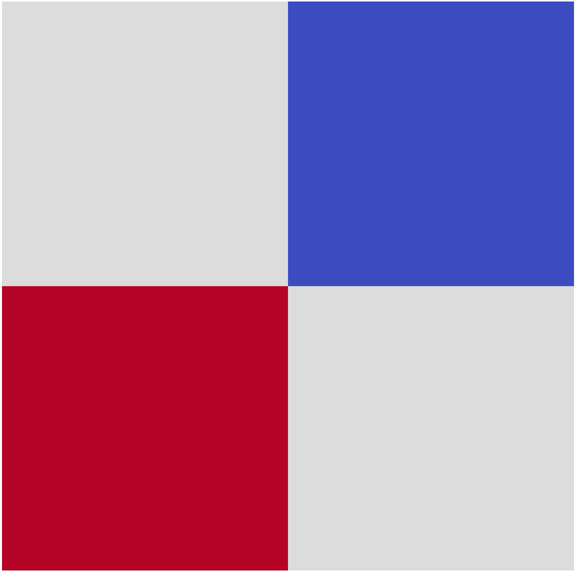

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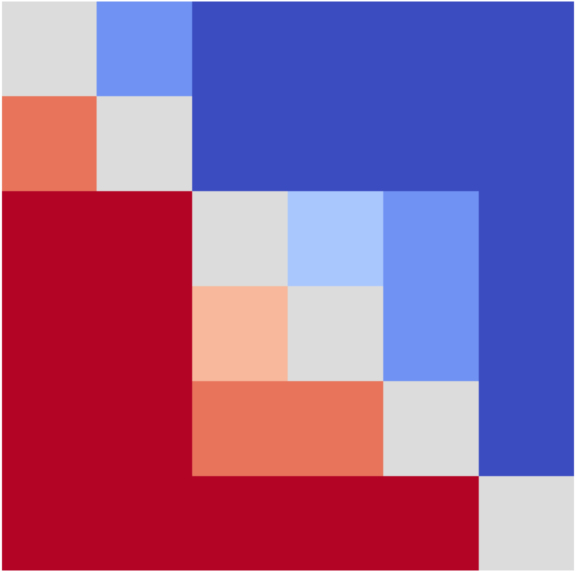

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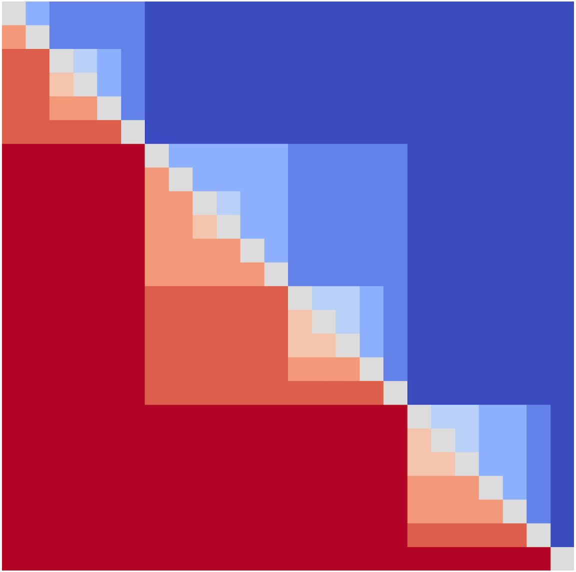

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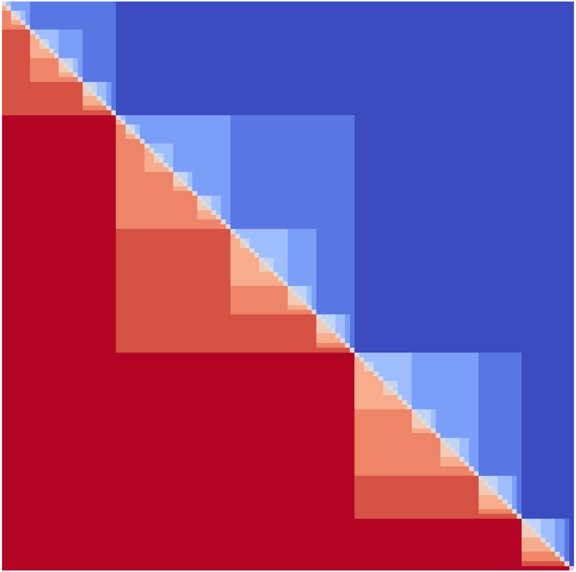

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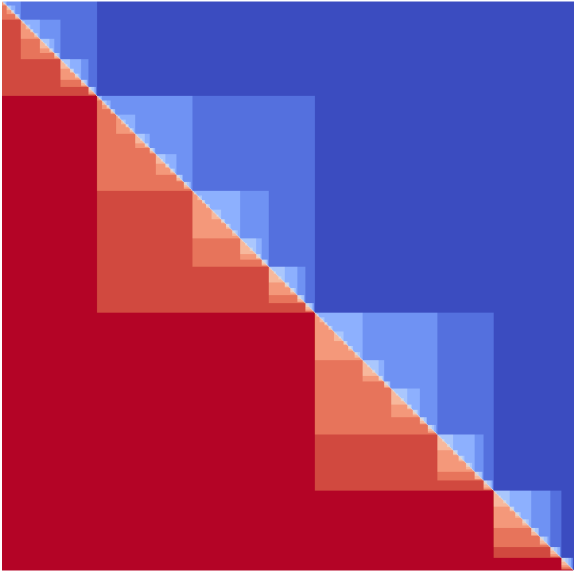

Proof sketch

UA from comparison

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Key consequence of weak comparison:
There is an elementary embedding $\pi : M \to V$ such that any internal ultrapowers $j_U : M \to M_U$ and $j_W : M \to M_W$ have a *close comparison*: a pair of close $i^U : M_U \to N$ and $i^W : M_W \to N$ with

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This suffices to prove UA.

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By Kechris, it suffices to *re-embed the ultrapowers into* V.

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Under AD^+, however, there is a technique for replacing ultrapowers with fully elementary *generic* ultrapowers.

Woodin's generic ultrapowers

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Theorem (Woodin, $\mathrm{AD}^+ + V = L(P(\mathbb{R})) + \Theta$ is regular)

For any measure U below Θ, there is a precipitous ideal J on $\mathbb{R}$ such that if $G \subseteq P(\mathbb{R})/J$ is generic, then in $V[G]$:

- *The ultrapower $j_G : V \to V_G$ is elementary.*
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Almost nothing is known even when U is the club measure on ω_1.

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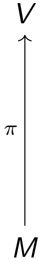

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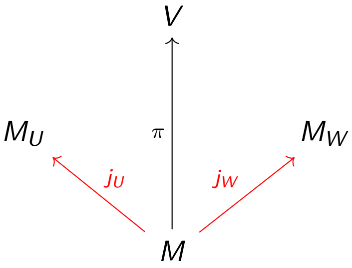

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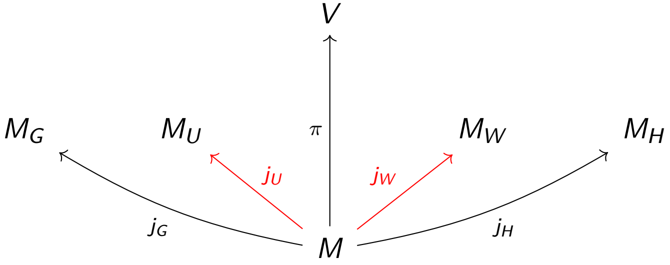

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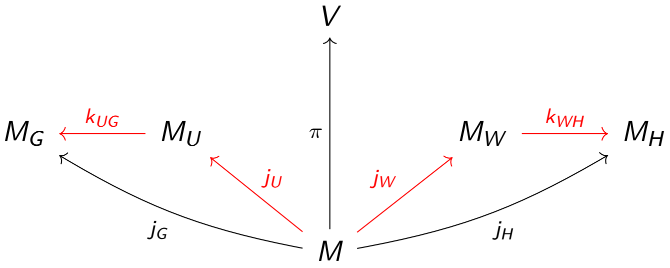

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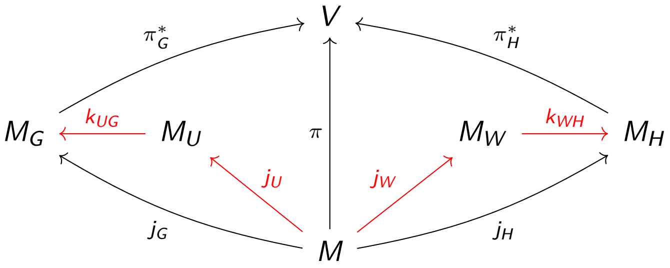

Conclusions

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The proof is local, so:

Corollary (G., AD^+)

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Corollary (AD^+)

If μ is a probability measure on an ordinal below Θ, then μ is a weighted sum of countably many measures.

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If $L(V_{\lambda+1})$ satisfies UA, *then ω_1 is measurable.*

Future directions

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- Global classification of measures
- Woodin's generic ultrapowers
- Measures on sets that are not well-orderable
- Connections with I_0 and stronger hypotheses
- Unification with HOD
- Strong and Woodin cardinals in determinacy models

Thanks!