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Set Theory
(online meeting)

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Abstract. Set theory continues to experience dramatic progress, both in pure set theory, with its fundamental techniques of forcing, large cardinals, and inner model theory, and in applied set theory, with its deep connections to other areas of mathematics. Specific topics include: (Pure Set Theory) Forcing axioms, iteration theorems for various classes of forcings, cardinal characteristics and descriptive set theory of the continuum and of generalized Baire spaces, HOD (the hereditarily ordinal definable sets), inner model theory and the core model induction, singular cardinal combinatorics and cardinal arithmetic (pcf theory), partition theorems, Borel reducibility; (Applied Set Theory) Borel and measurable combinatorics, structural Ramsey theory, set theory and operator algebras, topological dynamics and ergodic theory, set theory and Banach spaces, metric structures.

Mathematics Subject Classification (2010): 03Exx.

Introduction by the Organizers

In this workshop we intended to explore topics in both pure and applied set theory which have experienced the most exciting developments over the recent years. The goal was to bring together researchers in set theory from over 10 countries. Unfortunately, the workshop had to be cancelled due to the SARS-CoV-2 pandemic. This report is an attempt to capture the potential content of the non-existent meeting as well as possible.

Let us mention some of the recent major breakthroughs in the area of set theory which would have found a stage for being presented at that meeting.

A major progress in pure set theory would have been reported in D. Asperó's talk. In joint work with R. Schindler, they proved that the strong version of Martin's Maximum, namely MM^{++} implies Woodin's $\mathbb{P}_{\max}$ axiom $(*)$. Until now it was not known even whether these axioms were jointly consistent. This result provides unification of forcing axioms and Woodin's canonical $\mathbb{P}_{\max}$ model in which the Π_2 theory of the structure $H(\aleph_2)$, of all sets whose transitive closure has cardinality at most $\aleph_1$, is maximized. Building on this result, M. Viale defined a natural extension of ZFC with built-in absoluteness and proved an extension of Woodin's Π_2-maximality results in this context.

In a technical tour de force, A. Vignati proved that forcing axioms imply that all isomorphisms between coronas of separable, non-unital C^*-algebras are trivial. This confirms the rigidity conjecture posed by Coskey and Farah in 2013.

M. Gitik developed a novel way to violate the singular cardinal hypothesis (SCH). His forcing has the advantage that it preserves cardinals and cofinalities and can also be used to obtain failure of SCH (an instance of non compactness) together with failure of weak square and even the tree property, which are compactness type principles. Until now the only known way to get failure of SCH and failure of weak square (or with the tree property) simultaneously involved singularizing cardinals. The new construction answers an old combinatorial question and opens up a promising direction of solving other well known open problems, including a question of Woodin from the 80s.

There are also major advances in the theory of large cardinals. G. Goldberg expands his list of truly remarkable insights about strongly compact cardinals. He shows that above a strongly compact cardinal, the theory of large cardinals is, surprisingly, much more tractable than the slew of independence results would suggest. Some of his theorems in particular draw a stark parallel with inner model-like behavior of the large cardinal structure above a super compact cardinal. F. Schlutzenberg would have reported on striking results on Reinhardt cardinals, rank-into-rank embeddings, and new developments in the theory of iterated ultrapowers and extenders under ZF. In particular, he provides significant constraints on the possible existence of rank-into-rank embeddings just in ZF.

In descriptive set theory, Gao's contribution is particularly notable. In joint work with Etedadialiabadi, La Maître, and Melleray, Gao verified a conjecture of Vershik and proved that Hall's universal countable locally finite group can be embedded as a dense subgroup in the isometry group of the Urysohn space and in the automorphism group of the random graph. This is the culmination of a work of many hands.

The proposed participants are a mix of both established mathematicians and some very promising junior people. We hope that in the not too distant future we can bring them together to facilitate discussion and research collaboration for another Oberwolfach meeting of this type.

Workshop (online meeting): Set Theory

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Abstracts

Martin's Maximum^{++} and $(*)$

David Asperó
(joint work with Ralf Schindler)

Classical forcing axioms are natural maximality principles asserting some degree of saturation of the universe relative to forcing axioms: if σ is a simple enough statement that can be forced via a (nice enough) forcing, then σ is in fact true. The strongest such axiom at the level of ω_1 is Martin's Maximum^{++} (MM^{++}). If κ is a supercompact cardinal, then there is a semiproper forcing of size κ, obtained as the limit of an iteration of length κ, and which forces MM^{++} ([2]).

Another maximality principle, of a somewhat different flavour, is Woodin's $\mathbb{P}_{max}$ axiom $(*)$ ([4]). This is the assertion that AD holds in $L(\mathbb{R})$ and the inner model $L(\mathcal{P}(\omega_1))$ is a $\mathbb{P}_{\max}$-extension of $L(\mathbb{R})$. While $\mathrm{AD}^{L(\mathbb{R})}$ follows from large cardinals, the assertion that $L(\mathcal{P}(\omega_1))$ is a $\mathbb{P}_{\max}$-extension of $L(\mathbb{R})$ does not. Although $(*)$ may look prima facie as a minimality assumption about $L(\mathcal{P}(\omega_1))$, it turns out that this axiom implies remarkable forms of maximality for this inner model. For example:

Theorem 1. *(Woodin) Suppose $(*)$ holds and there is a proper class of Woodin cardinals. If A is a set of reals in $L(\mathbb{R})$, σ is a Π_2 sentence, and there is a set-forcing $\mathbb{P}$ such that $\Vdash_{\mathbb{P}} (H_{\omega_2}; \in, \mathrm{NS}_{\omega_1}, A^{\dot{G}_{\mathbb{P}}}) \models \sigma$,[1] then $(H_{\omega_2}; \in, \mathrm{NS}_{\omega_1}, A) \models \sigma$.*

$\mathbb{P}_{\max} \in L(\mathbb{R})$ is a weakly homogeneous forcing which is definable in $L(\mathbb{R})$ without parameters. It follows that, in the presence of large cardinals, $(*)$ completely decides the theory of $L(\mathcal{P}(\omega_1))$ modulo forcing:

Theorem 2. *(Woodin) Suppose there is a proper class of Woodin cardinals. If $\mathbb{P}$ and $\mathbb{Q}$ are partial orders, G is $\mathbb{P}$-generic over V, H is $\mathbb{Q}$-generic over V, $V[G] \models (*)$, and $V[H] \models (*)$, then $L(\mathcal{P}(\omega_1))^{V[G]}$ and $L(\mathcal{P}(\omega_1))^{V[H]}$ have the same theory.*

Despite its nice properties, in order for $(*)$ to be a convincing candidate for a natural axiom, it would have to be compatible with all consistent large cardinal axioms. While $L(\mathbb{R})[g]$ is trivially a model of $(*)$ if g is $\mathbb{P}_{\max}$-generic over $L(\mathbb{R})$, $L(\mathbb{R})[g]$ cannot even have measurable cardinals. In fact, prior to our work it was open whether $(*)$ is compatible with large cardinals beyond the level of Woodin cardinals ([4]).

Looking back at MM^{++}, it turned out that all natural questions about H_{ω_2} seemed to be decided by this axiom[2] and, moreover, that the answers MM^{++} gave to these question seemed to be the same as those provided by $(*)$. For example,

[1] By the presence of the Woodin cardinals, A is universally Baire and therefore there is a canonical interpretation A^G of A in any set-generic extension $V[G]$.

[2] This is not the case for a slight weakening of MM^{++} denoted by MM$^{+\omega}$ (s. [3]).

both axioms have as a consequence a certain Π_2 sentence about H_{ω_2} implying that $L(\mathcal{P}(\omega_1)) \models \mathrm{ZFC}$ and $2^{\aleph_0} = \aleph_2$.

This agreement between MM^{++} and $(*)$ made it natural to conjecture that MM^{++} implies $(*)$. However, the actual connection between classical forcing axioms, whose models are typically obtained by means of iterated over models of ZFC with large cardinals, and $(*)$, whose models were produced by forcing over models of determinacy, remained unknown for a while.

In 2019 we proved that the above conjecture is in fact true.

Theorem 3. *(Asperó-Schindler)* MM^{++} *implies* $(*)$.

This unifying result shows that $(*)$ is compatible with all large cardinals in V as it can in fact be set-forced if there is a supercompact cardinal, and thereby renders $(*)$ a convincing candidate for a natural axiom extending ZFC. It is notable that this natural axiom decides the cardinality of the continuum and in fact implies $2^{\aleph_0} = \aleph_2$.

A few words on the proof of Theorem 3: Given any $A \subseteq \omega_1$ such that $\omega_1 = \omega_1^{L[A]}$, we may define Γ_A to be the set of $\mathbb{P}_{\max}$-conditions $p = (M, I, a)$ such there is a correct iteration of p sending a to A. It was well-known that, assuming MM^{++} (in fact much less) and given any $A \subseteq \omega_1$ as above, Γ_A is a filter of $\mathbb{P}_{\max}$ such that every subset of ω_1 is in $L(\mathbb{R})[\Gamma_A]$. Hence it was enough to prove that Γ_A is generic over $L(\mathbb{R})$. In other words, given a dense subset D of $\mathbb{P}_{\max}$ in $L(\mathbb{R})$, which by our hypothesis is essentially a universally Baire sets, it was enough to produce, using MM^{++}, a correct iteration of a condition $(M, I, a) \in D$ sending a to A. There was a natural scenario for doing this using $\mathcal{L}$-forcing, i.e., adding the desired objects by finite approximations which, in some suitable outer model W, provide finite pieces of information about a certain object in W with properties mirroring the properties we want the desired objects to have. The main technical problem was in showing that some $\mathcal{L}$-forcing $\mathcal{P}$ doing the above also preserves stationary subsets of ω_1. This was finally accomplished through the incorporation of side conditions in the forcing consisting of countable models external to V and, more crucially, the construction of $\mathcal{P}$ as the union of a certain recursively defined sequence of forcing notions.[3]

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[3]This sequence is not an iteration.

Lower Bounds for Mutual Stationarity principles with an application to the theory of iterated Distributive Forcings

Omer Ben-Neria, Martin Zeeman
(joint work with Domink Adolf and Ralf Schindler)

The purpose of the talk is to present new lower bounds for the consistency strength of mutual stationarity principles at the first uncountable cardinals and for the theory of iterated forcing.

The notion of mutually stationary sets was introduced by Foreman and Magidor in [1]. The precise formulation for instance at $\aleph_\omega$ reads that, given some uncountable $\gamma = \aleph_k$ and a sequence S_n such that each S_n is a stationary subset of $\aleph_n$ concentrating on ordinals of cofinality γ there is a stationary set of substructures X of H_θ (θ large given in advance) such that $X \cap \aleph_n \in S_n$ on a tail-end of n's. Foreman and Magidor have shown in [1] that every sequence of stationary sets S_n consisting of ordinals of cofinality ω, is mutually stationary.

In this talk, we will address the extension of this principle to sequences of stationary sets S_n consisting of ordinals of a fixed uncountable cofinality. It is known that such principle is consistent relative to the existence of ω-many supercompact cardinals, and lower bounds at the level of measurable cardinals have been obtained by Koepke and Welch in [2], and Ben-Neria and Zeman. Our first main result improves the lower bound for the mutually stationary principle to the existence of a Woodin cardinal.

The second main result centers around the theory of iterated distributive forcings on different cardinals. We consider sequences of (names) of posets where each $\mathbb{Q}_n$ is a name (with respect to the finite iteration by $\mathbb{Q}_k$, $k < n$) of an $\aleph_n$ distributive posets of size $\aleph_n$. We study the forcing iteration principle which asserts that every such sequence of posets, there exists a cardinal preserving generic extension which contains generic filters for each $\mathbb{Q}_n$. Extending our lower bound methods for the mutually stationary principle, we show that the forcing iteration principle has a similar lower bound. Finally, by building on the iteration theory of Prikry-type forcings, developed by Gitik (cf. [3]), we prove that the forcing iteration principle is consistent relative to ω-many supercompact cardinals.

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Combinatorics of ultrafilters on Boolean algebras

Jörg Brendle
(joint work with Francesco Parente)

Combinatorial properties of free ultrafilters on ω, that is, ultrafilters on the Boolean algebra $\mathcal{P}(\omega)/\mathrm{fin}$, have been extensively studied for the past half century. Central questions have been for example existence of ultrafilters with additional properties like P-points (whose existence was shown to be independent by Shelah), the Rudin-Keisler ordering on ultrafilters, or cardinal invariants related to ultrafilters. Much less is known about ultrafilters on general Boolean algebras, though a strong interest in such ultrafilters has developed in recent years in the wake of the work of Malliaris and Shelah in model theory. We investigate combinatorial aspects of ultrafilters on complete ccc Boolean algebras, with particular focus on the following two closely related topics:

(1) existence and nonexistence of not Tukey maximal ultrafilters
(2) the ultrafilter number

Our results are mainly (but not exclusively) about Cohen and random algebras.

1. (Non-)Existence of non-maximal ultrafilters

Let $\langle D, \leq \rangle$ and $\langle E, \leq \rangle$ be directed sets. We say that $\langle D, \leq \rangle$ is *Tukey reducible* to $\langle E, \leq \rangle$ ($\langle D, \leq \rangle \leq_T \langle E, \leq \rangle$ in symbols) if there are maps $f : D \to E$ and $g : E \to D$ such that for all $d \in D$ and $e \in E$, $f(d) \leq e$ implies $d \leq g(e)$. If $D \leq_T E$ and $E \leq_T D$ both hold, we say D and E are *Tukey equivalent* and write $D \equiv_T E$.

A classical result of Tukey says that if $\langle D, \leq \rangle$ is a directed sets and κ is a cardinal at least the size of D then $\langle D, \leq \rangle \leq_T \langle [\kappa]^{<\omega}, \subseteq \rangle$. Note that if U is an ultrafilter on a Boolean algebra $\mathbb{A}$ then $\langle U, \geq \rangle$ is a directed set. In particular, $\langle U, \geq \rangle \leq_T \langle [\mathbb{A}]^{<\omega}, \subseteq \rangle$. We call U *Tukey maximal* if $U \equiv_T [\mathbb{A}]^{<\omega}$. A simple characterization of maximality is

(1) [DT] An ultrafilter U on $\mathbb{A}$ is Tukey maximal if and only if there exists a subset $X \subseteq U$ with $|X| = |\mathbb{A}|$ such that every infinite $Y \subseteq X$ is unbounded in U.

Tukey reducibility of free ultrafilters over ω has been studied intensively for the past decade, see e.g. [DT]. It is well-known that P-points are not Tukey maximal. On the other hand, an old question of Isbell asking for non-maximal ultrafilters over ω in ZFC is still open. A connection to ultrafilters on complete ccc Boolean algebras is given by

(2) Let U be an ultrafilter on a complete ccc Boolean algebra $\mathbb{A}$. Then there is a free ultrafilter V on ω such that $V \leq_T U$.

Let $\mathbb{C}_\kappa$ ($\mathbb{B}_\kappa$, respectively) denote the algebra for adding κ many Cohen reals (random reals, resp.).

Theorem 1. *Assume $\kappa^{\aleph_0} = \kappa$. Then every ultrafilter on $\mathbb{C}_\kappa$ and $\mathbb{B}_\kappa$ is Tukey maximal.*

In fact this holds for a larger class of forcing notions defined as quotients of the Baire subsets of 2^κ by an ideal obtained from an index invariant σ-ideal on 2^ω as in Kunen's framework [Ku].

Non-maximal ultrafilters over complete ccc Boolean algebras of size $\mathfrak{c}$ are more difficult to obtain because the existence of such an ultrafilter implies the existence of a non-maximal ultrafilter over ω (by item 2), which, as mentioned, is still open in ZFC. The situation for $\mathcal{P}(\omega)/\text{fin}$ suggests we look for "P-point like" objects. Following Starý [St], we say an ultrafilter U on a complete ccc Boolean algebra $\mathbb{A}$ is a *coherent P-ultrafilter* if for every maximal antichain $\{p_i : i \in \omega\}$ in $\mathbb{A}$, the set $\{X \subseteq \omega : \bigvee\{p_i : i \in X\} \in U\}$ is a P-point over ω.

Theorem 2. *Let $\mathbb{A}$ be a complete ccc Boolean algebra whose density is strictly smaller than its size. Then any coherent P-ultrafilter on $\mathbb{A}$ is not Tukey maximal.*

Since Starý [St] proved the existence of coherent P-ultrafilters on complete ccc Boolean algebras of size $\mathfrak{c}$ under $\mathfrak{d} = \mathfrak{c}$, we obtain for example:

Corollary 3. *Assuming $\mathfrak{d} = \mathfrak{c}$ there is a non-maximal ultrafilter on $\mathbb{C}_\omega$.*

We believe this is also true for $\mathbb{B}_\omega$. However, the approach above does not work because the density of $\mathbb{B}_\omega$ is above $\mathfrak{d}$, and so far we only have the following result, which uses a much stronger assumption.

Theorem 4. *Assuming $\diamondsuit$ there exists a non-maximal ultrafilter on $\mathbb{B}_\omega$.*

2. Ultrafilter numbers

Let $\mathbb{A}$ be an infinite Boolean algebra. The *ultrafilter number* $\mathfrak{u}(\mathbb{A})$ of $\mathbb{A}$ is the least size of a basis of an ultrafilter on $\mathbb{A}$. So $\mathfrak{u} = \mathfrak{u}(\mathcal{P}(\omega)/\text{fin})$. The discussion about Tukey reducibility shows

(3) if $\mathfrak{u}(\mathbb{A}) < |\mathbb{A}|$ then there is a non-maximal ultrafilter on $\mathbb{A}$ (follows from item 1)
(4) if $\mathbb{A}$ is a complete ccc Boolean algebra, then $\mathfrak{u} \leq \mathfrak{u}(\mathbb{A})$ (follows from item 2)
(5) in particular, $\max\{\mathfrak{u}, \kappa\} \leq \mathfrak{u}(\mathbb{C}_\kappa), \mathfrak{u}(\mathbb{B}_\kappa) \leq \kappa^{\aleph_0}$

Other known lower bounds are

(6) $\mathfrak{d} \leq \mathfrak{u}(\mathbb{C}_\omega)$
(7) ([CKP] and [Bu]) $\mathsf{cof}(\mathcal{N}) \leq \mathfrak{u}(\mathbb{B}_\omega)$

Thus, using any model for $\mathfrak{u} < \mathfrak{d}$ (see e.g. [BS]) we see

Corollary 5. *$\mathfrak{u} < \mathfrak{u}(\mathbb{C}_\omega)$ and $\mathfrak{u} < \mathfrak{u}(\mathbb{B}_\omega)$ are consistent.*

By an ω_1-stage finite support iteration (fsi) of a σ-centered forcing over a model for large continuum we obtain

Theorem 6. *$\mathfrak{u}(\mathbb{C}_\omega) < \mathfrak{c}$ is consistent.*

This can be extended to $\mathfrak{u}(\mathbb{C}_{\omega_1})$ as well. By the σ-centeredness $\mathsf{non}(\mathcal{N})$ and thus $\mathsf{cof}(\mathcal{N})$ will be large in this model, and we additionally obtain the consistency of $\mathfrak{u}(\mathbb{C}_\omega) < \mathfrak{u}(\mathbb{B}_\omega)$ (see item 7). Another fsi gives

Theorem 7. *$\mathfrak{u}(\mathbb{B}_\omega) < \mathfrak{c}$ is consistent.*

Again this also works with $\mathfrak{u}(\mathbb{B}_{\omega_1})$. We do not know whether $\mathfrak{u}(\mathbb{B}_\omega)$ can be strictly smaller than $\mathfrak{u}(\mathbb{C}_\omega)$. Neither do we know whether $\mathfrak{u}(\mathbb{C}_\omega)$ ($\mathfrak{u}(\mathbb{B}_\omega)$, resp.) can be strictly smaller than $\mathfrak{u}(\mathbb{C}_{\omega_1})$ ($\mathfrak{u}(\mathbb{B}_{\omega_1})$, resp.).

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A universal characterization of standard Borel spaces

Ruiyuan Chen

A **standard Borel space** is a measurable space that is isomorphic to a Borel subspace of Cantor space $2^{\mathbb{N}}$. Standard Borel spaces and **Borel maps** (i.e., preimages of Borel sets are Borel) are ubiquitous in descriptive set theory as a basic model of "definable sets" and "definable functions" between them. The notion of "definability" here is a coarse one where, roughly speaking, all countable information is considered definable. As a result, standard Borel spaces are closed under many familiar set operations of countable arity, e.g., countable products, countable unions, Borel preimages, injective (or more generally countable-to-1) Borel images. In this work, we give an abstract characterization of the category **SBor** of standard Borel spaces and Borel maps as the *universal* category equipped with some countable-arity operations, including the ones above, subject to some simple compatibility axioms. This gives a precise formulation of the idea that standard Borel spaces are a "canonical" notion of "definable space". The proof combines methods from descriptive set theory, Boolean algebras, and categorical logic.

The operations on **SBor** are formalized in terms of categorical **limits** and **colimits**, which are defined in general categories as universal objects equipped with morphisms to/from a diagram. For example, the **product** of two objects X, Y in a category C is by definition a universal object $X \times Y \in \mathsf{C}$ equipped with two morphisms $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$. Other types of limits give categorical generalizations of such set operations as preimages of subsets, the equality binary relation on a set, and the kernel of a function. Similarly, colimits give categorical generalizations of disjoint unions (**coproducts**) and quotients by equivalence relations. A **countably complete, Boolean countably extensive** category is one equipped with countable limits and countable coproducts obeying a natural compatibility axiom satisfied by disjoint unions of sets[1] and in addition all of whose subobjects have complements. Thus, these are categories equipped with abstract versions of familiar countable-arity set operations.

Theorem 1. **SBor** *is the* ***initial*** *countably complete, Boolean countably extensive category: for any other such category* C*, there is a unique-up-to-unique-isomorphism functor* $\mathbf{SBor} \to \mathsf{C}$ *preserving countable limits and countable coproducts.*

In other words, any other category C admitting the same operations must contain an essentially unique "image" of **SBor** inside. We also have a generalization to higher cardinalities:

Theorem 2. *For any infinite regular cardinal* κ*, the dual of the category* $\kappa\mathbf{Bool}_\kappa$ *of* κ*-presented* κ*-complete Boolean algebras is the initial* κ*-complete, Boolean* κ*-extensive category.*

This result implies Theorem 1, because it follows from a classical theorem of Loomis-Sikorski (cf. [1]) that $\omega_1\mathbf{Bool}_{\omega_1}$ is dually equivalent to **SBor**. The proof of Theorem 2 is by presenting $\kappa\mathbf{Bool}_\kappa^{\mathrm{op}}$ as the syntactic category of a theory in a restricted subset of the infinitary logic $\mathcal{L}_{\kappa\kappa}$. The main ingredient is a quantifier-elimination lemma, based on a proof of the strong amalgamation property for κ-complete Boolean algebras due to LaGrange (cf. [2]).

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Variations on Cohen forcing

James Cummings

(joint work with Arthur Apter)

Cohen forcing is a flexible tool for manipulating the values of the continuum function. In situations where we want to manipulate the continuum function and preserve large cardinals, we have to deal with posets of the form $j(\mathrm{Add}(\kappa, \lambda))$ where $j : V \to M$ is an elementary embedding with $\mathrm{crit}(j) = \kappa$ and ${}^\kappa M \subseteq M$: this is a κ^+-closed poset whose properties depend on the nature of the embedding j.

In connection with his work on getting failure of the SCH from weak hypotheses, Woodin showed that if GCH holds and U is a normal measure on κ then $j_U(\mathrm{Add}(\kappa, \kappa^{++}))$ is equivalent to $\mathrm{Add}(\kappa^+, \kappa^{++})$. Analogous results hold for posets of the form $j_U(\mathrm{Add}(\kappa, \lambda))$ up to about $\lambda = \kappa^{+\kappa}$, at which point the argument breaks down.

[1]namely, morphisms $X \to \bigsqcup_i Y_i$ are equivalent to partitions $X = \bigsqcup_i X_i$ together with morphisms $X_i \to Y_i$

We introduce a natural forcing poset $\mathrm{Add}^*(\kappa, \lambda)$ which shares many of the pleasant features of $\mathrm{Add}(\kappa^+, \lambda)$: assuming GCH it is κ^+-closed, κ^{++}-cc and adds λ many mutually generic Cohen subsets of κ^+. It also has a universal property: if E is an appropriate short extender with critical point κ then $\mathrm{Add}^*(\kappa, \lambda)$ projects onto $j_E(\mathrm{Add}(\kappa, \lambda))$. Add^* forcing can be used to give an alternative proof of a result of Friedman and Honzik:

Theorem 1 (Friedman and Honzik). *Let GCH hold and let F be a locally definable Easton function. Then there is a class Reverse Easton forcing poset such that in the generic extension:*

(1) *Cardinals and cofinalities are preserved.*
(2) $2^\kappa = F(\kappa)$ *for all regular* κ.
(3) *Strong cardinals and supercompact cardinals from the ground model are preserved.*

On wide Aronszajn trees in the presence of MA

Mirna Džamonja

1. Introduction

We study the class $\mathcal{T}$ of trees of height and size $\aleph_1$, but with no uncountable branch. We call such trees *wide Aronszajn trees.* A particular instance of such a tree is a classical Aronszajn tree, so the class $\mathcal{A}$ of Aronszajn trees satisfies $\mathcal{A} \subseteq \mathcal{T}$.

Definition 1. For two trees T_1, T_2, we say that T_1 is *weakly embeddable* in T_2 and we write $T_1 \leq T_2$, if there is $f : T_1 \to T_2$ such for all $x, y \in T_1$

$$x <_{T_1} y \implies f(x) <_{T_2} f(y).$$

We are interested in the structure of $(\mathcal{T}, \leq)$ and $(\mathcal{A}, \leq)$.

Our first result is Theorem 2, which proves that under $MA(\omega_1)$ there is no universal element in $(\mathcal{A}, \leq)$. The is a result of Todorčević from [2] to which we now give another proof. The second result is Theorem 5, which shows that under $MA(\omega_1)$ every wide Aronszajn tree embeds into an Aronszajn tree. Putting the two results together, we obtain the main result of the paper, Theorem 8, which shows that under $MA(\omega_1)$ the class $(\mathcal{T}, \leq)$ has no universal element. This resolves a question raised by [1].

Our paper contains two main theorems. We state them and define the corresponding forcing notions used in the proof.

2. Embeddings between Aronszajn trees and the non-existence of a universal element under MA

The following theorem is due to Todorčević, [2]. We give a different proof.

Theorem 2. *For every tree $T \in \mathcal{A}$, there is a ccc forcing which adds a tree in $\mathcal{A}$ not weakly embeddable into T. In particular, under the assumption of $MA(\omega_1)$ there is no Aronszajn tree universal under weak embeddings.*

Our proof is obtained using the following notion of forcing.

Definition 3. Suppose that $T \in \mathcal{A}$, we shall define a forcing notion $\mathbb{Q} = \mathbb{Q}(T)$ to consist of all $p = (u^p, v^p, <_p, c^p)$ such that:

(1) $u^p \subseteq \omega_1 \cup \{\langle\rangle\}$, $v^p \subseteq T$ are finite and $\langle,\rangle \in v^p$,
(2) if $\alpha \in v^p$ then there $\beta \in u^p$ with $\mathrm{ht}(\alpha) = \mathrm{ht}(\beta)$,
(3) $<_p$ is a partial order on u^p such that $\alpha <_p \beta$ implies $\mathrm{ht}(\alpha) < \mathrm{ht}(\beta)$ and which fixes $\alpha \cap_{<_p} \beta \in u^p$ for every two different elements α, β of u^p and fixes a $\langle\rangle$ root of u^p,
(4) c^p is a function from $\bigcup_{\delta \in \omega_1 \cap \mathrm{Lim}} \mathrm{lev}_\delta(u^p) \times \mathrm{lev}_\delta(v^p)$ to ω such that:
if $c(x_1, y_1) = c(x_2, y_2)$ and $(x_1, y_1) \neq (x_2, y_2)$, then $\alpha(x_1, y_1) \neq \alpha(x_2, y_2)$, $x_1 \perp_{T_1} x_2$, $y_1 \perp_{T_2} y_2$ and
$$\mathrm{ht}(x_1 \cap_{T_1} x_2) > \mathrm{ht}(y_1 \cap_{T_2} y_2).$$

The order $p \leq q$ on $\mathbb{Q}$ is given by inclusion $u^p \subseteq u^q, v^p \subseteq v^q, <_p \subseteq <_q, c^p \subseteq c^q$ with the requirement that if $p \leq q$, then the intersection and the root given by $<_p$ are preserved in $<_q$.

Remark 4. Theorem 2 gives another proof of the main result of [3], which is that under $MA(\omega_1)$ all Aronszajn trees are special.

3. Embedding wide Aronszajn trees into Aronszajn trees

The proof of the following theorem is the main method.

Theorem 5. *For every tree $T \in \mathcal{T}$, there is a ccc forcing which adds a tree in $\mathcal{A}$ into which T weakly embeds. In particular, under the assumption of $MA(\omega_1)$ the class $\mathcal{A}$ is cofinal in the class $(\mathcal{T}, \leq)$.*

We give the definition of the forcing used to prove this theorem. The forcing is dual to the one in §2, in the sense that we now start with a tree T in $\mathcal{T}$ and generically add an Aronszajn tree that T weakly embeds to. We use the control function c to make sure that the generic tree does not have an uncountable branch. For the definition of the forcing, we represent every $T \in \mathcal{T}$ by an isomorphic copy which is a subtree of ${}^{<\omega_1}\omega_1$.

Definition 6. Suppose that $T \subseteq {}^{\omega_1 >}\omega_1$ is a tree of size $\aleph_1$ and with no uncountable branches, we define a forcing notion $\mathbb{P} = \mathbb{P}(T)$ to consist of all
$$p = (u^p, v^p, <_p, f^p, c^p)$$
such that:

(1) $u^p \subseteq T$, $v^p \subseteq \omega_1$ are finite and $\langle\rangle \in u^p$,
(2) u^p is closed under intersections,
(3) $<_p$ is a partial order on v^p,
(4) f^p is a surjective weak embedding from $(u^p, \subset)$ onto $(v^p, <_p)$,
(5) if $f(\rho) <_p f(\sigma)$, then there are $\rho' \subset \sigma'$ such that $f(\rho') = f(\rho)$, $f(\sigma) = f(\sigma')$ and $\rho' \subset \sigma'$,
(6) for every $\eta \in u^p$, we have $\mathrm{ht}(f^p(\eta)) = \lg(\eta)$,
(7) c^p is a function from v^p into ω such that

$$\alpha <_p \beta \implies c^p(\alpha) \neq c^p(\beta).$$

The order $p \leq q$ on $\mathbb{P}$ is given by inclusion $u^p \subseteq u^q, v^p \subseteq v^q, <_p \subseteq <_q$ and $c^p \subseteq c^q$.

We remark that putting Theorem 5 together with the results of [2], gives a nice consequence about the class of Lipschitz trees, as follows.

Corollary 7. *Under $MA(\omega_1)$ the class $\mathcal{L}$ of Lipschitz trees is cofinal in the class of wide Aronszajn trees $(\mathcal{T}, \leq)$.*

4. Conclusion

Putting the results of Section §2 and Section §3 together, we obtain our main theorem, as follows.

Theorem 8. *Under $MA(\omega_1)$, there is no wide Aronszajn tree universal under weak embeddings.*

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Gaps and Towers at uncountable cardinals

Vera Fischer
(joint work with Diana C. Montoya, Jonathan Schilhan and Daniel T. Soukup)

In this project we study pseudo-intersection and tower numbers on uncountable regular cardinals, and particular focus on the question if these two cardinal characteristics are equal. Let κ be a regular uncountable cardinal. We say that a family $\mathcal{F} \subseteq [\kappa]^\kappa$ has the strong intersection property (appreviated SIP) if for every $\mathcal{H} \in [\mathcal{F}]^{<\kappa}$, the cardinality of $\bigcap \mathcal{H}$ is κ. A set $A \subseteq \kappa$ is a pseudo-intersection of a family $\mathcal{F} \subseteq [\kappa]^\kappa$ if for each $F \in \mathcal{F}$, $A \subseteq^* F$, which means that $|A \backslash F| < \kappa$ for each $F \in \mathcal{F}$. We say that a family $\mathcal{T} \subseteq [\kappa]^\kappa$ is a κ-tower (or just tower when κ is clear from the context) if $\mathcal{T}$ is $\leq^*$ well-founded, $\mathcal{T}$ has the SIP, but $\mathcal{T}$ no pseudo-intersection of cardinality κ. Recall that $\mathfrak{p}(\kappa)$ is defined as the least cardinality of a family with SIP on κ, which does not have a pseudo-intersection and $\mathfrak{t}(\kappa)$ is defined as the least cardinality of a κ-tower. In the above paper, we introduce a natural higher analogue of the notion of a gap (see Definition 2.6 of [1]), which gives us the following interesting analogue of a theorem of Malliaris-Shelah (see [2]), namely the following:

Theorem 1. *Let κ be a regular cardinal such that $\kappa^{<\kappa} = \kappa$. Then either $\mathfrak{p}(\kappa) = \mathfrak{t}(\kappa)$ or there is $\lambda < \mathfrak{p}(\kappa)$ and a club-supported $(\mathfrak{p}(\kappa), \lambda)$-gap of slaloms.*

While the existence of gaps as in the above theorem is unclear, the result is a promising step in lifting the celebrated result of Malliaris-Shelah stating that $\mathfrak{p} = \mathfrak{t}$. As a result of our study on gaps of slaloms, we obtain:

Theorem 2. *If κ is a regular uncountable cardinal, then $\mathfrak{p}(\kappa)$ is regular.*

Moreover, we study the club variants $\mathfrak{p}_{cl}(\kappa)$ and $\mathfrak{t}_{cl}(\kappa)$ of $\mathfrak{p}(\kappa)$ and $\mathfrak{t}(\kappa)$ respectively, where $\mathfrak{p}_{cl}(\kappa)$ is defined as the least cardinality of a family of clubs on κ which has the SIP but no pseudo-intersection and $\mathfrak{t}_{cl}(\kappa)$ is the least cardinality of a κ-tower consisting of clubs. We show that $\mathfrak{p}_{cl}(\kappa) = \mathfrak{t}_{cl}(\kappa) = \mathfrak{b}(\kappa)$ and obtain the following result:

Theorem 3. *Let $\kappa < \lambda$ be regular uncountable cardinals, where $\kappa = \kappa^{<\kappa}$. Then there is a κ-closed, κ^+-cc forcing extension, in which $\mathfrak{p}(\kappa) = \kappa^+ < \mathfrak{p}_{cl}(\kappa) = \lambda = 2^\kappa$.*

The consistency of $\mathfrak{p}(\kappa) < \mathfrak{b}(\kappa)(= \mathfrak{p}_{cl}(\kappa))$ is originally due to Shelah and Spasojevic, [3]. Our techniques however, significantly differ from theirs: We add κ-Cohen reals and successively diagonalize the club filter, while preserving a Cohen witness to $\mathfrak{p}(\kappa) = \kappa^+$.

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Vershik's Conjecture for Ultraextensive Spaces

Su Gao

(joint work with Mahmood Etedadialiabadi, François La Maître, and Julien Melleray)

Extending previous work by Bhattacharjee, McPherson, Vershik, Pestov, Solecki, and Rosendal, we introduce a notion of ultraextensive metric spaces and state some properties of such spaces, including that their isometry groups all contain dense locally finite groups. Then we verify a conjecture of Vershik which states that Hall's universal countable locally finite group can be embedded as a dense subgroup in the isometry group of the Urysohn space and in the automorphism group of the random graph. In fact, we show the same for all automorphism groups of known infinite ultraextensive spaces. These include, in addition, the isometry group of the rational Urysohn space, the isometry group of the ultrametric Urysohn spaces, and the automorphism group of the universal K_n-free graph for all $n \geq 3$. Furthermore, we show that finite group actions on finite metric spaces or finite relational structures form a Fraïssé class, where Hall's group appears as the acting group of the Fraïssé limit. We also embed continuum many non-isomorphic countable universal locally finite groups into the isometry groups of various Urysohn spaces, and show that all dense countable subgroups of these groups are mixed identity free (MIF). Finally, we give a characterization of the isomorphism type of the isometry group of the Urysohn Δ-metric spaces in terms of the distance value set Δ.

Some applications of Extender based forcings with overlapping extenders.

Moti Gitik

Let $\mathfrak{a}$ be a set of regular cardinals, with $|\mathfrak{a}| < \min(\mathfrak{a})$, and let J be an ideal on $\mathfrak{a}$. If $f, g \in \prod \mathfrak{a}$, then $f <_J g$ iff $\{\nu \in \mathfrak{a} \mid f(\nu \geq g(\nu)\} \in J$.

Definition 1. (S. Shelah [11]) A regular cardinal λ is called $\operatorname{tcf}(\prod \mathfrak{a}, <_J)$ iff there exists an $<_J$-increasing sequence of functions $\langle f_\alpha \mid \alpha < \lambda \rangle$ in $\prod \mathfrak{a}$ such that for every $g \in \prod \mathfrak{a}$ there is $\alpha < \lambda$ with $g <_J f_\alpha$.

Definition 2. (S. Shelah [11]) Let κ be a singular cardinal.

$$\begin{aligned}\operatorname{pp}(\kappa) = \sup(\{\operatorname{tcf}(\prod \mathfrak{a}, <_J) \mid \mathfrak{a} \subseteq \kappa, \mathfrak{a} \text{ consists of regular cardinals, } \mathfrak{a} \text{ is unbounded}\\ \text{in } \kappa,\ |\mathfrak{a}| = \operatorname{cof}(\kappa),\ J \text{ is an ideal on a which includes}\\ \text{the ideal } J^{\text{bd}} \text{ of bounded subsets of } \mathfrak{a} \text{ and such that}\\ \operatorname{tcf}(\prod \mathfrak{a}, <_J) \text{ exists } \}\end{aligned}$$

Shelah introduced $\operatorname{pp}(\kappa)$ as a replacement for 2^κ , i.e. for the usual power set operation, for singular cardinals κ. By the König Lemma, $\operatorname{cof}(2^\kappa) > \kappa$. S. Shelah (Problem ($\epsilon$), Analytical Guide, [11]) asked:

Question 3. *(Shelah) Does the König Lemma remain true if we replace* 2^κ *by* $\operatorname{pp}(\kappa)$*, i.e., is* $\operatorname{cof}(\operatorname{pp}(\kappa)) > \kappa$*?*

It turns out that the answer is negative. For example it is possible, after a forcing with an extender based forcings with overlapping extenders, to have the following situation: *There is a cardinal* κ *of cofinality* ω *such that* $\operatorname{cof}(\operatorname{pp}(\kappa)) = \omega_1$.

In the early 80s H. Woodin asked the following two questions:

Question 4. *(Woodin)*

(*i*) *Is it possible to have a singular strong limit κ such that weak $\Box_\kappa$ fails and $2^\kappa > \kappa^+$?*
(*ii*) *Is it possible to have a singular strong limit κ such that the tree property over $\kappa^+ + 2^\kappa > \kappa^+$?*

Both questions were answered affirmatively, the first one by A. Sharon and myself in [7] (even for a bit stronger approachability property $\neg\mathrm{AP}_{\kappa^+}$) and the second by I. Neeman [8]. D. Sinapova extended this results to cardinals of uncountable cofinality. A. Sharon [10] solved a similar question for the reflection property Ref_{κ^+}. Recently, Ben-Neria, Hayut, Unger [1] gave a different construction and Poveda, Rinot, Sinapova [9] formulated a general framework. Using extenders based forcings with overlapping extenders, new constructions of models of:

(1) $\neg\mathrm{AP}_{\kappa^+} + 2^\kappa > \kappa^+$
(2) $\mathrm{TP}_{\kappa^+} + 2^\kappa > \kappa^+$
(3) $\mathrm{Ref}_{\kappa^+} + 2^\kappa > \kappa^+$

for a singular strong limit κ, are given. Actually all three properties can hold in the same model.

Preprints are available at: http://www.math.tau.ac.il/˜gitik/somepapers.html

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Structure theorems from strongly compact cardinals

Gabe Goldberg

Expanding on ideas due to Hamkins in the context of forcing and Woodin in the context of inner model theory, we prove several theorems which suggest that the large cardinal structure of the universe of sets above a strongly compact cardinal is more tractable than the ubiquity of independence results at this level would suggest. The main idea behind our results is the following improvement on a result of Woodin, who used a supercompact cardinal instead of a strongly compact one.

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Theorem 1. *Suppose κ is a strongly compact cardinal and U is a countably complete ultrafilter in V_κ. Then the ultrapower of V by U has the κ-approximation and cover properties.*

The proof of this result has a number of applications. The first concerns the notion of a cardinal preserving embedding, introduced by Caicedo [1]. If M is an inner model, an elementary embedding $j : V \to M$ is said to be cardinal preserving if every cardinal of M is a cardinal in V. Caicedo asked whether there can be such an embedding. We use the theorem above to prove the nonexistence of cardinal preserving embeddings assuming large cardinals.

Theorem 2. *Suppose there is a proper class of strongly compact cardinals. Then there are no cardinal preserving embeddings.*

This result can be viewed as a generalization of the Kunen Inconsistency Theorem, but the proof is quite different from all of the known proofs of Kunen's theorem.

Second, we give a partial answer to an old question of Silver [2], in spite of an independence result due to Sheard [3]. A nonprincipal ultrafilter U on a cardinal λ is indecomposable if for any descending sequence $\{A_\alpha\}_{\alpha<\eta}$ of sets in U with $\omega_1 \leq \eta < \lambda$, $\bigcap_{\alpha<\eta} A_\alpha \in U$. Indecomposability is roughly "λ-completeness minus ω_1-completeness".

If there are infinitely many measurable cardinals $\langle\kappa_n\rangle_{n<\omega}$, there is an indecomposable ultrafilter on $\lambda = \sup_{n<\omega}\kappa_n$ that is not λ-complete: if D is a nonprincipal ultrafilter on ω and for each $n < \omega$, U_n is a nonprincipal κ_n-complete ultrafilter on κ_n, then

$$D - \lim_{n<\omega} U_n = \{A \subseteq \lambda \;:\; \{n < \omega \;:\; A \cap \kappa_n \in U_n\} \in D\}$$

is an indecomposable ultrafilter on λ. Can there be indecomposable ultrafilters on cardinals that are neither measurable nor the limit of countably many measurable cardinals? This question, posed by Silver [2], cannot be resolved in ZFC: in the canonical inner models, the answer is no, while in a forcing extension constructed by Sheard [3], the answer is yes. Once one reaches a strongly compact cardinal, however, inner model-like behaviour wins out:

Theorem 3. *Suppose κ is strongly compact and $\lambda \geq \kappa$ carries an indecomposable ultrafilter. Then λ is either a measurable cardinal or a countable cofinality limit of measurable cardinals.*

In fact, one can completely characterize the indecomposable ultrafilters above the first strongly compact cardinal as exactly those ultrafilters resulting from the construction above.

Theorem 4. *Suppose κ is strongly compact, $\lambda \geq \kappa$, and U is an indecomposable ultrafilter on λ. Then U is either λ-complete or else for some ultrafilter D on ω, some sequence $\langle\kappa_n\rangle_{n<\omega}$ of distinct measurable cardinals, and some sequence $\langle U_n\rangle_{n<\omega}$ of κ_n-complete ultrafilters on κ_n, $U = D - \lim_{n<\omega} U_n$.*

Our last result concerns the partial forms of strong compactness defined by Bagaria-Magidor [4]. A cardinal κ is almost strongly compact if for all $\nu < \kappa$, every κ-complete filter on κ extends to a ν-complete ultrafilter on κ. In [5], we prove that assuming an inner model principle, the first almost strongly compact cardinal is strongly compact (and in fact supercompact). Whether this is provable outright is an open question, posed by Boney and Brooke-Taylor. Our techniques give the following partial answer:

Theorem 5. *Assume the Singular Cardinals Hypothesis. If the least almost strongly compact cardinal has uncountable cofinality, it is strongly compact.*

It is not true in general that every almost strongly compact cardinal is strongly compact, since the almost strongly compact cardinals form a closed class, while every strongly compact cardinal is measurable. However, for successor almost strongly compacts, one can almost prove the equivalence of the two concepts:

Corollary 6. *For any ordinal α, if the $(\alpha+1)$-st almost strongly compact cardinal has uncountable cofinality, it is strongly compact.*

The corollary follows by applying the preceding theorem in $V[G]$ where $G \subseteq \mathrm{Col}(\omega, \kappa)$ is generic for the collapse of the α-th almost strongly compact cardinal. Note that the Singular Cardinals Hypothesis holds in this model.

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Bi-interpretation of weak set theories

Joel D. Hamkins

(joint work with Alfredo R. Freire)

Set theory exhibits a truly robust mutual interpretability phenomenon: In any model of one set theory we can define models of diverse other set theories and vice versa. In any model of ZFC, we can define models of ZFC + GCH and also of ZFC + ¬CH and so on in hundreds of cases. And yet, it turns out, in no instance do these mutual interpretations rise to the level of bi-interpretation. Ali Enayat proved that distinct theories extending ZF are never bi-interpretable, and models of ZF are bi-interpretable only when they are isomorphic. So there is no nontrivial bi-interpretation phenomenon in set theory at the level of ZF or above. Nevertheless, for natural weaker set theories, we prove, including ZFC^- without power set and Zermelo set theory Z, there are nontrivial instances of bi-interpretation. Specifically, there are well-founded models of ZFC^- that are bi-interpretable, but not isomorphic – even $\langle H_{\omega_1}, \in\rangle$ and $\langle H_{\omega_2}, \in\rangle$ can be bi-interpretable – and there are distinct bi-interpretable theories extending ZFC^-. Similarly, using a construction of Mathias, we prove that every model of ZF is bi-interpretable with a model of Zermelo set theory in which the replacement axiom fails.

Entangledness in Suslin lines and trees

John Krueger

This talk is concerned with the property of entangledness in Suslin lines and trees. The idea of an entangled linear order was originally introduced by Abraham and Shelah in the context of ω_1-dense sets of reals. Recall that an uncountable linear order L is *n-entangled*, where n is a positive integer, if for any pairwise disjoint sequence $\langle (a_{\xi,0}, \ldots, a_{\xi,n-1}) : \xi < \omega_1 \rangle$ of increasing n-tuples of L and any function $g : n \to 2$, there exist $\xi, \beta < \omega_1$ such that for all $i < n$, $a_{\xi,i} <_L a_{\beta,i}$ iff $g(i) = 1$. And L is *entangled* if it is n-entangled for all positive integers n. The concept of entangledness is closely tied to topological properties of the linear order L. For example, if L is 2-entangled then L has the countable chain condition, and if L is 3-entangled then L is separable. So any 3-entangled dense linear order is order isomorphic to a set of reals. Todorcevic proved that if there exists an entangled linear order, then there exist c.c.c. forcing posets $\mathbb{P}$ and $\mathbb{Q}$ such that $\mathbb{P} \times \mathbb{Q}$ is not c.c.c. It follows that Martin's axiom together with the negation of the continuum hypothesis implies that there does not exist an entangled linear order.

Recall that a Suslin line is a linear order with the countable chain condition which is not separable. By the remarks above, a Suslin line cannot be 3-entangled. In this talk we introduce a natural weakening of the property of entangledness which can consistently be satisfied by a Suslin line. For any positive integer n, we say that a linear order L is *weakly n-entangled* if the property described in the first paragraph holds, except only for pairwise disjoint sequences $\langle (a_{\xi,0}, \ldots, a_{\xi,n-1}) : \xi < \omega_1 \rangle$ of increasing n-tuples of L which have the property that there exist $c_0 <_L \ldots <_L c_{n-1}$ such that for all $\xi < \omega_1$ and $i < n-1$, $a_{\xi,i} <_L c_i <_L a_{\xi,i+1}$.

We proved that it is consistent for a Suslin line to be weakly n-entangled for all positive integers n. Any dense c.c.c. linear order L is weakly 2-entangled iff it is 2-entangled, so it is consistent for a Suslin line to be 2-entangled. However, this equivalence fails if L is not dense. If L is dense and separable, then L is n-entangled iff L is weakly n-entangled. Thus, we have found a natural weakening of entangledness which coincides with entangledness for dense separable linear orders, but can consistently be satisfied by Suslin lines.

It is a reasonable question to ask whether the concept of entangledness has any significance for partial orders other than linear orders. In this talk, we introduce a natural definition of entangledness in the class of ω_1-trees. Recall that an ω_1-tree is a tree of height ω_1 all of whose levels are countable, and a Suslin tree is an ω_1-tree which has no uncountable chains or antichains. As is well-known, there exists a Suslin line iff there exists a Suslin tree.

Let $(T, <_T)$ be an ω_1-tree. For any distinct nodes x and y of T, define $\Delta(x, y)$ as the order type of the set $\{z \in T : z <_T x, y\}$. For any positive integer n, we say that an ω_1-tree T is *n-entangled* if for all sequences $\langle (a_{\xi,0}, \ldots, a_{\xi,n-1}) : \xi < \omega_1 \rangle$ of injective n-tuples which satisfy that the set of ordinals

$$\{\Delta(a_{\xi,i}, a_{\xi,j}) : i < j < n,\ \xi < \omega_1\}$$

is bounded in ω_1, for all $g : n \to 2$ there exist $\xi < \beta < \omega_1$ such that for all $i < n$, $a_{\xi,i} <_T a_{\beta,i}$ iff $g(i) = 1$. The restriction on Δ is required, since any Suslin tree fails to have the property without this restriction. It turns out that an ω_1-tree T is 1-entangled iff it is Suslin, and more generally, T is n-entangled iff all of its derived trees of dimension n are Suslin. We also proved that for any positive integer n, it is consistent for a Suslin tree to be n-entangled, but all of its derived trees of dimension $n + 1$ are special.

Universally measurable sets may all be $\underset{\sim}{\boldsymbol{\Delta}}^1_2$

PAUL B. LARSON

(joint work with Saharon Shelah)

A subset of a Polish space X is said to be *universally measurable* if it is measured by the completion of any σ-additive Borel measure on X. Equivalently, $A \subseteq X$ is universally measurable if and only if $f^{-1}[A]$ is Lebesgue measurable whenever $f : \omega^\omega \to X$ is a Borel function. This characterization induces the corresponding notion for category : we say that a set $A \subseteq X$ is *universally categorical* if and only if $f^{-1}[A]$ has the property of Baire whenever $f : \omega^\omega \to X$ is a Borel function.

We identify a set $\mathbf{A}$ consisting of σ-algebras on ω^ω and prove the consistency of the following statement : for every $\mathcal{A} \in \mathbf{A}$, each $A \in \mathcal{A}$ is $\underset{\sim}{\Delta}^1_2$. The σ-algebras in $\mathbf{A}$ are induced by suitably coherent and absolute assignments of σ-ideals to the set of infinite branches through each finitely-branching tree of height ω. The set $\mathbf{A}$ contains the collection of universally measurable sets and the collection of universally categorial sets. The proof of our main theorem proceeds by a countable support iteration of (ω, ∞)-distributive partial orders forcing that every set of reals of cardinality $\aleph_1$ is $\underset{\sim}{\Delta}^1_2$.

The following theorem is a special case of our main theorem. In the case of the Lebesgue-null ideal, the theorem answers part of problem CG on David Fremlin's problem list.

Theorem 1. *If, for some $a \subseteq \omega$, $\mathbf{V}$=$\mathbf{L}[a]$, then there is a proper forcing extension in which every universally measurable subset of any Polish space is $\underset{\sim}{\Delta}^1_2$, and every universally categorical subset of any uncountable Polish space is $\underset{\sim}{\Delta}^1_2$.*

The decomposability conjecture

Andrew Marks
(joint work with Adam Day)

Assuming Π^1_2 determinacy, we prove the decomposability conjecture is true: a Borel function f is decomposable into a countable union of functions which are piecewise continuous on Δ^0_n domains iff the preimage of every Σ^0_n set under f is Σ^0_n. Our proof uses a new dichotomy characterizing when a set is Σ^0_n complete in terms of a Baire category criterion. A central tool in this proof is Antonio Montalbán's true stages machinery (cf. [1]). Our proof also relies a theorem of Leo Harrington in [2] that, assuming the axiom of determinacy, there are no definable ω_1 sequences of distinct Borel sets of bounded rank.

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Parametrised Miller Forcing

Heike Mildenberger
(joint work with Christian Bräuninger)

Let $\mathcal{F}$ be a filter over ω. Guzmán and Kalajdzievski introduced a parametrised version of Miller forcing called $\mathbb{PT}(\mathcal{F})$. We use $\mathbb{PT}(\mathcal{U})$ for a recursivly constructed sequence of ultrafilters. Using this type of iterands, we prove that we can specifically preserve certain P-points.

1. Brief outline

We use a new forcing notion introduced by Guzmán and Kalajdzievski in [5] and apply it with particularly chosen parameters. We use games in order to show that conditions with some blockstructure are dense. We transfer some key arguments in the evaluation of Blass Shelah-forcing to the new forcing. The aim is to work on the old conjecture that the existence of a simple $P_{\aleph_1}$-and simple $P_{\aleph_2}$-point is consistent relative to ZFC. In order to carry the construction over iteration steps of uncountable cofinality, absoluteness properties like the ones proved in [7, §3] for countable support iterations of Mathias forcing are needed.

Definition 1. Let κ be a regular uncountable cardinal.

(1) An ultrafilter $\mathcal{U}$ over ω is called a *P_κ-point* if for any $\gamma < \kappa$, any $\subseteq^*$-descending sequence $\langle A_\beta \ : \ \beta < \gamma \rangle$ of elements of $\mathcal{U}$ has a pseudointersection $B \in \mathcal{U}$, that is some B such that for $\beta < \gamma$, $B \subseteq^* A_\beta$. A $P_{\aleph_1}$-point is also just called a P-point.

(2) By $\mathcal{F}r$ we denote the *Fréchet filter* which is the filter of cofinite subsets of ω.
(3) Let $\mathcal{F}$ be a filter over ω that contains the Fréchet filter. We say just filter over ω. A subset $\mathcal{B} \subseteq \mathcal{F}$ is called a *basis* of $\mathcal{F}$ if for every $F \in \mathcal{F}$ there is some $B \in \mathcal{B}$ such that $B \subseteq F$.
(4) A P_κ-point is called *simple* if it has a basis $\mathcal{B} \subseteq \mathcal{U}$ such that $\mathcal{B}$ consists of a $\subseteq^*$-descending sequence $\langle A_\alpha \ : \ \alpha < \kappa \rangle$.
(5) The *character* of a filter $\mathcal{F}$ is the smallest size of a basis of $\mathcal{F}$.

Note that any P_κ-point with character κ is simple.

The space 2^ω is endowed with the product topology of the discrete space $2 = \{0, 1\}$. Any subset A of ω is a point in 2^ω via its characteristic function χ_A. Collections $\mathcal{C}$ of subsets of ω are said to be of descriptive complexity Γ if the set $\{\chi_A \ : \ A \in \mathcal{C}\}$ is contained in Γ.

Definition 2.

(1) The partial order $\mathbb{F}_\sigma$ is the forcing with F_σ-filters[1] over ω. Stronger filters are superfilters.
(2) If $\mathcal{F}$ is a filter, then $\mathbb{F}_\sigma(\mathcal{F})$ is the forcing with F_σ-filters that are compatible with $\mathcal{F}$, i.e. $\mathcal{G} \in \mathbb{F}_\sigma(\mathcal{F})$ iff $\mathcal{G}$ is an F_σ-filter and $\mathcal{G} \subseteq \mathcal{F}^+ = \{X \subseteq \omega \ : \ \forall (F \in \mathcal{F})(X \cap F \neq \emptyset)\}$.

Definition and Observation 3. Let G be an $\mathbb{F}_\sigma(\mathcal{F})$-generic filter. We let $\mathcal{U}$ be a $\mathbb{F}_\sigma(\mathcal{F})$-name for the union of G. By a density argument, the poset $\mathbb{F}_\sigma(\mathcal{F})$ forces that $\mathcal{U}$ is an ultrafilter that contains $\mathcal{F}$ as a subset.

The set of finite strictly increasing sequences of natural numbers is called $\omega^{\uparrow<\omega}$. The length of $s \in \omega^{\uparrow<\omega}$ is its domain. For $s, t \in \omega^{<\omega}$, we say "*$t$ extends s*" or "*s is an initial segment of t*" and write $s \trianglelefteq t$ if $\mathrm{dom}(s) \subseteq \mathrm{dom}(t)$ and $s = t \restriction \mathrm{dom}(s)$.

Definition 4. A subset $p \subseteq \omega^{\uparrow<\omega}$ that is closed under initial segments is called a *tree*. The elements of a tree are called *nodes*. Given any tree p, a node $s \in p$ is called a *splitting node of p* if s has more than one direct $\triangleleft$-successor in p and *ω-splitting node of p* if s has infinitely many direct $\triangleleft$-successors in p. The set of splitting nodes of p is denoted by $\mathrm{spl}(p)$ while $\omega\text{-}\mathrm{spl}(p)$ denotes the set of ω-splitting nodes of p.

The set of finite/infinite subsets of ω is denoted by $[\omega]^{<\omega}/[\omega]^\omega$.

Definition 5. For $\mathcal{E} \subseteq [\omega]^\omega$ such that for all $n \in \omega$ and $x_1, \ldots, x_n \in \mathcal{E}$ we have $x_1 \cap \cdots \cap x_n \in [\omega]^\omega$, we denote by $\mathrm{filter}(\mathcal{E})$ the filter generated by $\mathcal{E} \cup \mathcal{F}r$, i.e.

$$\mathrm{filter}(\mathcal{E}) = \{Y \subseteq \omega \ : \ \exists n \in \omega \exists x_1, \ldots, x_n \in \mathcal{E}(Y \supseteq^* x_1 \cap \cdots \cap x_n)\}.$$

In order to define a parametrised version of Miller-Forcing we will need some notions about blocks.

[1] Again, we consider only those filters that contain the Fréchet filter as a subset.

Definition 6.

(1) For any set A we write $[A]^{<\omega} = \{t : t \subseteq A, |t| < \omega\}$. The elements of $\mathrm{Fin} = [\omega]^{<\omega} \setminus \{\emptyset\}$ are called *blocks*.

(2) Let $\mathcal{F}$ be a filter over ω. We let

$$\mathcal{F}^{<\omega} = \{[A]^{<\omega} \setminus \{\emptyset\} : A \in \mathcal{F}\}$$
$$(\mathcal{F}^{<\omega})^+ = \{B \subseteq \mathrm{Fin} : \forall A \in \mathcal{F}([A]^{<\omega} \cap B \neq \emptyset)\}$$

Note that $\mathcal{F}^{<\omega}$ is a filter over Fin.

The following forcing notion was introduced by Guzmán and Kalajdzievski [5] in order to prove that the ultrafilter number $\mathfrak{u}$ may be smaller than the almost disjointness number $\mathfrak{a}$ without using large cardinals.

Definition 7. (See [5]) Let $\mathcal{F}$ be a filter over ω. The forcing $\mathbb{PT}(\mathcal{F})$ consists of all $p \subseteq \omega^{<\omega}$ such that for each $s \in p$ there is $t \trianglerighteq s$, such that $t \in \omega\text{-}\operatorname{spl}(p)$ and

$$\operatorname{sucspl}_p(t) := \{\operatorname{rge}(r) \setminus \operatorname{rge}(t) : r \text{ a } \triangleleft\text{-minimal infinitely splitting node of } p \text{ above } t\} \in (\mathcal{F}^{<\omega})^+.$$

Such a t is called an $\mathcal{F}$-splitting node. We furthermore require of p that each ω-splitting node is a $\mathcal{F}$-splitting node [2] and there is a unique $\triangleleft$-minimal ω-splitting node called the *trunk of* p, $\operatorname{tr}(p)$. The set of $\mathcal{F}$-splitting nodes of p is denoted by $\operatorname{spl}(p)$.

Consider t to be a function, e.g. $t \in \omega^{\uparrow<\omega}$. The symbol $\operatorname{rge}(t)$ denotes the range of a function, and vice versa $\operatorname{en}(r) \in \omega^{\uparrow<\omega}$ denotes the increasing enumeration of $r \in [\omega]^{<\omega}$. Note that in contrast to Guzmán and Kalajdzievski, we do not identify $t \in p \subseteq \omega^{\uparrow<\omega}$ with its range. The function sending $s \in \omega^{\uparrow<\omega}$ to its range is an isomorphism witnessing

$$(\omega^{\uparrow<\omega}, \trianglelefteq) \cong ([\omega]^{<\omega}, \sqsubseteq).$$

The main result is

Proposition 8. *We assume* CH*.*

(**A**) *There is a countable support iteration $\mathbb{P} = \langle \mathbb{P}_\gamma, \mathbb{Q}_\beta : \gamma \leq \aleph_1, \beta < \aleph_1 \rangle$ that is defined as follows:*

(1) *$\mathbb{P}_0 = \{0\}$, and*

(2) *For $\beta < \aleph_2$ we have the following: If*

- *for $\gamma < \beta$, r_γ is the $\mathbb{PT}(\mathcal{U}_\gamma)$-generic real over $\mathbf{V}^{\mathbb{P}_\gamma * \mathbb{F}_\sigma(\mathcal{F}_\gamma)}$,*
- *$\mathcal{F}_\beta = \operatorname{filter}(\{\operatorname{rge}(r_\gamma) : \gamma < \beta\})$ and*
- *$\mathcal{U}_\beta$ is the $\mathbb{F}_\sigma(\mathcal{F}_\beta)$-generic ultrafilter over $\mathbf{V}^{\mathbb{P}_\beta}$,*

then $\mathbb{P}_\beta \Vdash \mathbb{Q}_\beta = \mathbb{F}_\sigma(\mathcal{F}_\beta) * \mathbb{PT}(\mathcal{U}_\beta)$.

(**B**) *Any $\mathbb{P}$ as in (**A**) is proper, does not collapse $\aleph_2$, and forces that any P-point from the ground model generates still a P-point (and hence there is a simple $P_{\aleph_1}$-point) and forces that*

[2] We do not know whether the first condition can be waived. There might be finitely splitting nodes. The set of conditions without finitely splitting nodes is possibly not dense.

$$\text{filter}(\{\text{rge}(r_\gamma) \ : \ \gamma < \aleph_1\})$$

is a simple $P_{\aleph_1}$-point and is Canjar.

Of course, this is a CH model and the objects could be constructed without forcing at all. The hope is that the reflection arguments that leads to the Canjar property in the conclusion would give the analogous result at limits of uncountable cofinality of the same iteration of length $\aleph_2$.

Remark 9. The iteration given in (**A**) is not as uniform as it may look at first sight. For $\beta \leq \omega_2$ such that $\text{cf}(\beta) \geq \omega_1$ we have

$$\mathbb{P}_\beta \Vdash \mathcal{F}_\beta \text{ is an ultrafilter.}$$

Hence for $\beta < \omega_2$ such that $\text{cf}(\beta) = \omega_1$, the forcing $\mathbb{F}_\sigma(\mathcal{F}_\beta)$ is the one point forcing $\{\mathcal{F}_\beta\}$.

Definition 10. A filter $\mathcal{F}$ over ω is called a *Canjar filter* if for any sequence $\langle X_n \ : \ n < \omega\rangle$ of elements of $(\mathcal{F}^{<\omega})^+$ there is a sequence $s_n \in [X_n]^{<\omega}$ such that $\bigcup\{s_n \ : \ n < \omega\} \in (\mathcal{F}^{<\omega})^+$.

A filter is Canjar iff Mathias forcing with second components in the filter does not add a dominating real [6, Theorem 5]. There are more equivalent formulations, see, e.g., [1, 3, 4]. We use the following two known and crucial facts about Canjar filters.

Lemma 11.

(a) *The forcing $\mathbb{PT}(\mathcal{F})$ is proper for any Canjar Filter $\mathcal{F}$.*
(b) *The generic filter of the forcing $\mathbb{F}_\sigma$ is a Canjar ultrafilter.*

Proof. See Propositions 17 and 48 of [5]. □

We add a couple of new lemmas.

Lemma 12. *For $\text{cf}(\beta) \leq \omega$, $\mathbb{P}_\beta * \mathbb{F}_\sigma(\mathcal{F}_\beta)$ forces the following:*

(a) *$\mathcal{U}_\beta$ is Canjar.*
(b) *$\mathcal{U}_\beta$ is not nearly coherent to any P-point in $V^{\mathbb{P}_\beta}$.*

Lemma 13. *For $\alpha = \omega_1$, $\mathbb{P}_\alpha$ forces that $\mathcal{F}_\alpha$ is a Canjar ultrafilter that is not nearly coherent to any P-point in $\bigcup_{\gamma<\alpha} V^{\mathbb{P}_\gamma}$.*

The conjecture is that the countability and Π^1_1-absoluteness argument in the proof of the previous lemma would allow to prove: For $\alpha \leq \omega_2$, $\text{cf}(\alpha) \geq \omega_1$, $\mathbb{P}_\alpha$ forces that $\mathcal{F}_\alpha$ is a Canjar ultrafilter that is not nearly coherent to any P-point in $\bigcup_{\gamma<\alpha} V^{\mathbb{P}_\gamma}$. The Canjar property is open strictly above $\aleph_1$.

Definition 14. Let $\mathcal{F}$ be a filter, $p \in \mathbb{PT}(\mathcal{F})$ and let A be a $\mathbb{PT}(\mathcal{F})$-name for a subset of ω. We say p *decides A in pace* if

$$(\forall t \in \text{spl}(p))(\forall r \in \text{sucspl}_p(t))$$
$$(\forall i \leq \max(\text{rge}(t))(p \restriction (t^\frown \text{en}(r)) \text{ decides } i \in A)$$

Lemma 15. *Let $\mathcal{F}$ be a filter, $p \in \mathbb{PT}(\mathcal{F})$ and let A be a $\mathbb{PT}(\mathcal{F})$-name for a subset of ω. Then there exists a trunk-preserving extension $q \leq p$ that decides A in pace.*

Definition 16. Let $f\colon \omega \to \omega$ be a strictly increasing function with $f(0) = 0$. A condition $p \in \mathbb{PT}(\mathcal{F})$ is said to have *f-block structure* if

$$(\forall t \in \mathrm{spl}(p))(\forall r \in \mathrm{sucspl}_p(t))$$
$$(\exists k \in \omega)(\mathrm{rge}(r) \setminus \mathrm{rge}(t) \subseteq [f(k), f(k+1))).$$

With the help of block structure and decision in pace we prove a preservation theorem that builds on [2].

Lemma 17. (a) *If $\mathcal{U}$ is a Canjar ultrafilter that is not nearly coherent to a P-point $\mathcal{W}$, then forcing with $\mathbb{PT}(\mathcal{U})$ preserves $\mathcal{W}$.*
(b) *Let $\mathcal{W}$ be an P-point in the ground model. If a filter $\mathcal{F}$ is not almost ultra, then $\mathbb{F}_\sigma(\mathcal{F}) * \mathbb{PT}(\underset{\sim}{\mathcal{U}})$ preserves $\mathcal{W}$.*

Together with known preservation theorems for countable support iterations, by induction on $\alpha \leq \omega_1$ the lemmata yield a proof of a technically enhanced length-α version of the main theorem.

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The Feldman-Moore, Glimm-Effros, and Lusin-Novikov theorems over quotients

Benjamin D. Miller

We give countably-infinite bases of minimal counterexamples to generalizations of the results mentioned in the title to quotients spaces.

For all $k \geq 2$, let $\mathbb{F}_k$ denote the index k subequivalence relation of $\mathbb{E}_0$ given by $c \mathbin{\mathbb{F}_k} d \iff \exists n \in \mathbb{N} \forall m \geq n\ \sum_{\ell<m} c(\ell) \equiv \sum_{\ell<m} d(\ell) \mod k$.

A *partial transversal* of an equivalence relation E on X over a subequivalence relation F is a set $Y \subseteq X$ for which $E \restriction Y = F \restriction Y$.

Theorem 1. *Suppose that X is a Hausdorff space, E is an analytic equivalence relation on X, and F is a Borel equivalence relation on X for which every E-class is a countable union of $(E \cap F)$-classes. Then exactly one of the following holds:*

1. *The set X is a countable union of $(E \cap F)$-invariant Borel partial transversals of E over $E \cap F$.*
2. *There exists $\mathbb{F} \in \{\Delta(2^{\mathbb{N}})\} \cup \{\mathbb{F}_p \mid p \text{ is prime}\}$ for which there is a continuous embedding $\pi \colon 2^{\mathbb{N}} \hookrightarrow X$ of $(\mathbb{E}_0, \mathbb{F})$ into (E, F).*

A *partial uniformization* of a set $R \subseteq X \times Y$ over an equivalence relation F on Y is a subset of R whose vertical sections are contained in F-classes.

Theorem 2. *Suppose that X and Y are Hausdorff spaces, E is an analytic equivalence relation on X, F is a Borel equivalence relation on Y, and $R \subseteq X \times Y$ is an $(E \times \Delta(Y))$-invariant analytic set whose vertical sections are contained in countable unions of F-classes. Then exactly one of the following holds:*

1. *The set R is a countable union of $(E \times F) \restriction R)$-invariant Borel-in-$R$ partial uniformizations of R over F.*
2. *There exists $\mathbb{F} \in \{\Delta(2^{\mathbb{N}})\} \cup \{\mathbb{F}_p \mid p \text{ is prime}\}$ for which there are continuous embeddings $\pi_X \colon 2^{\mathbb{N}} \hookrightarrow X$ of $\mathbb{E}_0$ into E and $\pi_Y \colon 2^{\mathbb{N}} \hookrightarrow Y$ of $\mathbb{F}$ into F such that $(\pi_X \times \pi_Y)(\mathbb{E}_0) \subseteq R$.*

We say that a set $R \subseteq X \times X$ is a *graph of a partial injection* over an equivalence relation F on X if every horizontal and vertical section of R is contained in an F-class.

Theorem 3. *Suppose that X is a Hausdorff space, E is an analytic equivalence relation on X, F is a Borel equivalence relation on X, and every E-class is a countable union of $(E \cap F)$-classes. Then exactly one of the following holds:*

1. *The set E is a countable union of $((E \cap F) \times (E \cap F))$-invariant Borel-in-$E$ graphs of partial injections over $E \cap F$.*
2. *There exists $\mathbb{F} \in \{\Delta(2^{\mathbb{N}})\} \cup \{\mathbb{F}_p \mid p \text{ is prime}\}$ for which there is a continuous embedding $\pi \colon 2^{\mathbb{N}} \times 2 \hookrightarrow X$ of $(\mathbb{E}_0 \times I(2), (\mathbb{E}_0 \times \Delta(\{0\})) \cup (\mathbb{F} \times \Delta(\{1\})))$ into (E, F).*

A *transversal* of an equivalence relation E over a subequivalence relation F is a partial transversal of E over F that intersects every E-class, a *uniformization* of a set $R \subseteq X \times Y$ over an equivalence relation F on Y is a partial uniformization of R over F with the same projection onto X as R, and a *graph of a bijection* over an equivalence relation F on X is a graph of a partial injection over F whose horizontal and vertical sections are non-empty. If X is a Polish space and E is a countable Borel equivalence relation, then Theorems 1 and 2 easily imply their strengthenings in which the sets in condition (1) satisfy these stronger requirements, and it is not substantially more difficult to establish the analogous strengthening of Theorem 3, since every partial injection from a set to itself gives rise to a bijection with the same orbits in a canonical fashion.

If there is no continuous embedding of $\mathbb{E}_0$ into E, then condition (2) fails in all three theorems. In this case, one need only require that F is co-analytic, albeit at the cost that the corresponding sets do not enjoy the same level of invariance. The further special case of Theorem 2 where $E = \Delta(X)$ previously arose in unpublished work due to Conley-Miller. The still further special case where $E = F = \Delta(X)$ is essentially the Lusin-Novikov uniformization theorem.

If there is no continuous embedding of $\mathbb{E}_0$ into F, then the instances of condition (2) where $\mathbb{F} \in \{\mathbb{F}_p \mid p \text{ is prime}\}$ fail in all three theorems, so the only possibility is that $\mathbb{F} = \Delta(2^{\mathbb{N}})$, and even this fails in Theorem 3. The further special case of Theorem 1 where $F = \Delta(X)$ is essentially the Glimm-Effros dichotomy for countable analytic equivalence relations, the analogous special case of Theorem 2 answers a question posed by Kechris , and the analogous special case of Theorem 3 is essentially the Feldman-Moore theorem.

Finitely generated groups of piecewise linear homeomorphisms

Justin T. Moore

The group $\mathrm{PL}_o I$ consisting of all orientation preserving piecewise linear homeomorphisms of the unit interval has long been an interesting source of examples in group theory. On one hand, by work of Brin and Squier, $\mathrm{PL}_o I$ does not contain nonabelien free groups. On the other hand, it is not an elementary amenable group and contains a rich hierarchy of groups which are. It also contains Richard Thompson's group F, which itself has served as an important example in group theory since at least the early 1980s.

More recently a program has been initiated which attempts the understand the quasiorder of all finitely generated subgroups of $\mathrm{PL}_o I$, ordered by homomorphic embedding. Is there a substantial initial segment of this quasiorder which is *classifiable* is some suitable sense? Can one identify the point(s) at which the quasiorder becomes intractable? Some more precise questions are the following:

(1) (Brin) Which finitely generated subgroups G of $\mathrm{PL}_o I$ have the property that whenever H is a finitely generated subgroup of $\mathrm{PL}_o I$, then either G embeds into H or H embeds into G? Does F have this property?
(2) (Brin-Sapir) If a subgroup of $\mathrm{PL}_o I$ does not contain a copy of F, must it be elementary amenable?
(3) (Moore) Are the finitely generated subgroups of F well quasiordered?

At present it seems reasonable to conjecture that F is a *bottleneck* in the sense of (1) and that it provides the dividing line for nonelementary amenability in the sense of (2). Given this, it is natural to try to identify the obstructions for when a subgroup of $\mathrm{PL}_o I$ embeds into F. In joint work with James Hyde, I have isolated the notion of an *F-obstruction* and shown that, at least for one orbital groups, F-obstructions generate groups that do not embed into F and which moreover contain F. It is correctly unknown whether every subgroup of $\mathrm{PL}_o I$ which does not embed into F contains an F-obstruction, although we conjecture that this is not true — that our working notion of F-obstruction is incomplete.

The notion of an F-obstruction comes from Poincaré's rotation number associated to a homeomorphism of the circle. Specifically, if $f, g \in \mathrm{PL}_o I$ and

$s < f(s) \leq g(s) < f(g(s)) = g(f(s))$, then we can define the *rotation number of f modulo g at s* to be the rotation number of the map $\gamma : [s, f(g(s))) \to [s, f(g(s)))$ defined by $\gamma(t) = g^m(f(t))$ where $m \in \mathbf{Z}$ is such that $s \leq g^m(f(t) < f(g(s))$. If the rotation number of f modulo g at s is defined and irrational for some s, we say that (f, g) is an F-obstruction.

Theorem 1. *If $f, g \in \mathrm{PL}_o I$ is an F-obstruction and $\langle f, g\rangle$ has one component of support, then for every embedding $\phi : \langle f, g\rangle \to \mathrm{PL}_o I$, $\langle \phi(f), \phi(g)\rangle$ contains an F-obstruction.*

It is routine to show that the standard representations of F inside $\mathrm{PL}_o I$ do not contain F-obstructions. Thus the theorem implies that a subgroup of $\mathrm{PL}_o I$ generated by an F-obstruction is not embeddable into F. The next theorem gives some evidence for (1).

Theorem 2. *If $f, g \in \mathrm{PL}_o I$ is an F-obstruction, then $\langle f, g\rangle$ contains an isomorphic copy of F.*

In the direction of (3), there is the following theorem, which is joint work with Collin Bleak and Matthew G. Brin.

Theorem 3. *There is a transfinite sequence $(G_\xi \mid \xi < \varepsilon_0)$ of finitely generated elementary amenable subgroups of F such that:*

- *G_0 is the trivial group and $G_{\xi+1} \cong G_\xi + \mathbf{Z}$;*
- *G_ξ embeds into G_η if and only if $\xi \leq \eta$;*
- *Given $0 \leq \alpha < \varepsilon_0$ and $n < \omega$, let $\xi = \omega^{(\omega^\alpha)\cdot(2^n)}$. If $\alpha > 0$, then the EA-class of G_ξ is $\omega \cdot \alpha + n + 2$. If $\alpha = 0$, then the EA-class of G_ξ is $n + 1$.*

Σ-Prikry forcings and their iterations

Alejandro Poveda
(joint work with Assaf Rinot and Dima Sinapova)

In a series of papers [1] [2] we introduce the class of Σ-Prikry forcing, where $\Sigma := \langle \kappa_n \mid n < \omega\rangle$ is a non decreasing sequence of regular uncountable cardinals converging to some cardinal κ. In [1] we argue that this new concept yields an interesting class of forcing in the sense that many of the known Prikry-type posets that centers around singular cardinals of countable cofinality fall within this new paradigm. Among these forcing one can find, for instance, the standard Prikry forcing [3], Gitik-Sharon poset [4] or the Extender-Based Prikry forcing [5]. Also, in [1] a functor $\mathbb{A}(\cdot, \cdot)$ between the class of Σ-Prikry forcing and $\mathbb{P}$-names is defined. For each Σ-Prikry forcing $\mathbb{P}$ and each $\mathbb{P}$-name $\dot{T}$ for a non-reflecting stationary subset of $E^{\kappa^+}_\omega$, this functor produces a Σ-Prikry notion of forcing $\mathbb{A}(\mathbb{P}, \dot{T})$ that

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messes up the stationarity of $\dot{T}$. A key feature of this functor is that the projection from $\mathbb{A}(\mathbb{P},\dot{T})$ to $\mathbb{P}$ *splits*: that is, in addition to a projection map π from $\mathbb{A}(\mathbb{P},\dot{T})$ onto $\mathbb{P}$, there is a map $\pitchfork$ that goes in the other direction, and the two maps commute in a very strong sense. The exact details can be found in our definition of *forking projection* [1].

Our work is also narrowly tied with the broad program of finding viable iteration schemes for relevant families of forcings. The first successful transfinite iteration scheme was devised by Solovay and Tennenbaum in [6], who solved a problem concerning a particular type of linear orders of size $\aleph_1$ known as *Souslin lines*. The Solovay-Tennenbaum technique is very useful, but it admits no generalizations that allow to tackle problems concerning objects of size $> \aleph_1$. One crucial reason for the lack of generalizations has to do with the poor behavior of the higher analogues of *ccc* at the level of cardinals $> \aleph_1$ (see [7, 8, 9] for a discussion and counterexamples).

Still, various iteration schemes for posets having strong forms of the κ^+-chain-condition for κ regular were devised in [10, 11, 12, 13]. In contrast, there is a dearth of works involving iterations at the level of the successor of singular cardinals. A few ad-hoc treatments of iterations that are centered around a singular cardinal may be found in [14, §2], [15, §10] and [16, §1], and a more general framework is offered by [17, §3]. In [18], the authors took another approach in which they first pursue a forcing iteration along a successor of a regular cardinal κ, and at the very end they singularize κ by appealing to Prikry forcing. This was latter generalized to the context of Radin forcing in [19].

In our project, we propose yet another approach: we allow to put the Prikry-type forcing centered at κ as the very first step of our iteration, and then continue up to length κ^{++} without collapsing cardinals. In [2] we materialize this idea by developing a general scheme for iterating Σ-Prikry posets. The motivation for this new approach is as follows. Suppose that one would like to produce a generic extension where certain combinatorial principle holds at the successor of a singular cardinal κ. The first thing that one has to be concerned about is that the resulting forcing iteration $\mathbb{P}_{\kappa^{++}}$ enjoys the κ^{++}-chain condition. The arguments developed in [2] guarantee that, if $\mathbb{P}$ is a given Σ-Prikry notion of forcing, this property is preserved along the way of defining $\mathbb{P}_{\kappa^{++}}$, the κ^{++}-length iteration of $\mathbb{P}$. Thus, in particular, $\mathbb{P}_{\kappa^{++}}$ has the κ^{++}-cc and, actually, more than that (see [2, §1]).

Provided $2^{2^\kappa} = \kappa^{++}$ notice that, by using a bookkeeping enumeration, we have a way to ensure that all counterexamples for this hypothetical principle show up at some intermediate stage in the process of defining $\mathbb{P}_{\kappa^{++}}$. Thus, we fix a bookkeeping list $\langle z_\alpha \mid \alpha < \kappa^{++}\rangle$ of all these problems, and shall want that, for any $\alpha < \kappa^{++}$, $\mathbb{P}_{\alpha+1}$ will amount to force over the model $V^{\mathbb{P}_\alpha}$ to solve the problem suggested by z_α. The standard approach to achieve this is to set $\mathbb{P}_{\alpha+1} := \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a poset that takes care of z_α. However, the disadvantage of this approach is that if $\mathbb{P}$ is a notion of forcing that blows up 2^κ, then any typical poset $\mathbb{Q}_1$ in $V^{\mathbb{P}_1}$ which is designed to add a subset of κ^+ via bounded approximations will fail to have the κ^{++}-cc. To work around this, in our scheme, we set $\mathbb{P}_{\alpha+1} := \mathbb{A}(\mathbb{P}_\alpha, z_\alpha)$, where $\mathbb{A}(\cdot,\cdot)$ is a functor that to each Σ-Prikry poset $\mathbb{P}$ and a problem z, produces a Σ-Prikry poset $\mathbb{A}(\mathbb{P}, z)$ that projects onto $\mathbb{P}$ and solves the problem z. At the end of this process we will have defined a poset $\mathbb{P}_{\kappa^{++}}$ which will yield the desired generic extension. A special case of our main result from [2] may be roughly stated as follows.

Theorem 1. *Suppose that $\Sigma = \langle \kappa_n \mid n < \omega\rangle$ is a strictly increasing sequence of regular uncountable cardinals converging to a cardinal κ. For simplicity, let us say that a notion of forcing $\mathbb{P}$ is* nice *if $\mathbb{P} \subseteq H_{\kappa^{++}}$ and $\mathbb{P}$ does not collapse κ^+. Now, suppose that:*

- *$\mathbb{Q}$ is a nice Σ-Prikry notion of forcing;*
- *$\mathbb{A}(\cdot,\cdot)$ is a functor that produces for every nice Σ-Prikry notion of forcing $\mathbb{P}$ and every $z \in H_{\kappa^{++}}$, a corresponding nice Σ-Prikry notion of forcing $\mathbb{A}(\mathbb{P}, z)$ that admits a forking projection to $\mathbb{P}$;*
- *$2^{2^\kappa} = \kappa^{++}$, so that we may fix a bookkeeping list $\langle z_\alpha \mid \alpha < \kappa^{++}\rangle$.*

Then there exists a sequence $\langle \mathbb{P}_\alpha \mid \alpha \leq \kappa^{++}\rangle$ of nice Σ-Prikry forcings such that $\mathbb{P}_1$ is isomorphic to $\mathbb{Q}$, $\mathbb{P}_{\alpha+1}$ is isomorphic to $\mathbb{A}(\mathbb{P}_\alpha, z_\alpha)$, and, for every pair $\alpha \leq \beta \leq \kappa^{++}$, $\mathbb{P}_\beta$ projects onto $\mathbb{P}_\alpha$.

In [2, §5] we also present the very first application of our scheme. Here our aim is to obtain the consistency of finite simulatenous reflection of stationary subsets of κ^+ joint with a genuine failure of the SCH_κ. For this purpose we carry out an iteration of length κ^{++} where $\mathbb{P}$ is the Extender Based Prikry Forcing relative to Σ for making $2^\kappa = \kappa^{++}$. For the definition of the later steps we invoke the functor $\mathbb{A}(\mathbb{P}, z)$ from [1], which is devised to kill the nonreflecting stationary set z. As a corollary, we obtain a correct proof of one of the main result of A. Sharon's dissertation [20, §3].

Theorem 2. *Let $\langle \kappa_n \mid n < \omega\rangle$ be a strictly increasing sequence of supercompact cardinals. Set $\kappa := \sup_{n<\omega} \kappa_n$. Then there exists a cofinality-preserving forcing extension of the universe where κ remains strong limit, every finite collection of stationary subsets of κ^+ reflects simultaneously, and $2^\kappa = \kappa^{++}$.*

Corollary 3. *If* ZFC *is consistent with the existence of ω-many supercompact cardinals, then* ZFC *is also consistent with* $\mathrm{Refl}({<}\omega, \kappa^+) + \neg\mathrm{SCH}_\kappa$*, where κ is a strong limit singular cardinal with* $\mathrm{cof}(\kappa) = \omega$.

It is worth mentioned that the following question remains open:

Question 4. *Is it possible to obtain the above consistency result for $\kappa = \aleph_\omega$?*

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Transformations of the transfinite plane

Assaf Rinot

(joint work with Jing Zhang)

Ramsey's theorem [Ram30] asserts that every infinite graph contains an infinite subgraph which is either a clique or an anti-clique. In other words, for every function (or *coloring*, or *partition*, depending on one's perspective) $c : [\mathbb{N}]^2 \to 2$, there exists an infinite $X \subseteq \mathbb{N}$ which is *monochromatic* in the sense that, for some $i \in 2$, $c(x, y) = i$ for every pair $x < y$ of elements of X. A strengthening of Ramsey's theorem due to Hindman [Hin74] concerns the additive structure $(\mathbb{N}, +)$ and asserts that for every partition $c : \mathbb{N} \to 2$, there exists an infinite $X \subseteq \mathbb{N}$ which is monochromatic in the sense that, for some $i \in 2$, for every finite increasing sequence $x_0 < \cdots < x_n$ of elements of X, $c(x_0 + \cdots + x_n) = i$.

A natural generalization of Ramsey's and Hindman's theorems would assert that in any 2-partition of an uncountable structure, there must exist an uncountable monochromatic subset. However, this is not case. Already in the early 1930's, Sierpiński found a coloring $c : [\mathbb{R}]^2 \to 2$ admitting no uncountable monochromatic set [Sie33]. In contrast, a counterexample concerning the additive structure $(\mathbb{R}, +)$ was discovered only a few years ago [HLS17], by Hindman, Leader and Strauss.

In this work [RZ20], we study the existence of transformations of the transfinite plane that allow, among other things, to reduce the additive problem into to the considerably simpler Ramsey-type problem.

By convention, hereafter, κ denotes a regular uncountable cardinal, and θ, χ denote (possibly finite) cardinals $\leq \kappa$. The transformation of interest is captured by the following definition.

Definition 1. $\mathrm{P}\ell_1(\kappa)$ asserts the existence of a transformation $\mathbf{t} : [\kappa]^2 \to [\kappa]^2$ satisfying the following:

- for every $(\alpha, \beta) \in [\kappa]^2$, if $\mathbf{t}(\alpha, \beta) = (\alpha^*, \beta^*)$, then $\alpha^* \leq \alpha < \beta^* \leq \beta$;
- for every family $\mathcal{A}$ consisting of κ many pairwise disjoint finite subsets of κ, there exists a stationary $S \subseteq \kappa$ such that, for every pair $\alpha^* < \beta^*$ of elements of S, there exists a pair $a < b$ of elements of $\mathcal{A}$ with $\mathbf{t}[a \times b] = \{(\alpha^*, \beta^*)\}$.

Theorem 2. *If $\mathrm{P}\ell_1(\kappa)$ holds, then the following are equivalent:*

- *There exists a coloring $c : [\kappa]^2 \to \theta$ such that, for every $X \subseteq \kappa$ of size κ, and every $\tau \in \theta$, there exist $x \neq y$ in X such that $c(x, y) = \tau$;*
- *For every Abelian group $(G, +)$ of size κ, there exists a coloring $c : G \to \theta$ such that, for all $X, Y \subseteq G$ of size κ, and every $\tau \in \theta$, there exist $x \in X$ and $y \in Y$ such that $c(x + y) = \tau$.*

As the proof of Theorem 2 will make clear, the theorem remains valid even after relaxing Definition 1 to omit the first bullet and to weaken "stationary $S \subseteq \kappa$" into "cofinal $S \subseteq \kappa$". The reason we have added these extra requirements is to connect this line of investigation with other well-known problems, such as the problem of whether the product of any two κ-cc posets must be κ-cc (cf. [Rin14a]):

Theorem 3. *If $\mathrm{P}\ell_1(\kappa)$ holds, then there exists a κ-cc poset of size κ whose square does not satisfy the κ-cc.*

Now, let us consider a more informative variation of $\mathrm{P}\ell_1(\kappa)$.

Definition 4. $\mathrm{P}\ell_1(\kappa, \theta, \chi)$ asserts the existence of a function $\mathbf{t} : [\kappa]^2 \to [\kappa]^3$ satisfying the following:

- for all $(\alpha, \beta) \in [\kappa]^2$, if $\mathbf{t}(\alpha, \beta) = (\tau^*, \alpha^*, \beta^*)$, then $\tau^* \leq \alpha^* \leq \alpha < \beta^* \leq \beta$;
- for all $\sigma < \chi$ and a family $\mathcal{A} \subseteq [\kappa]^\sigma$ consisting of κ many pairwise disjoint sets, there exists a stationary $S \subseteq \kappa$ such that, for all $(\alpha^*, \beta^*) \in [S]^2$ and $\tau^* < \min\{\theta, \alpha^*\}$, there exist $(a, b) \in [\mathcal{A}]^2$ with $\mathbf{t}[a \times b] = \{(\tau^*, \alpha^*, \beta^*)\}$.

In [Rin12], by building on the work of Eisworth in [Eis13a, Eis13b], Rinot proved that $\mathrm{P}\ell_1(\lambda^+, \mathrm{cf}(\lambda), \mathrm{cf}(\lambda))$ holds for every singular cardinal λ.[1] The proof of that theorem was a combination of walks on ordinals, club-guessing considerations, applications of elementary submodels, and oscillation of *pcf* scales. In this work, we replace the last ingredient by the oscillation oracle $\mathrm{P}\ell_6(\ldots)$ from [Rin14b].

Our main result reads as follows:

Theorem 5. *For $\chi = \mathrm{cf}(\chi) \geq \omega$, $\mathrm{P}\ell_1(\kappa, \theta, \chi)$ holds in any of the following cases:*

(1) *$\chi < \chi^+ < \theta = \kappa$ and $\square(\kappa)$ holds;*
(2) *$\chi < \chi^+ < \theta = \kappa$ and $E^\kappa_{\geq\chi}$ admits a stationary set that does not reflect;*
(3) *$\chi < \chi^+ = \theta < \kappa$, κ is inaccessible, and $E^\kappa_{\geq\chi}$ admits a stationary set that does not reflect at inaccessibles.*

Note that the principle $\mathrm{P}\ell_1(\kappa, \theta, \chi)$ is strictly stronger than Shelah's principle $\mathrm{Pr}_1(\kappa, \kappa, \theta, \chi)$. Thus, Clause (1) improves the main result of [Rin14a] and Clause (2) improves the main result of [Rin14b]. Clause (2) is also consistently sharp, in the sense that it is consistent that for some strongly inaccessible cardinal κ, there exists a nonreflecting stationary subset of E^κ_ω, and yet, $\mathrm{P}\ell_1(\kappa, 1, \omega_1)$ fails.

The result of Clause (3) provides, in particular, an affirmative answer to a question posed by Eisworth to the first author at the *Set Theory* meeting in Oberwolfach, January 2014.

We also have some news on Shelah's classical principles:

Theorem 6. (1) *For any infinite regular cardinal μ such that $2^\mu = \mu^+$, if $\mathrm{Pr}_1(\mu^+, \mu^+, \mu^+, \mu)$ fails, then μ^+ is a Mahlo cardinal in L;*
(2) *For any infinite cardinal λ such that $2^{2^\lambda} = \lambda^{++}$, if $\mathrm{Pr}_0(\lambda^{++}, \lambda^{++}, \lambda^{++}, \lambda^+)$ fails, then λ^{++} is weakly compact in L.*

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How much choice is needed to construct a discontinuous homomorphism?

Christian Rosendal

The closed graph theorem of Banach and Schauder is originally formulated for linear operators between Banach space, but has a well-known formulation also for groups. Namely, it states that a homomorphism $G \xrightarrow{\phi} H$ between Polish groups whose graph is closed in $G \times H$ must also be continuous. I will present a recent generalisation of this result, relying on some ideas apparently inherent in unpublished work of Adian, which relaxes the condition of continuity at the identity of the homomorphism ϕ. More precisely, we show the following result.

Theorem. *Suppose $G \xrightarrow{\phi} H$ is a homomorphism between Polish groups so that, for all identity neighbouhoods $U \subseteq G$ and $V \subseteq H$, there is a finite set $F \subseteq U$ for which*

$$\bigcup_{f \in F} f \cdot \phi^{-1}(V) f^{-1}$$

is an identity neighbourhood in G. Then ϕ is continuous.

This result in turn will allow us to address two seemingly unrelated issues. Namely, on the one hand, it provides a positive answer to an old question of JPR Christensen regarding the continuity of universally measurable homomorphisms between Polish groups. And, on the other hand, it gives general lower bounds on the amount of the axiom of choice needed to construct a discontinuous homomorphism between Polish groups. In fact, under $\mathrm{ZF} + \mathrm{DC}$, we prove a quadrichotomy between various continuity properties of homomorphisms and colouring properties of the Hamming graph on products of finite spaces. These latter results are related to recent work by P. Larson and J. Zapletal.

The consistency of the failure of the convergence of K^c constructions

Grigor Sargsyan

We will outline the proof of a recent result that ZFC alone is not sufficient to prove the convergence of K^c constructions. More specifically we will show that the failure of both squares at ω_3 along with $\omega_2^\omega = \omega_3$ has a consistency strength weaker than a Woodin cardinal that is a limit of Woodin cardinals. Earlier it was shown by Jensen-Schimmerling-Schindler-Steel [1] that this particular combinatorial configuration implies that K^c has a superstrong cardinals, provided it converges.

The work combines techniques from $\mathbb{P}_{\max}$ forcing and HOD mice theory. Part of it is joint with Paul Larson.

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ZF rank-into-rank embeddings and non-definability

Farmer Schlutzenberg

Assume ZF throughout. A *Reinhardt cardinal*, introduced by William Reinhardt, is the critical point of an elementary embedding $j : V \to V$. We consider here such embeddings, and variants like elementary $j : V_\delta \to V_\delta$. Most of the results mentioned below can be seen in the notes [1] *Reinhardt cardinals and non-definability*, version v2 (will replace v1),[1] arxiv.org/abs/2002.01215.

Given a transitive structure M and $A \subseteq M$, we say that A is *definable over M from parameters* iff there is a formula φ in the language of set theory and $p \in M$ such that $A = \{x \in M \mid M \models \varphi(x, p)\}$.

Suzuki proved in *No elementary embedding from V into V is definable from parameters* what is stated by its title, working in ZF. We generalize this result as follows:

Theorem 1 (§3 of [1]). *Assume* ZF. *Let δ be an ordinal and $j : V_\delta \to V_\delta$ be Σ_1-elementary and definable over V_δ from the parameter $x \in V_\delta$, and $j \neq \mathrm{id}$. Then: $\delta = \beta + 1$ is a successor, and if j is fully elementary then* $\mathrm{rank}(x) = \beta$.

Recall that if $j : V_{\lambda+1} \to V_{\lambda+1}$ is elementary and λ a limit ordinal then j *is* definable over $V_{\lambda+1}$ from parameter $x = j \upharpoonright V_\lambda$, because given $A \subseteq V_\lambda$, we have $j(A) = \bigcup_{\alpha<\lambda} j(A \cap V_\alpha)$.

Assume ZF and let $\delta \leq \mathrm{OR}$ be a limit and $j : V_\delta \to V_\delta$ be Σ_1-elementary. Given $A \subseteq V_\delta$ we define

$$j(A) = \bigcup_{\alpha<\delta} j(A \cap V_\alpha).$$

The *finite iterates* $j^n : V_\delta \to V_\delta$ are defined by setting $j^1 = j$ and $j^{n+1} = j^n(j^n)$. In the proof of Suzuki's fact, it is useful that if $M, N \models \mathrm{ZF}$ are proper classes and $j : M \to N$ is Σ_1-elementary then j is fully elementary. A generalization:

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[1] Errata to [1] v1: In the development of extenders, the assertion on p.21 that "for every finite set a there is a finite set b with $a \subseteq b$, ..., and b extensional", is false; a counterexample will be given in v2, as will a corrected development. In the introduction, Theorem 3.2 is over-stated. The reference to definability of set-grounds in Footnote 7, p. 46, is incorrect.

Theorem 2 (§3 in [1]). *Assume* ZF *and let* $\delta \in \mathrm{OR}$ *be a limit and* $j : V_\delta \to V_\delta$ *be* Σ_1*-elementary. Then there is* $m < \omega$ *such that for all* $n \geq m$*,* j^n *is fully elementary, and in fact for all* $A \subseteq V_\delta$*,*

$$j^n : (V_\delta, A) \to (V_\delta, j^n(A))$$

is fully elementary in the expanded language with predicate for $A, j^n(A)$*.*

Together with Andreas Lietz, we show that the $m < \omega$ is necessary:

Theorem 3 (§3 in [1]). *Assume* ZF*. Suppose* $j : V_{\lambda^+} \to V_{\lambda^+}$ *is elementary and* $j \neq \mathrm{id}$*. Let* $m < \omega$*. Then there is a limit* $\eta < \lambda^+$ *such that letting* $k = j \restriction V_\eta$*, then* $k : V_\eta \to V_\eta$ *is* Σ_1*-elementary, but* $k^1, k^2, \ldots, k^m$ *are not* Σ_2*-elementary.*

Beyond direct definability over V_δ, we have:

Theorem 4 (§8 in [1]). *Assume* $\mathrm{ZF} + V = L(V_\delta)$ *for a limit* δ *of uncountable cofinality. Then there is no* Σ_1*-elementary* $j : V_\delta \to V_\delta$*.*

Goldberg proved this earlier under the further assumption that δ is inaccessible, via a different method. The situation is, however, more subtle when $\mathrm{cof}(\delta) = \omega$. The proof of the theorem above uses a development of the theory of ultrapowers by extenders under ZF; this is also used in [1] to show that if there is a proper class of weak Löwenheim-Skolem cardinals, then being the critical point of an elementary $j : V \to M$ with M transitive is first-order.

We next consider some connections between $\mathrm{HOD} = \mathrm{HOD}^V$ and the iterates of (V, j) when $(V, j) \models \mathrm{ZF}$ and $j : V \to V$ is elementary. Let $M_0 = (V, j)$ and $M_\alpha = (N_\alpha, j_\alpha)$ where $M_{\alpha+1} = (N_\alpha, j_\alpha(j_\alpha))$, and we take direct limits at limit α. Hamkins observed that the usual arguments show that all M_α are wellfounded, $(N_\alpha, j_\alpha) \models \mathrm{ZF}$ and $j_\alpha : M_\alpha \to M_\alpha$ is elementary. Let $\lambda = \sup_{n<\omega} \sup(\mathrm{crit}(j^n))$.

Theorem 5 (§9 in [1]). *Suppose* $(V, j) \models \mathrm{ZF}$ *where* $j : V \to V$ *is elementary and let* λ*, etc, be as above. Then:*

(1) $\lambda = \mathrm{crit}(j_\omega)$ *is inaccessible in* N_ω*,*
(2) $V_\lambda^{\mathrm{HOD}} = V_\lambda^{\mathrm{HOD}^{N_\omega}}$ *and* $V_{\lambda+1}^{\mathrm{HOD}} \subseteq V_{\lambda+1}^{\mathrm{HOD}^{N_\omega}}$ *and* $\mathrm{HOD} \subseteq \mathrm{HOD}_{j_\omega}^{N_\omega}$*,*
(3) $\mathrm{HOD} \not\subseteq \bigcap_{\alpha \in \mathrm{OR}} N_\alpha$*,*
(4) V *is not a set-generic extension of* N_α *for* $\alpha \geq \omega$*,*
(5) *every set* X *is contained in a set-generic extension of* N_α*, for each* α*,*
(6) *there is* $G \in N_\omega$ *which is set-generic over* HOD *and such that* $\mathrm{HOD}[G] \models$ *"*λ *is weakly compact" and* $V_\lambda^{\mathrm{HOD}[G]} = V_\lambda^{\mathrm{HOD}} = V_\lambda^{\mathrm{HOD}^{N_\omega}}$*,*

Question 6. *Is* λ *weakly compact in* HOD*?*

Using the analysis of the iterates M_α, one can deduce in second order set theory ZF_2, that if X is a set and A a class, then (i) if $V = \mathrm{HOD}(X)$ there is no Reinhardt cardinal, and (ii) if $V = \mathrm{HOD}_A(X)$ then V is not total Reinhardt and there is no Berkeley cardinal.[2]

[2]Goldberg and Usuba have independently proved stronger results (in particular that if there is a Reinhardt cardinal then AC is not set-forceable), via a quite different proof, which is more direct.

Finally recall that given a set X, the mouse $M_n(X)$ is the least proper class mouse over X with n Woodin cardinals, and $M_n^{\#}(X)$ is its sharp. The following will appear in v2 of [1]; the case $n = 0$ (all sets have sharps) is due to Goldberg:

Theorem 7. *Suppose* $(V, j) \models \mathrm{ZF}$ *and* $j : V \to V$ *is elementary. Then* $M_n^{\#}(X)$ *exists and is* OR*-iterable* (*above* X) *for every set* X.

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Transfinite sequences of topologies, descriptive complexity, and approximating equivalence relations

Sławomir Solecki

The aim of the present work is to describe the following general phenomenon: under appropriate topological conditions, increasing transfinite sequences of topologies interpolating between two given topologies $\sigma \subseteq \tau$ stabilize at τ and, under appropriate additional descriptive set theoretic conditions, the stabilization occurs at a countable stage of the interpolation. Increasing sequences of topologies play an important role in certain descriptive set theoretic considerations; see, for example, [8, Section 1], [2, Sections 5.1–5.2], [1, Section 2], [9, Section 2], [6, Chapter 6], [5, Section 3], [10, Sections 2–4], [7], [4], and, implicitly, [3, Sections 3–5]. In this context, such sequences of topologies are often used to approximate an equivalence relation by coarser, but more manageable, ones. We relate our theorems on increasing interpolations between two topologies to this theme. The results of this work are expected to have applications to a Scott-like analysis of quite general Borel equivalence relations.

Filtrations. *Unless otherwise stated, all topologies are assumed to be defined on a fixed set* X. We write

$$\mathrm{cl}_\tau \ \text{ and } \ \mathrm{int}_\tau$$

for the operations of closure and interior with respect to a topology τ. If τ is a topology and $x \in X$, by a **neighborhood of** x we understand a subset of X that contains x in its τ-interior. A **neighborhood basis of** τ is a family $\mathcal{A}$ of subsets of X such that for each $x \in X$ and each neighborhood B of x, there exists $A \in \mathcal{A}$ that is a neighborhood of x and $A \subseteq B$. So a neighborhood basis need not consist of open sets. A topology is called **Baire** if a countable union of nowhere dense sets has dense complement.

The notion of filtration defined below is the main new notion of the work. Let $\sigma \subseteq \tau$ be topologies and let ρ be an ordinal. A transfinite sequence $(\tau_\xi)_{\xi<\rho}$ of topologies is called a **filtration from σ to τ** if

$$\sigma = \tau_0 \subseteq \tau_1 \subseteq \cdots \subseteq \tau_\xi \subseteq \cdots \subseteq \tau \tag{1}$$

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and, for each $\alpha < \rho$, if F is τ_ξ-closed for some $\xi < \alpha$, then

$$\operatorname{int}_{\tau_\alpha}(F) = \operatorname{int}_\tau(F). \tag{2}$$

Note that if $F \subseteq X$ is an arbitrary set and $(\tau_\xi)_\xi$ is a transfinite sequence of topologies fulfilling (1), then for each α

$$\operatorname{int}_{\tau_\alpha}(F) \subseteq \operatorname{int}_\tau(F).$$

So condition (2) says that if F is simple from the point of view of τ_α, that is, if F is τ_ξ-closed for some $\xi < \alpha$, then $\operatorname{int}_{\tau_\alpha}(F)$ is as large as possible, in fact, equal to $\operatorname{int}_\tau(F)$. We write $(\tau_\xi)_{\xi \leq \rho}$ for $(\tau_\xi)_{\xi < \rho+1}$. Each filtration from σ to τ can be extended to all ordinals by setting $\tau_\xi = \tau$ for all $\xi \geq \rho$. For this reason, it will be harmless to assume that a filtration is defined on all ordinals.

Let $\sigma \subseteq \tau$ be two topologies. The first question is to determine whether a given filtration $(\tau_\xi)_\xi$ from σ to τ reaches τ, that is, whether there exists an ordinal ξ with $\tau_\xi = \tau$. Since all the topologies τ_ξ are defined on the same set, there exists an ordinal ξ_0 such that $\tau_\xi = \tau_{\xi_0}$ for all $\xi \geq \xi_0$; the question is whether $\tau_{\xi_0} = \tau$. If the answer happens to be positive, we want to obtain information on the smallest ordinal ξ for which $\tau_\xi = \tau$. We achieve these goals in Theorems 1 and 2 assuming that τ is regular and Baire and that it has a neighborhood basis consisting of sets that are appropriately definable with respect to σ. So, informally speaking, termination at τ of a filtration from σ to τ has to do with the attraction exerted by τ, which is expressed by τ being Baire, and with the distance from σ to τ, which is expressed by the complexity, with respect to σ, of a neighborhood basis of τ. Given an equivalence relation E on a set X, with X equipped with a topology τ, we can define a canonical equivalence relation that approximates E from above: make $x, y \in X$ equivalent when the τ-closures of the E equivalence classes of x and y are equal. Given a filtration, this procedure gives rise to a transfinite sequence of upper approximations of E. We consider the question of these approximations stabilizing to E, and answer it in Theorem 4. In addition to the results described above, we also define and study a canonical, slowest filtration from σ to τ.

Statements of results. Recall that **C-sets** with respect to a topology is the smallest σ-algebra of sets closed under the Souslin operation and containing all open sets with respect to this topology.

Theorem 1. *Let $\sigma \subseteq \tau$ be topologies. Assume that τ is regular, Baire, and has a neighborhood basis consisting of sets that are C-sets with respect to σ. Let $(\tau_\xi)_\xi$ be a filtration from σ to τ. If $\tau_{\xi_0} = \tau_{\xi_0+1}$ for some ξ_0, then $\tau_{\xi_0} = \tau$.*

Theorem 2 contains a more refined version of stabilization. It makes a connection with descriptive set theoretic complexity of neighborhood bases. Note that the assumptions of Theorem 2 ensure that Theorem 1 applies, but the conclusion of Theorem 2 gives an upper estimate on the smallest ξ_0 with $\tau_{\xi_0} = \tau$, which we do not get from Theorem 1.

Theorem 2. *Let $\sigma \subseteq \tau$ be topologies, with τ being regular and Baire. For an ordinal $\alpha \leq \omega_1$, let $(\tau_\xi)_{\xi \leq \alpha}$ be a filtration from σ to τ, with τ_ξ metrizable, for $\xi < \alpha$, and τ_α Baire.*

If τ has a neighborhood basis consisting of sets in $\bigcup_{\xi<\alpha} \mathbf{\Pi}^0_{1+\xi}$ with respect to σ, then $\tau_\alpha = \tau$.

Remark 3. *1.* Note that in Theorem 2 we do not make any separability assumptions.
2. One can relax the assumption of metrizability; it suffices to assume that τ_ξ are paracompact and that sets that are τ_ξ-closed are intersections of countably many sets that are τ_ξ-open, for all $\xi < \alpha$.
3. When $\alpha = \omega_1$, then, of course, $\bigcup_{\xi<\alpha} \mathbf{\Pi}^0_{1+\xi}$ is the family of all Borel sets with respect to σ.

Fix $(\tau_\xi)_{\xi<\rho}$, a transfinite sequence of topologies as in (1). Let E be an equivalence relation on X. There exists a natural way of producing a transfinite sequence of upper approximations of E using $(\tau_\xi)_{\xi<\rho}$. For each $\xi < \rho$ define the equivalence relation E_ξ on X by letting

$$xE_\xi y \text{ if and only if } \mathrm{cl}_{\tau_\xi}([x]_E) = \mathrm{cl}_{\tau_\xi}([y]_E).$$

Note that

$$E_0 \supseteq E_1 \supseteq \cdots \supseteq E_\xi \supseteq \cdots \supseteq E. \tag{3}$$

The main question is when the transfinite sequence of equivalence relations in (3) stabilizes at E.

Theorem 4. *Let $\sigma \subseteq \tau$ be topologies, with τ being Baire. Let $\alpha \leq \omega_1$, and let $(\tau_\xi)_{\xi<\alpha}$ be a filtration from σ to τ, with τ_ξ completely metrizable for each $\xi < \alpha$. Assume E is an equivalence relation whose equivalence classes are τ-open.*

If all E equivalence classes are in $\bigcup_{\xi<\alpha} \mathbf{\Pi}^0_{1+\xi}$ with respect to σ, then $E = \bigcap_{\xi<\alpha} E_\xi$.

Remark 5. *1.* Each E equivalence class being τ-open, as in Theorem 4, is equivalent to saying that E is a $(\tau \times \tau)$-open subset of $X \times X$.

2. In Theorem 4, if $\alpha < \omega_1$ is a successor, say $\alpha = \beta + 1$, then the conclusion reads: if all equivalence classes of E are in $\mathbf{\Pi}^0_{1+\beta}$ with respect to σ, then $E = E_\beta$.

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Ramsey degrees of products of infinite sets

Stevo Todorčević

We consider finite colourings of finite products $X_1 \times X_2 \times \cdots \times X_n$ of infinite sets and determine the minimal number of colours a subproduct $Y_1 \times Y_2 \times \cdots \times Y_n$ of infinite subsets could achieve. It is well known and easily seen that if $X_1 \times X_2 \times \cdots \times X_n$ is a finite sequence of countable infinite sets then there is a colouring of their product $\prod_{i=1}^n X_i$ with $n!$ colours each of which shows up in any subproduct $\prod_{i=1}^n Y_i$ with $Y_i \subseteq X_i$ are infinite. For example, letting $X_i = \mathbb{N}$ for all i and colouring a given one-to-one sequence $(k_1, k_2, ..., k_n)$ of integers by the permutation σ of $\{1, 2, ..., n\}$ such that $\sigma(i) < \sigma(j)$ is equivalent to $k_i < k_j$ for all $i < j$, it is clear that all permutations show up in any n-product of infinite subsets of $\mathbb{N}$. On the other hand, a simple application of Ramsey's theorem shows that for every finite colouring of $\prod_{i=1}^n X_i$ there exist infinite $Y_i \subseteq X_i$ such that the subproduct $\prod_{i=1}^n Y_i$ uses no more than $n!$ colours. As said above, we investigate this phenomenon in the case when some of the sets X_i are uncountable and in fact have different cardinalities. For example, we show that if one of the sets X_i is uncountable then we can find a subproduct of infinite sets that use no more than $(n-1)!$ colours and that this number in general cannot be lowered. on the other hand, if among the sets X_i one can find sets of three different cardinalities then the minimal number of colours a subproduct of infinite subsets could drops to $(n-1)!$, and so on. More precisely, we shall see that there is a general result of this kind that naturally fits in the classical set-theoretic study of the Ramsey degree phenomenon.

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Universal minimal flows of homeomorphism groups of high-dimensional manifolds are not metrizable

Todor Tsankov

The *universal minimal flow (UMF)* of a topological group G is a canonical object associated to the group which is of prime importance in abstract topological dynamics. For most classical groups (for example, infinite discrete and more generally, locally compact, non-compact), the UMF is a non-metrizable space that is difficult to describe explicitly. Somewhat surprisingly, for many large Polish groups of interest, the UMF is a metrizable compact space and a rather concrete object that carries interesting combinatorial and dynamical information.

The first interesting case of a non-trivial, metrizable UMF of a Polish group was computed by Pestov who proved that the UMF of the homeomorphism group of the circle is the circle itself. This naturally led to the question whether a similar result is true for homeomorphism groups of other manifolds (or more general topological spaces). A few years later, Uspenskij [1] proved that the action of a group on its UMF is never 3-transitive, thus giving a negative answer to the question for a vast collection of topological spaces. Still, the question of metrizability of their UMFs remained open and he asked specifically whether the UMF of the homeomorphism group of the Hilbert cube is metrizable. We give a negative answer to this question for the Hilbert cube and all closed manifolds of dimension at least 2, thus showing that metrizability of the UMF of a homeomorphism group is essentially a one-dimensional phenomenon. In dimension 3 or higher, we also prove that the universal minimal flow does not have a comeager orbit (which implies non-metrizability).

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Hyperfinite subequivalence relations of treed equivalence relations

Anush Tserunyan

(joint work with Robin Tucker-Drob)

A large part of measured group theory studies structural properties of countable groups that hold "on average". This is made precise by studying the orbit equivalence relations induced by free measurable actions of these groups on a standard probability space. In this vein, the amenable groups correspond to hyperfinite equivalence relations, and the free groups to the treeable ones. In joint work with R. Tucker-Drob, we give a detailed analysis of the structure of hyperfinite subequivalence relations of a treed equivalence relation on a standard probability space, deriving the analogues of structural properties of amenable subgroups (copies of $\mathbb{Z}$)

of a free group. Most importantly, just like every such subgroup is contained in a unique maximal one, we show that even in the non-pmp setting, every hyperfinite subequivalence relation is contained in a unique maximal one.

We now define all the notions mentioned in the previous paragraph and explain its content in more detail. Let (X, μ) be a standard probability space, which may as well be equal to $[0, 1]$ with Lebesgue measure. An equivalence relation E on X is said to be *Borel* if it is a Borel subset of X^2. We say that E is *countable* (resp. *finite*) if each E-class is countable (resp. finite).

Amenable groups $\rightleftarrows$ hyperfinite equivalence relations. Recall that a countable group is amenable if it admits an invariant *mean*, i.e. a finitely additive probability measure defined on all subsets of the group and invariant under (left) translation. An equivalence relation E on X is called *hyperfinite* (resp. *μ-hyperfinite*) if it is equal to an increasing union of finite Borel equivalence relations (resp. modulo a μ-null set).

It is a theorem of Slaman and Steel [8], [6](Theorem 6.6) that hyperfinite equivalence relations are precisely the orbit equivalence relations of Borel actions of $\mathbb{Z}$. In the measurable context, the Ornstein–Weiss theorem [7] states that in fact measurable actions of all countable amenable groups induce μ-hyperfinite equivalence relations.

Free groups $\rightleftarrows$ treeable equivalence relations. A *Borel graph G* on X is just an irreflexive symmetric Borel subset of X^2. An equivalence relation E is called *treeable* (resp. *μ-treeable*) if there is an acyclic Borel graph T whose connected components (trees) are precisely the E-classes; call such a T a *treeing* of E. It is clear that any free measurable action of the free group Fin_n on $n \leq \infty$ generators induce a μ-treeable equivalence relation E because the action of the standard generators of $\mathbb{F}_n$ provides a $2n$-regular treeing of E. Conversely, a theorem of Hjorth [4] says that up to, so-called, stable orbit equivalence, all probability measure preserving (pmp) μ-treeable equivalence relations arise in this fashion. (We call a Borel equivalence relation E on (X, μ) *probability measure preserving* (*pmp*) if every Borel automorphism γ of X with $\mathrm{graph}(\gamma) \subseteq E$ preserves the measure μ.)

Hyperfinite inside treeable. We study hyperfinite subequivalence relations F of a treeable equivalence relation E on (X, μ) and their interaction with a fixed treeing T of E. An analogy to keep in mind is: a copy of $\mathbb{Z}$ inside $\mathbb{F}_2$. The following was proven in [2] for not just treeable, but more generally, for equivalence relations acting on a bundle of hyperbolic spaces:

Theorem 1 (Bowen)**.** *Let E be a treeable equivalence relation on (X, μ). If E is **pmp**, then every μ-hyperfinite subequivalence relation $F \subseteq E$ admits a **unique** maximal μ-hyperfinite extension $\overline{F} \subseteq E$.*

The proof of this, and in general, any analysis of μ-hyperfinite subequivalence relations F of a treeable equivalence relation E is done using *end selection*: a result of Adams [1] and Jackson, Kechris, and Louveau [5] (Lemma 3.21) that given a treeing T of E, each F-class measurably selects zero, one or two ends of T, and there is a maximum such selection.

Since end selection holds without the pmp assumption, it is natural to ask whether Theorem 1 is true more generally for *all* E (not necessarily pmp). Towards answering this question, we first realized that Theorem 1 easily follows from the following observation, which we vaguely state here:

Lemma 2. *Let E be a treeable equivalence relation on (X, μ), T a treeing of E, and $F \subseteq E$ a μ-hyperfinite subequivalence relation. If E is pmp, then each F-class spans exactly the ends it maximally selects.*

However, this lemma is false without the pmp assumption: the action of Fin_2 on its boundary induces a hyperfinite equivalence relation F and a natural 4-regular treeing of it, so each F-class spans all of the continuum-many ends, yet selecting only one. Nevertheless, using other methods, we answer the question positively:

Theorem 3 (Ts.–Tucker-Drob)**.** *Let E be a treeable equivalence relation on (X, μ). Every μ-hyperfinite subequivalence relation $F \subseteq E$ admits a **unique** maximal μ-hyperfinite extension $\overline{F} \subseteq E$.*

This is corollary of our main result: a complete structural analysis of F with respect to the geometry of a given treeing T of E and the Radon–Nikodym cocycle on E associated with μ (assuming without loss of generality that μ is E-quasi-invariant).

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Tameness for Set Theory

Matteo Viale

This brief report accounts on the main results of [4, 5, 6] where it is shown that there is a *recursive signature* τ extending the signature $\{\in\}$ for set theory and a *definable recursive extension* T in signature τ of the $\in$-theory ZFC such that:

- The universal fragment of T is provably invariant across set-sized forcing extensions of any of its models (cfr. Thm. 1).
- T admits a model companion which is the provable fragment of the τ-theory of H_{ω_2} in any model of MM^{++} (cfr. Thm. 4).

Note that the model companion of a τ-theory T is the unique τ-theory S which satisfies exactly the same universal sentences of T and is model complete (i.e. given models $\mathcal{M}, \mathcal{N}$ of S with $\mathcal{M}$ a substructure of $\mathcal{N}$, $\mathcal{M} \prec \mathcal{N}$). The relevance of the above results is that they show that the notion of forcibility and consistency for Π_2-properties in signature τ overlap (cfr. Thm. 4).

Let ZFC$^-$ denote the theory ZFC without the powerset axiom. Let τ_{ST} be a signature containing predicate symbols R_ψ of arity m for all bounded $\in$-formulae $\psi(x_1, \ldots, x_m)$, function symbols f_θ of arity k for for all bounded $\in$-formulae $\theta(y, x_1, \ldots, x_k)$, constant symbols ω and $\emptyset$. $\mathrm{ZFC}_{\mathsf{ST}} \supseteq \mathrm{ZFC}$ is the τ_{ST}-theory obtained adding axioms which force in each of its τ_{ST}-models $\emptyset$ to be interpreted by the empty set, ω to be interpreted by the first infinite ordinal, each R_ψ as the class of k-tuples defined by the bounded formula $\psi(x_1, \ldots, x_k)$, each f_θ as the l-ary class function whose graph is the extension of the bounded formula $\theta(x_1, \ldots, x_l, y)$ (whenever θ defines a functional relation), see [5, Notation 2] for details.

We supplement [5, Notation 2] with the following:

Notation 1.

- $\tau_{\mathbf{NS}_{\omega_1}}$ is the signature $\tau_{\mathsf{ST}} \cup \{\omega_1\} \cup \{\mathbf{NS}_{\omega_1}\}$ with ω_1 a constant symbol, $\mathbf{NS}_{\omega_1}$ a unary predicate symbol.
- $T_{\mathbf{NS}_{\omega_1}}$ is the $\tau_{\mathbf{NS}_{\omega_1}}$-theory given by T_{ST} together with the axioms

$$\omega_1 \text{ is the first uncountable cardinal},$$

$$\forall x\ [(x \subseteq \omega_1 \text{ is non-stationary}) \leftrightarrow \mathbf{NS}_{\omega_1}(x)].$$

- $\mathrm{ZFC}^-_{\mathbf{NS}_{\omega_1}}$ is the $\tau_{\mathbf{NS}_{\omega_1}}$-theory

$$\mathrm{ZFC}^-_{\mathsf{ST}} + T_{\mathbf{NS}_{\omega_1}}.$$

- Accordingly we define $\mathrm{ZFC}_{\mathbf{NS}_{\omega_1}}$.

We can immediately formulate our first main result:

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Theorem 1. *Assume*[1] $(V, \tau^V_{\mathbf{NS}_{\omega_1}})$ *models* $\mathrm{ZFC}_{\mathbf{NS}_{\omega_1}}$ + there are class many Woodin cardinals. *Then the* Π_1*-theory of* V *for the language* $\tau_{\mathbf{NS}_{\omega_1}} \cup \mathsf{UB}$ *is invariant under set sized forcings.*

To formulate our second result we need more notations and definitions. Let UB denote the family of universally Baire sets (see for details [5, Section 4.2]), and $L(\mathsf{UB})$ denote the smallest transitive model of ZF which contains UB.

We briefly introduce the key definitions of $\mathbf{MAX}(\mathsf{UB})$ and $(*)$-UB which are preliminary to the formulation of our main results.

Definition 2. $\mathbf{MAX}(\mathsf{UB})$: There are class many Woodin cardinals in V, and for all G V-generic for some forcing notion $P \in V$:

(1) Any subset of $(2^\omega)^{V[G]}$ definable in $(H_{\omega_1}^{V[G]} \cup \mathsf{UB}^{V[G]}, \in)$ is universally Baire in $V[G]$.
(2) Let H be $V[G]$-generic for some forcing notion $Q \in V[G]$. Then[2]:

$$(H_{\omega_1}^{V[G]} \cup \mathsf{UB}^{V[G]}, \in) \prec (H_{\omega_1}^{V[G][H]} \cup \mathsf{UB}^{V[G][H]}, \in).$$

We observe that $\mathbf{MAX}(\mathsf{UB})$ is a (slightly weaker) form of sharp for the family of universally Baire sets which holds if V has class many Woodin cardinals and is a generic extension obtained by collapsing a supercompact cardinal to become countable (see [3, Thm 3.4.17]). Moreover if $\mathbf{MAX}(\mathsf{UB})$ holds in V, it remains true in all further set forcing extensions of V. It is open whether $\mathbf{MAX}(\mathsf{UB})$ is a direct consequence of suitable large cardinal axioms.

We now turn to the definition of $(*)$-UB, a natural maximal strengthening of Woodin's axiom $(*)$. Key to all results of this report is an analysis of the properties of generic extensions by $\mathbb{P}_{\max}$ of $L(\mathsf{UB})$. In this analysis $\mathbf{MAX}(\mathsf{UB})$ is used to argue (among other things) that all sets of reals definable in $L(\mathsf{UB})$ are universally Baire, so that most of the results established in [2] on the properties of $\mathbb{P}_{\max}$ for $L(\mathbb{R})$ can be also asserted for $L(\mathsf{UB})$. Here we will not define the $\mathbb{P}_{\max}$-forcing; our reference on this topic is [2].

Definition 3. Let $\mathcal{A}$ be a family of dense subsets of $\mathbb{P}_{\max}$.

- $(*)$-$\mathcal{A}$ holds if $\mathbf{NS}_{\omega_1}$ is saturated[3] and there exists a filter G on $\mathbb{P}_{\max}$ meeting all the dense sets in $\mathcal{A}$.
- $(*)$-UB holds if $\mathbf{NS}_{\omega_1}$ is saturated and there exists an $L(\mathsf{UB})$-generic filter G on $\mathbb{P}_{\max}$.

Woodin's definition of $(*)$ [2, Def. 7.5] is equivalent to $(*)$-$\mathcal{A}$+*there are class many Woodin cardinals* for $\mathcal{A}$ the family of dense subsets of $\mathbb{P}_{\max}$ existing in $L(\mathbb{R})$.

[1]We follow the convention introduced in [5, Notation 2.1] to define $(V, \tau^V_{\mathbf{NS}_{\omega_1}})$.

[2]Elementarity is witnessed via the map defined by $A \mapsto A^{V[G][H]}$ for $A \in \mathsf{UB}^{V[G]}$ and the identity on $H_{\omega_1}^{V[G]}$ (See [5, Notation 4.6] for the definition of $A^{V[G][H]}$).

[3]See [3, Section 1.6, pag. 39] for a discussion of saturated ideals on ω_1.

Notation 2.

- σ_{ST} is the signature containing a predicate symbol S_ϕ of arity n for any τ_{ST}-formula ϕ with n-many free variables.
- $\sigma_{\omega,\mathbf{NS}_{\omega_1}}$ is the signature $\tau_{\mathsf{ST}} \cup \sigma_{\mathsf{ST}}$.
- $T_{\text{l-UB}}$ is the $\sigma_{\omega,\mathbf{NS}_{\omega_1}}$-theory given by the axioms

  $$\forall x_1 \dots x_n \, [S_\psi(x_1,\dots,x_n) \leftrightarrow (\bigwedge_{i=1}^n x_i \subseteq \omega^{<\omega} \wedge \psi^{L(\mathsf{UB})}(x_1,\dots,x_n))]$$

  as ψ ranges over the τ_{ST}-formulae.
- $\mathrm{ZFC}^{*-}_{\text{l-UB}}$ is the σ_ω-theory

  $$\mathrm{ZFC}^-_{\mathsf{ST}} \cup T_{\text{l-UB}};$$
- $\mathrm{ZFC}^{*-}_{\text{l-UB},\mathbf{NS}_{\omega_1}}$ is the $\sigma_{\omega,\mathbf{NS}_{\omega_1}}$-theory

  $$\mathrm{ZFC}^-_{\mathbf{NS}_{\omega_1}} \cup T_{\text{l-UB}};$$
- Accordingly we define $\mathrm{ZFC}^*_{\text{l-UB}}$, $\mathrm{ZFC}^*_{\text{l-UB},\mathbf{NS}_{\omega_1}}$.

A key observation is that $\mathrm{ZFC}^-_{\mathsf{ST}}$, $\mathrm{ZFC}^-_{\mathbf{NS}_{\omega_1}}$, $\mathrm{ZFC}^{*-}_{\text{l-UB}}$, $\mathrm{ZFC}^{*-}_{\text{l-UB},\mathbf{NS}_{\omega_1}}$ are all *definable* extension of ZFC; more precisely any $\in$-structure (M, E) of ZFC^- admits a unique extension to a τ-structure satisfying the extra axioms outlined in the above items for τ among the signatures written above (for $\tau_{\mathsf{ST}} \cup \{\omega_1, \mathbf{NS}_{\omega_1}\}$ the $\in$-model must satisfy the sentence stating the existence of a smallest uncountable cardinal). The same considerations apply to $\mathrm{ZFC}_{\mathsf{ST}}$, $\mathrm{ZFC}_{\mathbf{NS}_{\omega_1}}$, $\mathrm{ZFC}^*_{\text{l-UB}}$, $\mathrm{ZFC}^*_{\text{l-UB},\mathbf{NS}_{\omega_1}}$.

Theorem 4. *Let T be any $\sigma_{\omega,\mathbf{NS}_{\omega_1}}$-theory extending*

$\mathrm{ZFC}^*_{\text{l-UB},\mathbf{NS}_{\omega_1}} + \mathbf{MAX}(\mathsf{UB}) +$ *there is a supercompact cardinal and class many Woodin cardinals*

Then T has a model companion T^*.

Moreover TFAE for any for any Π_2-sentence ψ for $\sigma_{\omega,\mathbf{NS}_{\omega_1}}$:

(A) $T^* \vdash \psi$;

(B) *For any complete theory*

$$S \supseteq T,$$

$S_\forall \cup \{\psi\}$ *is consistent;*

(C) *T proves*[4]

$$\exists P\,(P \text{ is a partial order} \wedge \Vdash_P \psi^{\dot H_{\omega_2}});$$

(D) *T proves*

$$L(\mathsf{UB}) \models [\mathbb{P}_{\max} \Vdash \psi^{\dot H_{\omega_2}}];$$

(E)

$$T_\forall + \mathrm{ZFC}^*_{\text{l-UB},\mathbf{NS}_{\omega_1}} + \mathbf{MAX}(\mathsf{UB}) + (*)\text{-}\mathsf{UB} \vdash \psi^{H_{\omega_2}}.$$

[4] $\dot H_{\omega_2}$ denotes a canonical P-name for H_{ω_2} as computed in generic extension by P.

Crucial to the proof of Theorem 4 is the recent breakthrough of Asperó and Schindler [1] establishing that $(*)$-UB follows from MM^{++}.

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Rigidity conjectures in C*-algebras

Alessandro Vignati

We study automorphisms' groups of corona C*-algebras. C*-algebras are Banach self-adjoint subalgebras of $\mathcal{B}(H)$, the algebra of bounded operators on a complex Hilbert space H. Via Gelfand's transform, abelian C*-algebras arise as algebras of continuous functions on locally compact spaces, so the study of C*-algebra can be viewed as noncommutative topology.

In the same way from a locally compact space X one associates its Čech-Stone compactification βX and its remainder $\beta X \setminus X$, to a nonunital C*-algebra A one associates its multiplier algebra $\mathcal{M}(A)$ and its corona $\mathcal{Q}(A) = \mathcal{M}(A)/A$. If $A = C_0(X)$, then $\mathcal{M}(A) = C(\beta X)$ and $\mathcal{Q}(A) = C(\beta X \setminus X)$, hence coronas provide noncommutative analogues of Čech-Stone remainders. As automorphisms of $C(\beta X \setminus X)$ correspond to homeomorphisms of $\beta X \setminus X$, the study of automorphisms of commutative coronas feeds back into topology.

The interest on homeomorphisms' groups of Čech-Stone remainders takes its origin from the work of Rudin, Shelah, and Veličokić among others ([6, 7, 8]), who proved that the existence of a nontrivial homeomorphism of $\beta\omega \setminus \omega$ depends on set theory. This intuition was later brought to the setting C*-algebras when Phillips and Weaver, and Farah ([5, 3]) showed that whether all automorphisms of the Calkin algebra $\mathcal{Q}(H)$ are inner depends on set theory. ($\mathcal{Q}(H)$ is the quotient of $\mathcal{B}(H)$ by the ideal of compact operators $\mathcal{K}(H)$, when H is a separable Hilbert space. $\mathcal{Q}(H)$ is the corona algebra of $\mathcal{K}(H)$.) A topological notion of triviality for automorphisms of general coronas was given in [1]. (Other, algebraic, notions of triviality where introduced, discussed, and linked with Ulam stability phaenomena in [9] and [4].)

Conjecture 1 ([1]). Let A be a separable nonunital C*-algebra. Then

- CH implies that there exist $2^{\aleph_1}$ automorphisms of $\mathcal{Q}(A)$ that are not topologically trivial;
- PFA implies that all automorphisms of $\mathcal{Q}(A)$ are topologically trivial.

We confirmed the rigidity part of the conjecture. A crucial step was obtained in [4], where the noncommutative version the OCA lifting theorem of [2] was proved.

Theorem 2 ([9]). $\mathrm{OCA} + \mathrm{MA}_{\aleph_1}$ *imply that if A is a separable* C*-*algebra then all automorphisms of $\mathcal{Q}(A)$ are topologically trivial.*

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Weak Vopěnka cardinals

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Vopěnka's Principle, in one of its several equivalent formulations, says that for every proper class of structures in a common signature with any number of finitary function and relation symbols,[1] there is a homomorphism between two structures in that class. Adámek, Rosický, and Trnková [1] observed that Vopěnka's Principle is equivalent to the statement that no sequence of structures $\langle M_\alpha : \alpha \in \mathrm{Ord}\rangle$ has both of the following properties: (1) whenever $\alpha \leq \beta$ there is a unique homomorphism $M_\alpha \to M_\beta$, and (2) whenever $\alpha < \beta$ there is no homomorphism $M_\beta \to M_\alpha$. They then defined *Weak Vopěnka's Principle* as the dual statement (obtained by reversing the arrows) which says that no sequence of structures $\langle M_\alpha : \alpha \in \mathrm{Ord}\rangle$ has both of the following properties: (1) whenever $\alpha \leq \beta$ there is a unique homomorphism $M_\beta \to M_\alpha$, and (2) whenever $\alpha < \beta$ there is no homomorphism

[1]It is equivalent to consider structures with just one binary relation, i.e. graphs, but we will not need this fact.

$M_\alpha \to M_\beta$. They showed that this dual statement is in fact a consequence of Vopěnka's Principle, justifying its "weak" designation.

Similar to the notion of a Vopěnka cardinal, we may define the notion of a *weak Vopěnka cardinal* as a cardinal below which the local version of Weak Vopěnka's Principle holds. First we define:

Definition 1. A *weak Vopěnka sequence* for a regular cardinal κ is a sequence of structures $\langle M_\alpha : \alpha < \kappa \rangle$ in a common signature with fewer than κ function and relation symbols such that

(1) whenever $\alpha \leq \beta < \kappa$ there is a unique homomorphism $M_\beta \to M_\alpha$, and
(2) whenever $\alpha < \beta < \kappa$ there is no homomorphism $M_\alpha \to M_\beta$.

Example 2. The sequence of unital rings $\langle \mathbb{Z}/2^i\mathbb{Z} : i < \omega \rangle$ is a weak Vopěnka sequence for ω: the only (unital) homomorphisms among these rings are the ones mapping $n + 2^j\mathbb{Z}$ to $n + 2^i\mathbb{Z}$ for $i \leq j$.

Definition 3. A regular cardinal κ is a *weak Vopěnka cardinal* (or has the *weak Vopěnka property*) if there is no weak Vopěnka sequence for κ.

Every supercompact cardinal is a weak Vopěnka cardinal (Wilson [5]). Since the least supercompact cardinal is not a Vopěnka cardinal, this result showed the inequivalence of Weak Vopěnka's Principle with Vopěnka's Principle, which was an open problem. It can be sharpened as follows.

Theorem 4 (Wilson [6])**.** *Let κ be an inaccessible cardinal. Then κ has the weak Vopěnka property if and only if it is a Woodin cardinal.*

Although every Vopěnka cardinal is strong limit and therefore inaccessible, the same is not necessarily true for weak Vopěnka cardinals. Indeed, the weak Vopěnka property seems to be the "algebraic essence" of Woodinness, similar to how the super tree property ITP is the combinatorial essence of supercompactness as argued by Weiss [4].[2]

One can show (using AC) that the weak Vopěnka property must fail at ω_1, and that under CH it must fail at ω_2 also. More generally, one can use a $\square^*_\nu$ sequence to construct a weak Vopenka sequence for ν^+. On the other hand, the proof that every supercompact cardinal has the weak Vopěnka property can be modified to show:

Theorem 5. *For every regular cardinal κ, if ITP(κ) holds, then κ has the weak Vopěnka property.*

Because ITP(ω_2) holds under PFA (Weiss [3]) and ITP(ν^+) holds whenever ν is a countable cofinality limit of supercompact cardinals (Hachtman and Sinapova [2]), we thereby obtain two different examples of the weak Vopěnka property at successor cardinals. Naturally, the weak Vopěnka property is weaker than ITP and we have the expected upper bound on its consistency strength:

[2]Perhaps amusingly, this idea suggests that Example 2 is the essential reason that ω is not a Woodin cardinal.

Theorem 6. *If κ is a Woodin cardinal, then it retains the weak Vopěnka property after the Mitchell forcing to make κ equal to ω_2 (or to the double successor of any other regular cardinal less than κ.)*

For consistency strength lower bounds, the picture is somewhat murky. It seems difficult to obtain anything beyond virtual large cardinals in L from the hypothesis that ω_2 is a weak Vopěnka cardinal. However, it is not hard to show that if there is a weak Vopěnka cardinal larger than ω_2, then $0^\sharp$ exists. Moreover, if there is a weak Vopěnka cardinal that is countably closed, it should be possible to show that there is an inner model with a Woodin cardinal, although the details of this have not yet been worked out. Nevertheless, the idea of the weak Vopěnka property as the algebraic essence of Woodinness is supported by the following result:

Theorem 7. *If κ is a weak Vopěnka cardinal and W is an inner model with the countable approximation property in which κ is inaccessible, then κ is a Woodin cardinal in W.*

We end with the remark that neither the weak Vopěnka property nor the tree property implies the other, which is no surprise since neither Woodinness nor weak compactness implies the other.

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Coloring algebraic hypergraphs without choice

Jindrich Zapletal

An algebraic hypergraph of arity n is a subset of $[\mathbb{R}^k]^n$ defined by an algebraic equation with integer coefficients, for some dimension $k \geq 1$. The chromatic numbers of such hypergraphs have been studied for many years, notably by Erdős, Hajnal, and Komjáth. Several years ago, Schmerl completely characterized algebraic hypergraphs of countable chromatic number. He showed that for each such hypergraph Γ, exactly one of the following holds. Either ZFC proves that $\chi(\Gamma) \leq \aleph_0$, or ZFC proves that $\chi(\Gamma) > \aleph_0$, or there is a natural number $m \geq 1$ such that ZFC proves that $\chi(\Gamma) \leq \aleph_0$ is equivalent to $2^{\aleph_0} \leq \aleph_m$. Moreover, there is a computer algorithm which determines the appropriate slot of this multichotomy for each algebraic hypergraph Γ.

In a similar spirit, we attempt to compare algebraic hypergraphs by their chromatic number in the weaker theory ZF+DC. Such a task must result in a chart much more complex and informative than that of Schmerl. We describe the first general result and the first independence results of this program.

Definition 1. A hypergraph Γ on a set X is *redundant* if for each set $a \subset X$, the set $\{x \in X : a \cup \{x\} \in \Gamma\}$ is finite.

Redundant algebraic hypergraphs include the hypergraph on $\mathbb{R}^2$ of arity 3 consisting of all triples of vertices of equilateral triangles, the hypergraph on $\mathbb{R}^n$ of arity 4 consisting of all quadruples of vertices of squares, the hypergraph on $\mathbb{R}$ of arity 3 consisting of solutions to $x^3 + y^3 + z^3 - 3xyz = 0$. Non-algebraic redundant hypergraphs include the hypergraph on G consisting of all solutions to $x_0 x_1^{-1} x_2 x_3^{-1} = 1$ for any Polish group G. An example of a hypergraph which is not redundant: the triples of vertices of isosceles triangles in $\mathbb{R}^2$.

Algebraic redundant hypergraphs are relatively easy to color in choiceless set theory. We construct a balanced forcing which adds a coloring to each such hypergraph over the symmetric Solovay model and obtain the following:

Theorem 2. *Let Γ be a redundant algebraic hypergraph. It is consistent relative to an inaccessible cardinal that ZF+DC holds, the chromatic number of Γ is countable, and*

(1) *there is no uncountable sequence of pairwise distinct Borel sets of bounded rank;*
(2) *there is no discontinuous homomorphism between Polish groups;*
(3) *no turbulent orbit equivalence relation has a selector.*

Comparing chromatic numbers of specific hypergraphs, we get theorems such as

Theorem 3. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, the square hypergraph in $\mathbb{R}^2$ is countably chromatic, and the equilateral triangle hypergraph in $\mathbb{R}^2$ is uncountably chromatic.*

Theorem 4. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, the square hypergraph in $\mathbb{R}^2$ is countably chromatic, and the square hypergraph in $\mathbb{R}^3$ is uncountably chromatic.*

Theorem 5. (Joint with Paul Larson) *It is consistent relative to an inaccessible cardinal that ZF+DC holds, the equilateral triangle hypergraph in $\mathbb{R}^2$ is countably chromatic, and the equilateral triangle hypergraph in $\mathbb{R}^3$ is uncountably chromatic.*

As a final word of caution we point out that there are algebraic hypergraphs for which countable coloring provides objects very close to a well-ordering of the reals.

Theorem 6. *(ZF) Let Γ be the right triangle hypergraph in $\mathbb{R}^2$. If the chromaic number of Γ is countable, then there is a countable-to-one map from $\mathbb{R}$ to ω_1.*

Topological dynamics beyond Polish groups

Andy Zucker
(joint work with Gianluca Basso)

To each topological group G, one can construct its *universal minimal flow* $M(G)$, a minimal G-flow which admits a G-map onto every other minimal flow. This property characterizes $M(G)$ up to isomorphism. In the past two decades, much work has gone into the case where G is a Polish group, i.e. a topological group whose underlying topological space is a separable, completely metrizable space. For a number of Polish groups, $M(G)$ turns out to be trivial, for instance when $G = U(H)$ for an infinite-dimensional Hilbert space, or when $G = \mathrm{Aut}(\mathbb{Q})$, the group of order-preserving bijections of the rationals with the pointwise topology. Other times, $M(G)$ is non-trivial, but still metrizable, for instance when $G = \mathrm{Sym}(\omega)$ or $G = \mathrm{Homeo}(2^\omega)$. In the remaining cases, $M(G)$ is extremely large, for instance whenever G is a locally compact, non-compact Polish group.

For Polish groups, the works of Kechris, Pestov, and Todorčevič [4]; Melleray, Nguyen Van Thé, and Tsankov [5]; Zucker [6]; and Ben Yaacov, Melleray, and Tsankov [2] provide an almost complete understanding of when $M(G)$ is metrizable and what $M(G)$ looks like if so. In the case that $G = \mathrm{Aut}(\mathbf{K})$ for $\mathbf{K}$ a countable ultrahomogeneous structure, $M(G)$ is trivial iff $\mathrm{Age}(\mathbf{K})$ is a Ramsey class, and $M(G)$ is metrizable iff $\mathrm{Age}(\mathbf{K})$ has finite Ramsey degrees. In the latter case, there is a canonical expansion of the class $\mathrm{Age}(\mathbf{K})$ so that $M(G)$ is the associated space of expansions of $\underline{\mathrm{K}}$. As an example, if $\underline{\mathrm{K}}$ is the Random graph, then $M(G)$ is the space of all linear orders of $\mathbf{K}$. If G is a general Polish group with $M(G)$ metrizable, then there is a closed, co-precompact subgroup H with $M(H)$ trivial and with $M(G) = \widehat{G/H}$, the completion of G/H with respect to the metric inherited from any compatible right-invariant metric on G. Hence in the case that G is a Polish group, the property of having $M(G)$ metrizable is a natural dividing line, capturing those groups with "nice" dynamics.

When we move beyond the class of Polish groups, far less is known. The first effort in this direction is due to Bartošová [1], who considers groups of the form $\mathrm{Aut}(\mathbf{K})$ for $\mathbf{K}$ an uncountable, *ω-homogeneous* structure. Endowed with the pointwise topology, $\mathrm{Aut}(\mathbf{K})$ is a topological group, and Bartošová extends many of the results of [4] to this uncountable setting. For instance, $M(\mathrm{Aut}(\mathbf{K}))$ is trivial iff $\mathrm{Age}(\mathbf{K})$ is a Ramsey class, and if $\mathbf{K}$ is an uncountable, ω-homogeneous graph which embeds every finite graph, then $M(G)$ is the space of linear orders of $\mathbf{K}$. This is no longer a metrizable space, but it is still somehow "nice." So if we seek to extend our dynamical dividing line to all topological groups, a new criterion is needed.

In this work, we propose a dividing line which makes sense for any topological group. If G is a topological group and X is a G-flow, then the set of *almost periodic* points of X is the set $\mathrm{AP}(X) := \{x \in X : \overline{Gx} \text{ is minimal}\}$. We say that a topological group is *CAP*, for "closed AP," if for every G-flow X, the set $\mathrm{AP}(X) \subseteq X$ is closed. While this appears to have nothing to do with our earlier discussion, one can show that when G is Polish, then G is CAP iff $M(G)$ is metrizable.

There are a number of equivalent ways of saying that a topological group G is CAP. While the definition is the easiest to state, the most useful formulation refers to how copies of $M(G)$ can sit inside $S(G)$, the *Samuel compactification* of G. While $S(G)$ comes with a compact topology, one can also equip $S(G)$ with a finer topology called the *UEB* topology. This in turn equips $M(G) \subseteq S(G)$ with a finer topology; one can show that this will not depend on the choice of minimal subflow of $S(G)$. We show that G is CAP precisely when these two topologies on $M(G)$ coincide, generalizing [2].

We show that the class of CAP groups is closed under arbitrary products, surjective inverse limits, and group extensions. If G is CAP and H is arbitrary, we have that $M(G \times H) = M(G) \times M(H)$; if $\{G_i : i \in I\}$ is a family of CAP groups and $G = \prod_i G_i$, then $M(G) = \prod_i M(G_i)$. We use this to compute the universal minimal flow of the group $G = \mathrm{Homeo}(\omega_1)$, a group recently investigated by Gheysens [3]. When $G = \mathrm{Aut}(\mathbf{K})$ for $\mathbf{K}$ an uncountable, ω-homogeneous structure, we show that G is CAP iff $\mathrm{Age}(\mathbf{K})$ has finite Ramsey degrees, generalizing [6].

We also have a weak version of the result from [5], namely, if $H \subseteq G$ is a closed, co-precompact subgroup with $M(H)$ trivial and $\widehat{G/H}$ a minimal flow, then G is CAP and $M(G) = \widehat{G/H}$. The converse remains open, and is related to a question asked in [1]. As an example of this question, suppose that $\mathbf{K}$ is an uncountable, ω-homogeneous graph which embeds every finite graph. Then is there some linear order on $\mathbf{K}$ so that $\langle \mathbf{K}, < \rangle$ is also ω-homogeneous?

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