

Two generalizations of the Ultrapower Axiom

Gabriel Goldberg

UC Berkeley

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- ▶ (1967) Lévy-Solovay Theorem: large cardinals *cannot* settle the Continuum Hypothesis

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Dream: these structural principles for large cardinals also answer classical set theory questions

The Ultrapower Axiom

Suppose M is a model of ZFC.

- ▶ A point $U \in M$ is a *countably complete ultrafilter* of M if $M \models "U \text{ is a countably complete ultrafilter}"$
- ▶ An *internal ultrapower embedding* of M is an elementary embedding $j : M \rightarrow N$ such that $j = (j_U)^M$ for some countably complete ultrafilter U of M

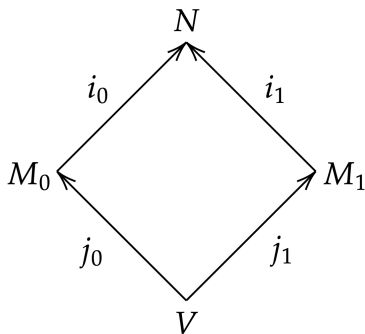
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Ultrapower Axiom: For all (internal) ultrapower embeddings $j_0 : V \rightarrow M_0, j_1 : V \rightarrow M_1$, there exist internal ultrapower embeddings $i_0 : M_0 \rightarrow N, i_1 : M_1 \rightarrow N$ such that $i_0 \circ j_0 = i_1 \circ j_1$

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Question

Is UA consistent with a supercompact cardinal?

The consistency of UA

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To try to answer this question, let's look at the status of generalizations of UA that don't follow from the Comparison Lemma

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Definition

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Definition

An elementary embedding $j : V \rightarrow M$ is an *extender embedding* if it is generated by a set, i.e., is λ -generated for some cardinal λ .

The Extender Power Axiom

If M is a model of ZFC, then $j : M \rightarrow N$ is an *internal extender embedding* of M if j is a definable class of M and $M \models j$ is an extender embedding

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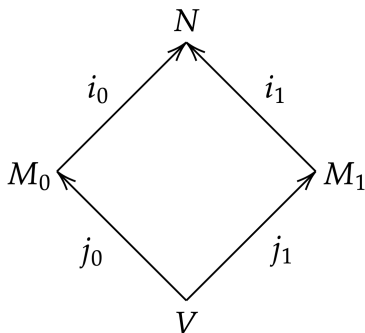
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Consistency of EPA

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Theorem

EPA is consistent with the existence of a strong cardinal.

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EPA is consistent with the existence of a strong cardinal.

An elementary embedding $j : V \rightarrow M$ is λ -closed if $M^\lambda \subseteq M$.

Theorem

The restriction of EPA to countably closed extender embeddings is consistent with a Woodin cardinal.

Instances of EPA from UA

An elementary embedding $i : M \rightarrow N$ is *close* to M if the inverse image of any set in N under i belongs to M .

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Theorem (UA)

Suppose $j_0 : V \rightarrow M_0$ is an ultrapower embedding and $j_1 : V \rightarrow M_1$ is an elementary embedding. Then there is a close embedding $i_0 : M_0 \rightarrow N$ and an internal ultrapower embedding $i_1 : M_1 \rightarrow N$ such that $i_0 \circ j_0 = i_1 \circ j_1$.

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Corollary (UA)

Suppose $j_0 : V \rightarrow M_0$ is a λ -closed ultrapower embedding and $j_1 : V \rightarrow M_1$ is λ^+ -generated. Then EPA holds for j_0 and j_1 .

The failure of EPA, Part I

Turns out EPA is false:

Theorem (Woodin, G.)

If there is a supercompact cardinal, then EPA is false. In fact, EPA fails for a pair of \aleph_1 -generated embeddings.

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Under UA, EPA holds for \aleph_0 -generated embeddings, and even for pairs of embeddings, one \aleph_0 -generated and the other \aleph_1 -generated

The proof only requires a cardinal κ that is 2^κ -supercompact, but even this seems like overkill

The failure of EPA, Part II

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Theorem

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It seems likely that EPA can be refuted from a Woodin cardinal, but the following is the best we can do for now:

Theorem

If there is a Woodin cardinal, then there is a canonical inner model with a Woodin cardinal in which EPA is false.

The Complete Ultrapower Axiom

The second generalization of UA we consider concerns the structure of supercompact cardinals: is the first supercompact “structurally different” from the others?

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κ -Complete Ultrapower Axiom (UA_κ): For all ultrapowers $j_0 : V \rightarrow M_0$, $j_1 : V \rightarrow M_1$ with critical points at least κ , there are internal ultrapowers $i_0 : M_0 \rightarrow N$, $i_1 : M_1 \rightarrow N$ with critical points at least κ such that $i_0 \circ j_0 = i_1 \circ j_1$.

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Complete Ultrapower Axiom: UA_κ holds for all cardinals κ .

UA in generic extensions

Proposition

For any λ , UA_{λ^+} holds if and only if UA holds in $V^{\text{Col}(\omega, \lambda)}$.

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For any λ , $UA_{\lambda+}$ holds if and only if UA holds in $V^{\text{Col}(\omega, \lambda)}$.

If κ is measurable, then UA_{κ} reduces to UA_{λ} for smaller $\lambda < \kappa$:

Proposition

If κ is measurable, the following are equivalent:

1. UA_{κ} holds.
2. UA_{λ} holds for all sufficiently large $\lambda < \kappa$.
3. UA_{λ} holds for cofinally many $\lambda < \kappa$.

Consistency of UA_κ

UA_κ is consistent with very large cardinals (if UA is):

Theorem (UA)

Suppose $j_0 : V \rightarrow M_0$, $j_1 : V \rightarrow M_1$ are λ -closed ultrapowers. Then there are λ -closed internal ultrapowers $i_0 : M_0 \rightarrow N$, $i_1 : M_1 \rightarrow N$ such that $i_0 \circ j_0 = i_1 \circ j_1$.

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So the version of UA_κ where κ -completeness is replaced by κ -closure is just *true* assuming UA

Corollary (UA)

If no cardinal $\kappa < \lambda$ is λ -supercompact, then UA_λ holds. Thus if κ is the least cardinal that is supercompact past a measurable cardinal, Complete UA holds in V_κ .

The least supercompact

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Also the UA proof that the least strongly compact is supercompact is totally different from the proof that the second strongly compact is supercompact

Complete UA, on the other hand, says there are no “structural propositions” provable from UA about the first supercompact that are not also provable for the second supercompact

Conjecture (UA)

Let κ be the least supercompact cardinal. Then $UA_{\kappa+}$ is false.