Ordinal definability and the structure of large cardinals

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Outline

1. The Jensen covering lemma
2. Canonical models and the HOD dichotomy
3. A case for/against the HOD conjecture

The universe of sets

- Two sources of uncountability (Cantor):
  - Powerset operation: $P(X) = \{A : A \subseteq X\}$
  - Transfinite ordinals: $\text{Ord} = \{0, 1, 2, 3, \ldots$ $\omega, \omega + 1, \ldots \omega + \omega, \ldots \omega \times \omega, \ldots\ \omega_1, \ldots \omega_2, \ldots \omega_\omega, \ldots\}$
- Classical set theory: can they be reconciled?
  - E.g., continuum problem: is $|P(\mathbb{N})| = \omega_1$?

Definition

Cumulative hierarchy:

- $V_0 = \emptyset$.
- $V_{\alpha+1} = P(V_\alpha)$.
- $V_\gamma = \bigcup_{\beta<\gamma} V_\beta$ for limit ordinals γ.

Universe of sets: $V = \bigcup_{\alpha \in \text{Ord}} V_\alpha$.

Many universes

- ZFC axioms of set theory attempt to describe V in the same way Peano axioms describe $\mathbb{N}$.
- But Cohen's forcing technique shows there are many models ZFC. Rough rough idea:
  - Start with a countable model M of ZFC.
  - Pick a real number x at random.
  - With probability 1, one can adjoin x to M preserving ZFC.
- Modern set theory: the model theory of ZFC?
  - ZFC does not pin down the structure of the universe of sets.
  - Different models of ZFC have different versions of the transfinite ordinals, the powerset operation, the cumulative hierarchy...
  - Answers to classical questions depend on choice of model.

The constructible universe

Granted one model of ZFC, forcing shows there are many others...
without providing a single explicit example.

Denote the set of first-order definable subsets of M by $\mathrm{def}(M)$.

Definition (Gödel)

Constructible hierarchy:

- $L_0 = \emptyset$.
- $L_{\alpha+1} = \mathrm{def}(L_\alpha, \in)$.
- $L_\gamma = \bigcup_{\beta<\gamma} L_\beta$ for limit ordinals γ.

Constructible universe: $L = \bigcup_{\alpha\in\mathrm{Ord}} L_\alpha$.

L is the minimum model of ZFC containing every ordinal.

The constructible universe, continued

The constructible universe L is *canonical* in various ways:

- If M_0 and M_1 are models of ZFC containing the same ordinal numbers, $L^{M_0} = L^{M_1}$.
- Every question about L can be answered in ZFC except those that concern "how many ordinals exist"
  - (Gödel) L satisfies the generalized continuum hypothesis
  - Independent: is there an ordinal α large enough that $L_\alpha \models \text{ZFC}$?
- In general, ZFC does not decide which *large cardinal axioms* are true in L.
  - Such axioms just reveal more levels of constructible hierarchy?
  - Assuming these levels exist, all questions about them have answers, too.

The Jensen covering lemma

Does $V = L$? This is independent of ZFC. Can arbitrary sets be approximated by constructible ones?

Definition

Suppose κ is an infinite cardinal. A class of sets M has the *κ-cover property* if for every $<\kappa$-sized set $A \subseteq M$, there is a $<\kappa$-sized set $B \in M$ such that $A \subseteq B$.

Theorem (Jensen)

One of the following holds:

(1) *For all cardinals $\kappa \geq \omega_2$, L has the κ-cover property.*

(2) *There is an elementary embedding from L to L.*

If (1) holds, we say *L covers V*.

One side of the Jensen dichotomy

Definition

A cardinal κ is *regular* if every $<\kappa$-sized set of cardinals less than κ is bounded below κ. (E.g., $\omega, \omega_1, \omega_2, \ldots$) Otherwise κ is *singular*. (E.g., $\omega_\omega, \omega_{\omega+\omega}, \ldots$) A cardinal κ is a *strong limit cardinal* if for all $\lambda < \kappa$, $|P(\lambda)| < \kappa$. E.g. $|V_{\omega+\omega}|$.

If L covers V, then V resembles L:

- Many instances of the Generalized Continuum Hypothesis must hold. In fact, $|P(\lambda)| = \lambda^+$ for all singular strong limit cardinals λ.
- If λ is a singular cardinal, then λ is singular in L. Moreover L correctly computes the successor of λ.
  - Equivalently, if a constructible set A has cardinality λ, then there is a constructible bijection between A and λ.

The other side of the Jensen dichotomy

Definition

A cardinal is *inaccessible* if it is a regular strong limit cardinal.

If there is an elementary embedding from L to L:

- $V \neq L$.
- For all ordinals α, $|P(\alpha) \cap L| = |\alpha|$.
  - L does not compute any successor cardinals correctly.
  - Every infinite cardinal is inaccessible in L.
- In fact, if κ is an uncountable cardinal, then in L, κ has *every* large cardinal property that can hold in L.

The existence of an embedding from L to L is the *minimum large cardinal axiom beyond L.* Widely believed to be consistent.

The inner model hierarchy

If there is an elementary embedding from L to L, there is a *canonical model beyond L*:

- It's the class $L[j]$ of sets constructible relative to a carefully chosen elementary embedding $j : L \to L$.
- $L[j]$ satisfies an analogous covering lemma; i.e., either $L[j]$ covers V or there is an embedding from $L[j]$ to $L[j]$. The latter yields yet another canonical model.
- Proceeding this way, one obtains a transfinite hierarchy of canonical models of ZFC, satisfying stronger and stronger large cardinal axioms.

Some open problems

(1) How far does the inner model hierarchy go?

- Do all large cardinal axioms have canonical models?
  - The known canonical models can satisfy large cardinal axioms up to a Woodin cardinal and somewhat beyond.

Definition

A cardinal κ is *strongly compact* if every κ-complete filter can be extended to a κ-complete ultrafilter.

The known canonical models of ZFC cannot contain strongly compact cardinals.

(2) Is there a canonical model of ZFC that covers V (unconditionally)? I.e., does the universe of sets admit a deeper analysis...

Ordinal definability

The definable elements of a model of ZFC usually do not themselves form a definable set. (Otherwise consider the least undefinable ordinal, contradiction.)

Definition (Gödel)

A set is ordinal definable (OD) if it is definable in the structure $(V, \in)$ from ordinal parameters.

Equivalently x is OD if for some ordinal α, x is definable in $(V_\alpha, \in)$ without parameters. So ordinal definability is a first-order expressible property.

Hereditary ordinal definability

Definition

A set x is *hereditarily ordinal definable* if x is ordinal definable, every element of x is ordinal definable, every element of every element of x is ordinal definable, etc.

The class HOD of all hereditarily ordinal definable sets is a model of ZFC.

But HOD is *not* a canonical model of ZFC in the sense that L is.

- Different models of ZFC can have wildly different HODs.
- ZFC proves nothing about the internal structure of HOD.
- All known large cardinal axioms are consistent with $V = \mathrm{HOD}$.

Covering for HOD

Canonical models of ZFC are ordinal definable. Therefore if V is covered by a canonical inner model, it is covered by HOD.

Is there an unconditional covering lemma for HOD? The answer is consistently no, but the following possibility remains:

Question

Do large cardinal axioms imply that HOD covers V?

Extendible cardinals

If $j : M \to N$ is a nontrivial elementary embedding between models of ZFC, then there must be an ordinal $\alpha \in M$ such that $j(\alpha) \neq \alpha$. The *critical point* of j is the least such ordinal.

Definition

A cardinal κ is *extendible* if for all $\alpha \geq \kappa$, for some $\beta \geq \alpha$, there is an elementary embedding $j : V_\alpha \to V_\beta$ with critical point κ.

Extendible cardinals are very large. For example, every extendible cardinal is a limit of inaccessible cardinals and strongly compact cardinals.

Woodin's HOD dichotomy

Theorem (Woodin)

Assume κ is extendible. Then exactly one of the following holds:

(1) *For all $\lambda \geq \kappa$,* HOD *has the λ-cover property.*

(2) *Every regular cardinal $\lambda \geq \kappa$ is inaccessible in* HOD.

Looks just like the situation with L. Except no large cardinal axiom can imply (2)?

Conjecture (Woodin)

If there is an extendible cardinal, then (1) *holds.*

This is a version of the *HOD conjecture.*

A case against the HOD conjecture

The HOD conjecture is counterintuitive: given the Jensen covering lemma, almost no one conjectures that L (provably) covers V.

To argue against the HOD conjecture, one might try to strengthen the analogy with Jensen's covering lemma.

- Is the failure of the HOD conjecture equivalent to the existence of large cardinals and "canonical structures" beyond HOD?
- If so, can one present a compelling case that these large cardinals are consistent?

Embeddings into HOD

Question

Can one show the failure of the HOD conjecture is equivalent to the existence of a nontrivial elementary embedding from HOD to HOD?

Theorem

Assume κ is extendible. Then the following are equivalent:

- *The* HOD *conjecture fails.*
- *For all ordinals $\alpha \geq \kappa^+$, there is a model of set theory containing α that elementarily embeds into* $\text{HOD} \cap V_\alpha$.

Constructibly Ramsey cardinals

The failure of Jensen covering for L is equivalent to a partition property of constructible functions.

A cardinal λ is *Ramsey* if for any function $f : [\lambda]^{<\omega} \to \{0,1\}$, there is a set $H \subseteq \lambda$ of cardinality λ such that $f \restriction [H]^n$ is constant for all $n < \omega$.

Theorem (Gloede)

The following are equivalent:

- *There is an elementary embedding from L to L.*
- *For all cardinals $\lambda \geq \omega_1$, if $f : [\lambda]^{<\omega} \to \{0,1\}$ is constructible, there is a set $H \subseteq \lambda$ of cardinality λ such that $f \restriction [H]^n$ is constant for all $n < \omega$.*

Definably Jónsson cardinals

The failure of the HOD conjecture turns out to be equivalent to an infinitary partition property of *ordinal definable* functions.

A cardinal λ is ω-*Jónsson* if for every $f : [\lambda]^\omega \to \lambda$, there is a proper subset H of λ of cardinality λ that is closed under f: i.e., if s is a subset of H of ordertype ω, then $f(s) \in H$.

Theorem

Assume κ is strongly compact. Then the following are equivalent:

- *The* HOD *conjecture fails.*
- *For all cardinals $\lambda \geq \kappa^+$, if $f : [\lambda]^\omega \to \lambda$ is ordinal definable, there is a proper subset of λ of cardinality λ that is closed under f.*

The Kunen inconsistency theorem

"Disanalogy" between Ramsey and ω-Jónsson properties:

Theorem (Erdös-Hajnal)

There are no ω-Jónsson cardinals.

Closely related to the Kunen inconsistency theorem:

Theorem (Kunen)

There is no elementary embedding from V to V.

The existence of ω-Jónsson cardinals *is* consistent if the Axiom of Choice (AC) is dropped.

What about elementary embeddings from V to V?

Large cardinals beyond choice

In the analogy between HOD dichotomy and Jensen covering, could embeddings from V to V play role of embeddings from L to L?

Theorem (Woodin)

If it is consistent that there is an elementary embedding from V to V and a proper class of extendible cardinals, then the HOD *conjecture is false.*

Are large cardinals beyond choice a reason to believe the HOD conjecture is false? Or vice versa?

Structure theory for large cardinals beyond choice

Theorem (G., Schlutzenberg)

An elementary embedding $j : V_\alpha \to V_\alpha$ is definable from parameters over V_α if and only if α is odd.

Let θ_α denote the least ordinal that is not the surjective image of V_β for any $\beta < \alpha$.

Theorem

Assume α is even and there is an elementary $j : V_{\alpha+2} \to V_{\alpha+2}$ with critical point κ.

- *There are fewer than κ regular cardinals between θ_α and $\theta_{\alpha+1}$.*
- (κ-DC) *There are more than κ regular cardinals between $\theta_{\alpha+1}$ and $\theta_{\alpha+2}$.*

Consistency of large cardinals beyond choice

There is a first-order sharpening of Kunen's theorem:

Theorem (Kunen)

For any ordinal λ, there is no elementary embedding from $V_{\lambda+2}$ to $V_{\lambda+2}$.

Theorem (Schlutzenberg)

Assuming very large cardinals, it is consistent with ZF *that for some ordinal λ there is an elementary embedding from $V_{\lambda+2}$ to $V_{\lambda+2}$.*

How far can Schlutzenberg's theorem be extended?

A case for the HOD conjecture

Woodin's approach to the HOD conjecture: assuming large cardinals, build a canonical model satisfying an unconditional covering theorem.

- If this is possible, the HOD conjecture is true.
- It is plausible that there are canonical models for all large cardinal axioms.
- Woodin argues that if there is a canonical model with a supercompact, it must be the "ultimate inner model," containing *all large cardinals*. In particular, it must cover V.

We discuss a more direct approach.

The uniqueness of elementary embeddings

Suppose δ is a regular cardinal and $\alpha < \beta$ are ordinals above δ.

Question

If $j_0, j_1 : V_\alpha \to V_\beta$ are elementary embeddings such that $\sup j_0[\delta] = \sup j_1[\delta]$ and $j_0(\delta) = j_1(\delta)$, must $j_0 \restriction \delta = j_1 \restriction \delta$?

If so, say *embeddings from V_α to V_β are unique on δ.*

If embeddings from V_α to V_β are unique on δ, then for any $j : V_\alpha \to V_\beta$, $j \restriction \delta$ is ordinal definable.

The uniqueness of elementary embeddings, continued

Theorem

Suppose κ is extendible. Then the following are equivalent:

- *For all regular $\delta \geq \kappa$, for all sufficiently large ordinals $\alpha < \beta$, embeddings from V_α to V_β are unique on δ.*
- *The* HOD *conjecture holds.*

Uniqueness problem looks (sort of) tractable.

The uniqueness of elementary embeddings, continued

Most large cardinal properties (including extendibility) can be formulated in terms of elementary embeddings $j : V \to M$ where $M \subseteq V$ is a transitive model of ZFC.

Assuming the HOD Conjecture, Woodin proved that if $j_0, j_1 : V \to M$ are elementary embeddings with the same target model, then $j_0 \restriction \mathrm{Ord} = j_1 \restriction \mathrm{Ord}$.

Very recently:

Theorem

If $j_0, j_1 : V \to M$ are elementary embeddings with the same target model, then $j_0 \restriction \mathrm{Ord} = j_1 \restriction \mathrm{Ord}$.

HOD dichotomy from strongly compact cardinals

Cardinals significantly smaller than extendibles suffice for a version of the HOD dichotomy.

Theorem

Suppose κ is strongly compact. Then one of the following holds:

- *All sufficiently large regular cardinals are inaccessible in* HOD.
- *For all strong limit cardinals $\lambda \geq \kappa$,* HOD *has the λ-cover property.*

In the second case, every singular strong limit cardinal $\lambda \geq \kappa$ is singular in HOD and $(\lambda^+)^{\text{HOD}} = \lambda^+$.

Definability from ultrafilters

Connection between strongly compact cardinals and HOD dichotomy suggests a closer look at the definability theory of κ-complete ultrafilters.

Definition

A set is *κ-completely definable* if it is definable in $(V, \in)$ allowing κ-complete ultrafilters on ordinals as parameters.

The class $\mathrm{HCD}(\kappa)$ of *hereditarily κ-completely definable sets* is a model of ZF.

Theorem

If κ is strongly compact, then $\mathrm{HCD}(\kappa)$ *satisfies* ZFC *and* $\mathrm{HCD}(\kappa)$ *covers V.*

Actually V is a forcing extension of $\mathrm{HCD}(\kappa)$.

Definability from ultrafilters, continued

Definition

Let $\mathrm{HCD} = \bigcap_{\kappa \in \mathrm{Ord}} \mathrm{HCD}(\kappa)$.

Building on work of Usuba:

Theorem

If κ is extendible, then $\mathrm{HCD}(\kappa) = \mathrm{HCD}$.

For any embeddings $j_0, j_1 : V \to M$, $j_0 \restriction \mathrm{HCD} = j_1 \restriction \mathrm{HCD}$.

Theorem

If $j_0, j_1 : V \to M$ are elementary embeddings with critical point above the least extendible cardinal, then $j_0 = j_1$.

A guess

Maybe... embeddings from V to V are consistent, but not with very large cardinals.

Maybe the HOD conjecture consistently fails, but is provable from strong enough large cardinal axioms.

Thanks!