The Ultrapower Axiom

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Tarski's question

Definition

- An uncountable cardinal κ is *measurable* if there is a nonprincipal κ-complete ultrafilter on κ.
- An uncountable cardinal κ is *strongly compact* if every κ-complete filter extends to a κ-complete ultrafilter.

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- An uncountable cardinal κ is *strongly compact* if every κ-complete filter extends to a κ-complete ultrafilter.

Question (Tarski)

Is the least strongly compact cardinal larger than the least measurable cardinal?

Normal ultrafilters

Definition

An ultrafilter U on an ordinal κ is

- *fine* if for all $\alpha < \kappa$, $[\alpha, \kappa] \in U$
- *normal* if any regressive function on κ (i.e., $f(\alpha) < \alpha$ for nonzero α) is constant on a set in U.

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Theorem (Scott, Keisler-Tarski)

If κ is an uncountable cardinal, the following are equivalent:

- *κ is measurable.*
- *There is a κ-complete fine ultrafilter on κ.*
- *There is a κ-complete normal fine ultrafilter on κ.*

Combinatorics of $P_\kappa(\lambda)$

If $\kappa \leq \lambda$ are cardinals, $P_\kappa(\lambda) = \{\sigma \subseteq \lambda : |\sigma| < \kappa\}$.

Definition

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Theorem (Solovay)

An uncountable cardinal κ is strongly compact if and only if for all $\lambda \geq \kappa$, there is a κ-complete fine ultrafilter on $P_\kappa(\lambda)$.

Supercompactness and Solovay's conjecture

Definition

A cardinal κ is *supercompact* if for every $\lambda \geq \kappa$, there is a κ-complete normal fine ultrafilter on $P_\kappa(\lambda)$.

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If κ is supercompact then the set of measurable cardinals less than κ is stationary in κ.

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Conjecture (Solovay)

Every strongly compact cardinal is supercompact.

Counterexamples to Solovay's conjecture

Theorem (Menas)

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- *It is consistent with* ZFC *that the least strongly compact cardinal is supercompact.*

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Theorem (Magidor)

- *It is consistent with* ZFC *that the least strongly compact cardinal is supercompact.*
- *It is consistent with* ZFC *that the least measurable cardinal is strongly compact.*

Magidor called this the *identity crisis* for strongly compact cardinals.

Refining Tarski's question

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Are there "natural hypotheses" that resolve Tarski's question?

The theory of countably complete ultrafilters

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On the theory of countably complete ultrafilters:

> *One can investigate all the questions [from the theory of ultrafilters on ω], hoping that the additional property of countable completeness will give more usable information. Unfortunately this has not yet turned out to be the case.*

The Rudin-Frolík order

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Definition (Rudin-Frolík)

If U and W are countably complete ultrafilters, then $U \leq_{\text{RF}} W$ if there is a countably complete ultrafilter Z of M_U such that $(M_Z)^{M_U} = M_W$ and $j_W = (j_Z)^{M_U} \circ j_U$.

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[Really just a preorder, but if $U \leq_{\text{RF}} W$ and $W \leq_{\text{RF}} U$ then their ultrapowers are the same; i.e., $M_U = M_W$ and $j_U = j_W$]

The Rudin-Frolík order

The analogous order on ultrafilters on ω is highly pathological. It is tree-like (Rudin); it contains chains order-isomorphic to $\mathbb{R}$ (Booth); it has $2^{2^{\aleph_0}}$ minimal elements (Kunen). In particular it is not directed.

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It is consistent, however, that the Rudin-Frolík order on countably complete ultrafilters is quite simple (yet nontrivial). This can be seen by considering canonical inner models.

The Ultrapower Axiom

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We conjecture that the Ultrapower Axiom will hold in all canonical inner models that will ever be constructed, since it follows from the methodologically central Comparison Lemma.

Conjecture

The Ultrapower Axiom is consistent with all large cardinal axioms.

Strongly compact cardinals and UA

Theorem (UA)

Every strongly compact cardinal is either supercompact or a limit of supercompact cardinals.

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Remainder of talk: aspects of the proof of this theorem.

Interesting feature: potential structural *dissimilarities*

- between the least supercompact cardinal and all other supercompact cardinals
- betweem local versions of strong compactness and supercompactness

Ketonen's Theorem

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Theorem (Ketonen)

A cardinal κ is strongly compact if and only if every regular cardinal $\delta \geq \kappa$ carries a κ-complete uniform ultrafilter.

The Ketonen order

Ketonen's proof uses the following order on ultrafilters:

Definition (Ketonen)

If U and W are countably complete ultrafilters on ordinals, $U <_{\Bbbk} W$ if $j_W[U]$ is covered by a countably complete ultrafilter of M_W that contains the ordinal $[\mathrm{id}]_W$.

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Assuming every regular $\lambda \geq \kappa$ carries a κ-complete uniform ultrafilter, Ketonen shows that any $<_{\mathbb{k}}$-*minimal* κ-complete uniform ultrafilter on a regular λ is Rudin-Keisler equivalent to a κ-complete fine ultrafilter on $P_\kappa(\lambda)$.

The Ketonen order and UA

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Definition (UA)

If $\kappa \leq \lambda$ are uncountable cardinals, let $\mathcal{K}_{\kappa,\lambda}$ denote the $<_{\Bbbk}$-least κ-complete uniform ultrafilter on λ.

The least supercompact cardinal

Theorem (UA)

Suppose λ is a successor cardinal and κ is least such that there is a κ-complete fine ultrafilter on $P_\kappa(\lambda)$. Then $\mathcal{K}_{\kappa,\lambda}$ is isomorphic to a κ-complete normal fine ultrafilter on $P_\kappa(\lambda)$.

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Theorem (UA)

Suppose κ is inaccessible and no cardinal is supercompact in V_κ. Then the Rudin-Frolík order is directed on κ-complete ultrafilters.

Irreducible ultrafilters

The proof that the first strongly compact κ is supercompact does not generalize. Instead, we propagate the supercompactness of $\mathcal{K}_{\kappa,\lambda}$ to a far more general class of ultrafilters:

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- Normal fine ultrafilters are irreducible
- Irreducibles are dense in the Rudin-Frolík order
- (UA) Countably complete ultrafilters factor into irreducibles

Lemma (UA)

If λ is regular and $\mathcal{K}_{\kappa,\lambda}$ exists, either $\mathcal{K}_{\kappa,\lambda}$ is irreducible or for arbitrarily large $\bar{\kappa} < \kappa$, $\mathcal{K}_{\bar{\kappa},\lambda}$ is irreducible.

Higher supercompact cardinals (cont)

Theorem (Kunen)

A cardinal κ is supercompact if and only if for all $\lambda \geq \kappa$, there is an ultrafilter U that is exactly κ-complete such that M_U is closed under λ-sequences.

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Theorem (UA)

Suppose λ is a successor cardinal and U is a uniform irreducible ultrafilter on λ. Then M_U is closed under λ-sequences.

Corollary (UA)

If κ is strongly compact, either κ is supercompact or κ is a limit of supercompact cardinals.

Thanks!