Two generalizations of the Ultrapower Axiom

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Table of Contents

The Ultrapower Axiom

Generalizations of UA, I

Generalizations of UA, II

Independence and large cardinals, Part I

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- (1965-1990) Hundreds of problems in classical descriptive set theory settled in this way
- (1967) Lévy-Solovay Theorem: large cardinals *cannot* settle the Continuum Hypothesis

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Goal: find basic structural principles answering these questions in a uniform, reasonable way
Dream: these structural principles for large cardinals also answer classical set theory questions

The Ultrapower Axiom

Suppose M is a model of ZFC.

- A point $U \in M$ is a *countably complete ultrafilter of M* if $M \vDash$ "U is a countably complete ultrafilter"
- An *internal ultrapower embedding of M* is an elementary embedding $j : M \to N$ such that $j = (j_U)^M$ for some countably complete ultrafilter U of M

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Ultrapower Axiom: For all (internal) ultrapower embeddings $j_0 : V \to M_0$, $j_1 : V \to M_1$, there exist internal ultrapower embeddings $i_0 : M_0 \to N$, $i_1 : M_1 \to N$ such that $i_0 \circ j_0 = i_1 \circ j_1$

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Question

Is UA consistent with a supercompact cardinal?

The consistency of UA

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To try to answer this question, let's look at the status of generalizations of UA that don't follow from the Comparison Lemma

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Definition

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Definition

An elementary embedding $j : V \to M$ is an *extender embedding* if it is generated by a set, i.e., is λ-generated for some cardinal λ.

The Extender Power Axiom

If M is a model of ZFC, then $j : M \to N$ is an *internal extender embedding* of M if j is a definable class of M and $M \vDash j$ is an extender embedding

- Equivalently, there is an extender E of M such that $j = (j_E)^M$

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Extender Power Axiom: For all extender embeddings $j_0 : V \to M_0$ and $j_1 : V \to M_1$, there are internal extender embeddings $i_0 : M_0 \to N$ and $i_1 : M_1 \to N$ such that $i_0 \circ j_0 = i_1 \circ j_1$.

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Consistency of EPA

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Theorem

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An elementary embedding $j : V \to M$ is λ-*closed* if $M^\lambda \subseteq M$.

Theorem

The restriction of EPA *to countably closed extender embeddings is consistent with a Woodin cardinal.*

Instances of EPA from UA

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Theorem (UA)

Suppose $j_0 : V \to M_0$ is an ultrapower embedding and $j_1 : V \to M_1$ is an elementary embedding. Then there is a close embedding $i_0 : M_0 \to N$ and an internal ultrapower embedding $i_1 : M_1 \to N$ such that $i_0 \circ j_0 = i_1 \circ j_1$.

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Corollary (UA)

Suppose $j_0 : V \to M_0$ is a λ-closed ultrapower embedding and $j_1 : V \to M_1$ is λ^+-generated. Then EPA *holds for j_0 and j_1.*

The failure of EPA, Part I

Turns out EPA is false:

Theorem (Woodin, G.)

If there is a supercompact cardinal, then EPA *is false. In fact,* EPA *fails for a pair of* $\aleph_1$*-generated embeddings.*

The proof uses towers of measures arising in Woodin's counterexample to the Unique Branches Hypothesis

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Under UA, EPA holds for $\aleph_0$-generated embeddings, and even for pairs of embeddings, one $\aleph_0$-generated and the other $\aleph_1$-generated

The proof only requires a cardinal κ that is 2^κ-supercompact, but even this seems like overkill

The failure of EPA, Part II

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A more recent result refutes EPA from a much weaker hypothesis (but not a genuine large cardinal hypothesis):

Theorem

If there is an iterably Woodin cardinal, then EPA *is false.*

It seems likely that EPA can be refuted from a Woodin cardinal, but the following is the best we can do for now:

Theorem

If there is a Woodin cardinal, then there is a canonical inner model with a Woodin cardinal in which EPA is false.

The Complete Ultrapower Axiom

The second generalization of UA we consider concerns the structure of supercompact cardinals: is the first supercompact "structurally different" from the others?

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κ-**Complete Ultrapower Axiom** (UA_κ): For all ultrapowers $j_0 : V \to M_0$, $j_1 : V \to M_1$ with critical points at least κ, there are internal ultrapowers $i_0 : M_0 \to N$, $i_1 : M_1 \to N$ with critical points at least κ such that $i_0 \circ j_0 = i_1 \circ j_1$.

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Complete Ultrapower Axiom: UA_κ holds for all cardinals κ.

UA in generic extensions

Proposition

For any λ, UA_{λ^+} *holds if and only if* UA *holds in* $V^{\mathrm{Col}(\omega,\lambda)}$.

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If κ is measurable, then UA_κ reduces to UA_λ for smaller $\lambda < \kappa$:

Proposition

If κ *is measurable, the following are equivalent:*

1. UA_κ *holds.*
2. UA_λ *holds for all sufficiently large* $\lambda < \kappa$.
3. UA_λ *holds for cofinally many* $\lambda < \kappa$.

Consistency of UA_κ

UA_κ is consistent with very large cardinals (if UA is):

Theorem (UA)

Suppose $j_0 : V \to M_0$, $j_1 : V \to M_1$ are λ-closed ultrapowers. Then there are λ-closed internal ultrapowers $i_0 : M_0 \to N$, $i_1 : M_1 \to N$ such that $i_0 \circ j_0 = i_1 \circ j_1$.

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So the version of UA_κ where κ-completeness is replaced by κ-closure is just *true* assuming UA

Corollary (UA)

If no cardinal $\kappa < \lambda$ is λ-supercompact, then UA_λ *holds. Thus if κ is the least cardinal that is supercompact past a measurable cardinal,* Complete UA *holds in V_κ.*

The least supercompact

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Also the UA proof that the least strongly compact is supercompact is totally different from the proof that the second strongly compact is supercompact

Complete UA, on the other hand, says there are no "structural propositions" provable from UA about the first supercompact that are not also provable for the second supercompact

Conjecture (UA)

Let κ be the least supercompact cardinal. Then UA_{κ^+} is false.