Strong compactness and the ω-club filter

Gabriel Goldberg

UC Berkeley

2021

Outline

1. Background: the Jensen covering lemma
2. Optimal versions of the HOD dichotomy
3. Strong compactness in canonical models
4. Embeddings from HOD to HOD

The universe of sets

Definition (von Neumann)

Cumulative hierarchy:

- $V_0 = \emptyset$.
- $V_{\alpha+1} = P(V_\alpha)$.
- $V_\gamma = \bigcup_{\beta<\gamma} V_\beta$ for limit ordinals γ.

Universe of sets: $V = \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha$.

The constructible hierarchy

Definition (Gödel)

Constructible hierarchy:

- $L_0 = \emptyset$.
- $L_{\alpha+1} = \text{def}(L_\alpha, \in)$.
- $L_\gamma = \bigcup_{\beta<\gamma} L_\beta$ for limit ordinals γ.

Constructible universe: $L = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha$.

The constructible universe

The constructible universe L is *canonical* in various ways:

- L is the minimum model of ZFC containing all ordinals.
- If M_0 and M_1 are models of ZFC containing the same ordinals, $L^{M_0} = L^{M_1}$.
- Every question about L can be answered.

Theorem (Gödel)

The constructible universe satisfies the Generalized Continuum Hypothesis and the principle $V = L$.

The Jensen covering lemma

Does $V = L$? This is independent of ZFC. Can arbitrary sets be approximated by constructible ones?

Theorem (Jensen)

One of the following holds:

(1) *For any set $S \subseteq L$, there is a set $T \in L$ of cardinality $\max(|S|, \aleph_1)$ such that $S \subseteq T$.*

(2) *There is an elementary embedding from L to L.*

If (1) holds, we say *L covers V.*

The Jensen dichotomy

If L covers V, V resembles L:

- Many instances of the Generalized Continuum Hypothesis must hold: $2^\lambda = \lambda^+$ for all singular strong limits λ.
- If λ is a singular cardinal, then λ is singular in L. Moreover L correctly computes the successor of λ, i.e., $\lambda^{+L} = \lambda^+$. (Known as weak covering.)

If there is an elementary $j : L \to L$, V is far from L:

- For all ordinals α, $|P(\alpha) \cap L| = |\alpha|$, so L does not compute any successor cardinals correctly.
- Every infinite cardinal is inaccessible, Mahlo, weakly compact, indescribable in L.
- $j : L \to L$ is a large cardinal axiom that cannot hold in L.

The inner model hierarchy

If there is an elementary embedding from L to L, there is a *canonical model beyond* L: the class $L[j]$ of sets constructible relative to a (carefully chosen) elementary embedding $j : L \to L$.

- $L[j]$ satisfies an analogous covering lemma; i.e., either $L[j]$ covers V or there is an embedding from $L[j]$ to $L[j]$. The latter yields yet another canonical model.
- Continuing, one obtains a transfinite hierarchy of canonical models of ZFC, satisfying stronger and stronger large cardinal axioms.

Do all large cardinal axioms have canonical models? Is there always a canonical model that approximates V in some sense?

Ordinal definability

A set x is *ordinal definable* if it is definable in $(V, \in)$ from ordinal parameters and *hereditarily ordinal definable* if every y admitting a finite sequence $y \in x_1 \in x_2 \in \cdots \in x_{n-1} \in x$ is ordinal definable.

Definition (Gödel)

HOD is the class of hereditarily ordinal definable sets.

HOD is a model of ZFC, but unlike L, *not* a canonical model:

- Different models of ZFC can have wildly different HODs.
- ZFC proves nothing about the internal structure of HOD.
- All known large cardinal axioms are consistent with $V = \text{HOD}$.

Covering for HOD

Canonical models of ZFC are ordinal definable. Therefore if V is covered by a canonical inner model, it is covered by HOD.

Is there an unconditional covering lemma for HOD? The answer is consistently no, but the following possibility remains:

Question

Do large cardinal axioms imply that HOD covers V?

Extendible cardinals

If $j : M \to N$ is a nontrivial elementary embedding between models of ZFC, then there must be an ordinal $\alpha \in M$ such that $j(\alpha) \neq \alpha$. The *critical point* of j is the least such ordinal.

Many modern large cardinals are formulated in terms of elementary embeddings and their critical points.

- κ is *measurable* if there is a $j : V \to M$ such that $\kappa = \text{crit}(j)$.
- κ is *extendible* if for all $\alpha \geq \kappa$, for some $\beta \geq \alpha$, there is an elementary embedding $j : V_\alpha \to V_\beta$ with critical point κ.

Woodin's HOD dichotomy

Theorem (Woodin)

Assume κ is extendible. Then exactly one of the following holds:

(1) *For any set $S \subseteq \mathrm{HOD}$, there is a set $T \in \mathrm{HOD}$ of cardinality $\max(|S|, \kappa)$ such that $S \subseteq T$.*

(2) *Every regular cardinal $\lambda \geq \kappa$ is measurable in* HOD.

Looks just like the situation with L. Except no large cardinal axiom can imply (2)?

Conjecture (Woodin)

If there is an extendible cardinal, then (1) *holds.*

Version of the *HOD conjecture.*

Strong compactness

A cardinal κ is *strongly compact* if for all $\lambda \geq \kappa$, there is a $j : V \to M$ with critical point κ such that every set $S \subseteq M$ with $|S| \leq \lambda$ is contained in a set $T \in M$ with $|T|^M < j(\kappa)$.

- Equivalently, every κ-complete filter extends to a κ-complete ultrafilter.

Strong compactness blocks various forcings that destroy covering properties.

Theorem (Solovay)

If there is a strongly compact cardinal, then for all singular strong limit cardinals $\lambda \geq \kappa$, $2^\lambda = \lambda^+$.

The optimal hypothesis for the HOD dichotomy

Theorem

Suppose κ is strongly compact. Then one of the following holds:

(1) *Every singular $\lambda \geq \kappa$ is singular in* HOD *and $\lambda^{+\mathrm{HOD}} = \lambda^+$.*

(2) *All sufficiently large regular cardinals are measurable in* HOD.

- It is consistent that κ is supercompact, (1) holds, but there are unboundedly many singular cardinals less than κ that are measurable in HOD.
- If (2) is consistent with a supercompact κ, it is consistent with $V_\alpha \subseteq \mathrm{HOD}$ for any ordinal Δ_2-definable from κ.

A more optimal hypothesis

A cardinal κ is ω_1-strongly compact if for all $\lambda \geq \kappa$, there is a $j : V \to M$ such that every set $S \subseteq M$ with $|S| \leq \lambda$ is contained in a set $T \in M$ with $|T|^M < j(\kappa)$.

- Equivalently, every κ-complete filter extends to an ω_1-complete ultrafilter.

It is consistent that the least ω_1-strongly compact cardinal is singular... but supercompact in HOD.

Theorem

If κ is ω_1-strongly compact, then one of the following holds:

(1) *Every singular $\lambda \geq \kappa^+$ is singular in* HOD *and $\lambda^{+\mathrm{HOD}} = \lambda^+$.*

(2) *All sufficiently large regular cardinals are measurable in* HOD.

What about covering?

A model N has the λ-*cover property* if every set subset of N of cardinality less than λ is contained in a set in N of cardinality less than λ.

Theorem

Suppose κ is strongly compact. Then one of the following holds:

1. *For all strong limit cardinals $\lambda \geq \kappa$,* HOD *has the λ-cover property.*
2. *All sufficiently large regular cardinals are measurable in* HOD.

Conjecture: in case (1), one can prove every set $S \subseteq \text{HOD}$ is contained in a set $T \in \text{HOD}$ of cardinality $\max(|S|, \kappa)$.

A dichotomy for ω-club amenable models

- $S \subseteq \delta$ is *ω-club* if cofinal and closed under countable suprema.
- *ω-club filter* $\mathcal{C}_{\delta,\omega}$ = set of subsets of δ that contain an ω-club.
- N is *ω-club amenable* if for any ordinal δ of uncountable cofinality, $\mathcal{C}_{\delta,\omega} \cap N \in N$. E.g., $L[\langle C_\delta : \mathrm{cf}(\delta) > \omega\rangle]$.

Theorem

If κ is strongly compact and N is ω-club amenable, either:

(1) *For all singular $\lambda \geq \kappa$, λ is singular in N and $\lambda^{+N} = \lambda^+$.*

(2) *All sufficiently large regular cardinals are measurable in N.*

The dividing line

For regular δ, $\mathcal{C}_{\delta,\omega}$ is far from an ultrafilter: every $\mathcal{C}_{\delta,\omega}$-positive set splits into δ-many positive sets. Size of an ω-club amenable model N is determined by whether such splittings exist in N.

Case 1: For some γ, N cannot partition any δ into γ-many $\mathcal{C}_{\delta,\omega}$-positive sets.

- Then all regular $\delta \geq (2^\gamma)^+$ are measurable in N.

Case 2: Otherwise.

- For all γ, get ω_1-complete fine ultrafilters on $P_\kappa(\gamma) \cap N$.
- Cover countable sets by $<\kappa$-sized sets in N.
- This plus ω-club amenability implies for all singular λ, λ is singular in N, $\lambda^{+N} = \lambda^+$.

HOD and determinacy

- 1970s and 80s: determinacy of infinite two-player games of perfect information with definable payoff set. For example:
  - Open, closed, Borel.
  - Analytic, definable in $(V_{\omega+1}, \in)$.
  - Constructible over $V_{\omega+1}$.
- Axiom of Determinacy: every infinite two-player game of perfect information is determined.
  - Contradicts the Axiom of Choice.
- 1990s: In natural models of AD, HOD is a canonical inner model. E.g., HOD $\vDash$ GCH.
  - HOD extends Σ^2_1.
  - Inner model theory extends effective descriptive set theory.
  - Open how far this goes. Can HOD satisfy strong compactness principles?

Local covering and compactness properties

Definition

A model N has the the *κ-cover property below δ* if every subset of δ of cardinality less than κ is contained in a set in N of cardinality less than κ.

Under AD, HOD does not have the κ-cover property below κ^{+V} for any cardinal κ.

Definition

A cardinal κ is δ-strongly compact (resp. (ω_1, δ)-strongly compact) if every κ-complete filter generated by δ-many sets extends to a κ-complete (resp. ω_1-complete) ultrafilter.

Woodin's work suggests no κ is κ^{+V}-strongly compact in HOD.

Cover and compactness

The failure of global covering and global compactness for HOD is explained and unified by the following equivalence:

Theorem

Assume $\mathsf{AD}^+ + V = L(P(\mathbb{R}))$*. Suppose* κ *is regular and* $\delta \geq \kappa$ *is regular in* HOD*. The following are equivalent:*

- HOD *has the* κ*-cover property below* δ*.*
- κ *is* (ω_1, δ)*-strongly compact in* HOD *via countably complete ultrafilters.*
- κ *is* δ*-strongly compact in* HOD *via* $\mathcal{C}_{\delta,\omega}$*.*

Rigidity of HOD

Can there be an elementary embedding from HOD to HOD?

Question

Suppose κ is extendible. Must one of the following hold?

(1) For all singular $\lambda \geq \kappa$, λ is singular in HOD and $\lambda^{+\text{HOD}} = \lambda^+$.

(2) There is an elementary embedding from HOD to HOD.

Until recently it was open whether these are exclusive.

Theorem

Assume there is a strongly compact cardinal. If (1) holds, then (2) does not.

Rigidity of HOD, continued

While the converse seems very unlikely to be provable, an approximation is true:

Theorem

Suppose κ is strongly compact. Then exactly one of the following holds:

(1) *For all singular $\lambda \geq \kappa$, λ is singular in* HOD *and $\lambda^{+\mathrm{HOD}} = \lambda^+$.*

(2) *For all ordinals α, there is a nontrivial elementary embedding from an ω-club amenable model containing α into* $\mathrm{HOD} \cap V_\alpha$.

Conclusions

- The HOD dichotomy is a result of interactions between strong compactness and the ω-club filter.
- Strong compactness in canonical models is likely witnessed by the ω-club filter of a determinacy model.
- HOD is far from V if and only if there is something like an embedding $j : \mathrm{HOD} \to \mathrm{HOD}$.

Thanks!