The Σ_2-Potentialist Principle

Gabriel Goldberg

UC Berkeley

January 9, 2025

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No matter how far our mind may have progressed in the contemplation of God, it does not attain to what He is but only to what is beneath Him.

—St. Gregory, $\sim$600

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In fact, if φ is Σ_2, it is equivalent to the sentence

$$\exists\alpha\, V_\alpha \vDash (\varphi \wedge \forall\beta\, \beth_\beta \text{ exists})$$

Similarly, Π_2-sentences are *locally falsifiable*.

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It is motivated by *set-theoretic potentialism*, the view that the universe of sets forms a *potential totality*, not a completed one.

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Definition (Woodin)

The Σ_2-*Potentialist Principle* states that every V-satisfiable Σ_2-sentence is true.

Consequences of the Σ_2-Potentialist Principle

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Maybe the following analogy will explain my attitude; we use the standard American ethnic prejudice system, as it is generally familiar. So a typical universe of set theory is the parallel of Mr. John Smith, the typical American; my typical universe is quite interesting (even pluralistic), it has long intervals where GCH holds, but others in which it is violated badly, many λ's such that λ^+-Suslin trees exist and many λ's for which every λ^+-Aronszajn is special, and it may have lots of measurables, with a huge cardinal being a marginal case but certainly no supercompact.

—Saharon Shelah, "The Future of Set Theory"

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Our main theorem is:

Theorem (Ben Neria–G.–Kaplan)

If ZFC *plus a supercompact cardinal is consistent, so is* ZFC *plus the Σ_2-Potentialist Principle.*

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Proposition

If ZFC *is consistent, so is* ZFC *plus the* Σ_2-*Potentialist Scheme.*

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The following theorem rules out the most naive approach:

Theorem (Adolf–Ben Neria–Zeman)

If $V = L$, there is a forcing iteration $\langle \mathbb{P}_n, \dot{\mathbb{Q}}_n : n < \omega \rangle$ such that the following hold where $\kappa_n = \text{rank}(\mathbb{P}_n)$:

- $\mathbb{P}_n \Vdash \dot{\mathbb{Q}}_n$ *preserves* $V_{\kappa_n + \omega}$.
- $\varprojlim \mathbb{P}_n$ *forces* $\sup_{n<\omega} \kappa_n$ *to be countable.*

The Príkry property

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If $\mathbb{P} = (P, \leq)$ is a poset, a *direct extension order on* $\mathbb{P}$ is a partial order $\leq^*$, included in $\leq$, that has the *Príkry property*:

> *For any condition* $p \in \mathbb{P}$ *and any statemeng* φ *in the forcing language of* $\mathbb{P}$*, there is a* $\leq^*$*-extension of* p *that decides* φ*.*

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The most basic nontrivial example is the direct extension order on the Príkry forcing. But there are many other examples: Radin forcing, Magidor forcing, diagonal Príkry forcing, extender based Príkry forcing, diagonal supercompact extender-based Magidor-Radin forcing with interleaved collapses...

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If $\mathbb{P}$ admits a κ-complete direct extension order and $G \subseteq \mathbb{P}$ is V-generic, then $V[G]_\kappa = V_\kappa$.

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Lemma

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A *κ-complete Príkry type forcing* is a pair $(\mathbb{P}, \leq^*)$ where $\mathbb{P}$ is a poset and $\leq^*$ is a κ-complete direct extension order on $\mathbb{P}$.

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A sequence $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \theta, \beta < \theta \rangle$ is a *Magidor support iteration of κ-complete Príkry type forcings* if

- For $\beta < \theta$, $1_{\mathbb{P}_\beta} \Vdash \dot{\mathbb{Q}}_\beta$ is a κ-complete Príkry-type forcing.
- $\langle \mathbb{P}_\alpha, \overline{\mathbb{Q}}_\beta : \alpha \leq \theta, \beta < \theta \rangle$ is a forcing iteration, where $\overline{\mathbb{Q}}_\beta$ is a $\mathbb{P}_\beta$-name for the underlying poset of $\dot{\mathbb{Q}}_\beta$.
- For limits $\gamma \leq \theta$, $\mathbb{P}_\gamma$ is the set of $p : \gamma \to V$ with $p \restriction \beta \in \mathbb{P}_\beta$ and $p \restriction \beta \Vdash p_\beta \leq^* 1_{\dot{\mathbb{Q}}_\beta}$ for all but finitely many $\beta < \gamma$.

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For $\alpha \leq \theta$, the *induced direct extension order of* $\mathbb{P}_\alpha$ is given by $p \leq^* q$ if and only if for all $\beta < \alpha$, $p \restriction \beta \Vdash p_\beta \leq^* q_\beta$.

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Theorem (Gitik)

The poset $\mathbb{P}_\theta$, equipped with the induced direct extension order, is a κ-complete Príkry type forcing.

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Theorem (Ben Neria–G.–Kaplan)

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This improves a result of Gitik.

Forcing the Σ_2-Potentialist Principle

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At the n-th stage we will also have defined some $\lambda_n < \kappa$. Working in $V^{\mathbb{P}_n}$, we define $\dot{\mathbb{Q}}_n$. Let

$$\varphi_n = \text{“}\exists \alpha\, V_\alpha \models \psi\text{”}$$

be the least λ_n-completely satisfiable Σ_2-sentence. Choose a λ_n-complete Príkry type forcing $\dot{\mathbb{P}}_n \in V_\kappa^{\mathbb{P}_n}$ such that $(V^{\mathbb{P}_n})^{\dot{\mathbb{Q}}_n} \models \varphi_n$.

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Finally, working in V, let $\lambda_{n+1} \geq \lambda_n$ be least such that $(V^{\mathbb{P}_n})^{\dot{\mathbb{Q}}_n} \models \text{“}\exists \alpha \leq \lambda_{n+1}\, V_\alpha \models \psi_n\text{”}$.

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Does the Σ_2-Potentialist Principle imply that $0^\#$ exists?

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Using unpublished ideas of Adolf–Ben Neria–Zeman on mutual stationarity, we show:

Theorem (Ben Neria–G.–Kaplan)

The Σ_2-Potentialist Principle implies the consistency of ZFC *plus a Woodin cardinal.*