Predictions of the Ultrapower Axiom

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Outline

1. Inner model theory background
2. The Ultrapower Axiom
3. The HOD conjecture
4. The Kunen inconsistency theorem

Canonical models of set theory

Inner model program: Build canonical models of large cardinal axioms.

Example 1: The constructible universe L

- Inaccessibles, Mahlos, weakly compacts are downwards absolute to L.
- Scott's theorem: L has no measurable cardinals.

Example 2: The inner model $L[U]$

- If κ is measurable, the sets constructible with oracle access to a κ-complete ultrafilter U on κ form a canonical inner model with a measurable.
- Does not depend on the choice of U.

Surprisingly, there are canonical inner models satisfying containing much larger cardinals; e.g., many Woodin cardinals.

Open: Is there a canonical model with a supercompact cardinal?

The comparison lemma

Suppose α is an ordinal. If $M_0, M_1 \preceq L_\alpha$, either $M_0 \cong M_1$, $M_0 \cong N_0 \in M_1$, or $M_1 \cong N_1 \in M_0$.

- False for V_α if there is a measurable cardinal below α.

The correct generalization is the *comparison lemma:*

Theorem (Kunen)

Any elementary substructures M_0 and M_1 of $L_\alpha[U]$ have iterated ultrapowers N_0 and N_1 such that either $N_0 = N_1$, $N_0 \in N_1$, or $N_1 \in N_0$.

The comparison lemma generalizes to all known canonical models.

Open: Is the comparison lemma compatible with a supercompact cardinal?

The Ultrapower Axiom

- An *ultrapower* is an inner model that is isomorphic to $\mathrm{Ult}(V, U)$ for some ultrafilter U.
- A substructure M of a transitive model N is an *internal ultrapower of* N if N satisfies that M is an ultrapower.

Weak UA: Any two ultrapowers have a common internal ultrapower.

Ultrapower Axiom (UA): Weak UA + the diagram commutes:

$$\begin{tikzcd}
& N & \\
M_0 \arrow[ur, "i_0"] & & M_1 \arrow[ul, "i_1"'] \\
& V \arrow[ul, "j_0"] \arrow[ur, "j_1"'] &
\end{tikzcd}$$

Open: Is UA consistent with a supercompact cardinal?

The HOD conjecture

Theorem (Jensen's L dichotomy)

One of the following holds:

V is close to L: For all singular cardinals γ, $(\gamma^+)^L = \gamma^+$.

V is far from L: Every uncountable cardinal is inaccessible in L.

Assuming large cardinal axioms, V is far from L.

Theorem (Woodin's HOD dichotomy)

If κ is an extendible cardinal, one of the following holds:

V is close to HOD: For singular cardinals $\gamma \geq \kappa$, $(\gamma^+)^{\text{HOD}} = \gamma^+$.

V is far from HOD: Every regular above κ is measurable in HOD.

HOD Conjecture: Assuming large cardinals, V is *close* to HOD.

Goal of talk

Outline some results that constitute...

- evidence that UA is consistent with very large cardinals
- evidence for the HOD conjecture?
- evidence against the HOD conjecture??
- evidence that UA is true???

Themes:

- Analogies with determinacy
  - Definability and regularity properties of large cardinals
- Prediction and confirmation

Prediction and confirmation

Suppose δ is an ordinal. A function $f : 2^\delta \to 2^\delta$ is *Lipschitz* if for $x \in 2^\delta$ and $\alpha < \delta$, $f(x) \restriction \alpha$ depends only on $x \restriction \alpha$. If $A, B \subseteq 2^\delta$, $A \leq_L B$ if A is the preimage of B under a Lipschitz function.

- In general, $\leq_L$ is illfounded and has large antichains.

Theorem (Wadge, Martin)

If every real has a sharp, $\leq_L$ is wellfounded and semilinear on $\mathbf{\Sigma}^1_1$ subsets of 2^ω; that is, $\leq_L$-antichains have cardinality at most 2.

If $A \subseteq 2^\omega$ is open but not clopen, $\{A, 2^\omega \setminus A\}$ is an $\leq_L$-antichain.

Theorem (Louveau-Saint Raymond, Harrington)

- *The semilinearity of $\leq_L$ on Borel subsets of 2^ω is provable in second-order arithmetic.*
- *Assume that every $\mathbf{\Sigma}^1_1$ set that is not Borel is $\mathbf{\Sigma}^1_1$-complete. Then every real has a sharp.*

Lipschitz reducibility and ultrafilters

Since $P(\delta) \cong 2^\delta$, one can port $\leq_L$ to subsets of $P(\delta)$.

Ultrafilters on δ are subsets of $P(\delta)$!

- Even on ultrafilters, $\leq_L$ is illfounded with large antichains.

Theorem (UA)

The restriction of $\leq_L$ to ω_1-complete ultrafilters on δ is a wellorder.

It is natural in this context to consider only Lipschitz reductions $f : P(\delta) \to P(\delta)$ with the property that the inverse image of any ω_1-complete ultrafilter under f is again an ω_1-complete ultrafilter.

The resulting reducibility, called the *Ketonen order*, is wellfounded.

Theorem

UA *holds if and only if the Ketonen order is linear.*

Ordinal definability and UA

The linearity of the Ketonen order implies:

Theorem (UA)

1. *Every ω_1-complete ultrafilter on an ordinal is* OD.
2. *If κ is strongly compact, then* HOD *correctly computes cardinals, cofinalities, and the continuum function above κ^{++}.*

Aside: assuming AD + DC, Kunen proved (1) for ordinals below Θ, the least ordinal that is not the surjective image of $\mathbb{R}$.

Theorem (Woodin)

Assume κ is extendible and V is close to HOD. *If U is a κ-complete ultrafilter on an ordinal, $U \cap$* HOD *is ordinal definable.*

Theorem

The HOD *dichotomy only requires a strongly compact cardinal.*

The Kunen inconsistency theorem

The potential failure of the HOD conjecture is related to the consistency of large cardinals beyond the Axiom of Choice.

Definition

A cardinal λ is *rank Berkeley* if for all $\alpha < \lambda \leq \beta$, there is an elementary embedding $j : V_\beta \to V_\beta$ such that $\alpha < \mathrm{crit}(j) < \lambda$.

Theorem (Kunen)

There are no rank Berkeley cardinals.

Open: Can this be proved without the Axiom of Choice?

Dropping AC, an apparently endless hierarchy of *large cardinals beyond choice* emerges.

Theorem (Woodin, ZF + HOD Conjecture)

There is at most one rank Berkeley cardinal.

UA from large cardinals

The Lévy-Solovay theorem shows that nothing like UA follows from traditional large cardinal axioms.

What about large cardinals beyond choice?

- The ultrapower formulation of UA seems useless without AC.
- Instead, consider the linearity of the Ketonen order, which is equivalent to UA assuming AC.

Theorem (ZF)

If λ is rank Berkeley, there is a cardinal κ such that:

- *The Ketonen order on κ-complete ultrafilters is wellfounded and any set of $\leq_{\mathbb{k}}$-incomparables has cardinality at most λ.*
- *Consequently, if U is a κ-complete ultrafilter on an ordinal, U belongs to an* OD *set of size less than λ and $U \cap \mathrm{HOD}$ is* OD*.*

Idea: An elementary $j : P(\delta) \to P(\delta)$ is a Lipschitz reduction.

The uniqueness of elementary embeddings

Theorem (UA)

If $j_0, j_1 : V \to M$ are elementary, then $j_0 = j_1$.

Theorem

If $j_0, j_1 : V \to M$ are elementary, then j_0 and j_1 agree on the ordinals. If $\text{crit}(j_0)$ is above the least extendible cardinal, $j_0 = j_1$.

Theorem

If κ is an extendible cardinal, the following are equivalent:

V is close to HOD*:* *For singular cardinals $\lambda \geq \kappa$, $(\lambda^+)^{\text{HOD}} = \lambda^+$.*

Unique embeddings: *If $\delta \geq \kappa$ is regular, α is sufficiently large, $j_0, j_1 : V_\alpha \to M$ are elementary, $j_0(\delta) = j_1(\delta)$, and $\sup j_0[\delta] = \sup j_1[\delta]$, then $j_0 \restriction \delta = j_1 \restriction \delta$.*

Conclusions

- UA is a long determinacy principle for ω_1-complete ultrafilters.
- UA predicts consequences of the HOD conjecture.
- UA also predicts consequences of choiceless large cardinals.
- UA also makes predictions verified in ZFC.

Conjecture

If κ is extendible and U is a κ-complete ultrafilter on an ordinal, then $U \cap \mathrm{HOD}$ is ordinal definable.

Speculation

The HOD *conjecture is true but the existence of a rank Berkeley cardinal is consistent.*

Thanks!