Measurable cardinals and choiceless axioms

$j : V \to M$

Scott, Solovay - Reinhardt

non-identity

"There is an elementary $j : V \to V$"

Reinhardt cardinal

(Kunen)

Theorem There is no $j : V \to V$.

Requires AC. Q. Is a Reinhardt cardinal consistent if AC is dropped?

+DC. yes.

Q. Is NBG + Reinhardt consistency-wise stronger than ZFC + I_0?

$\boxed{j : L(V_{\lambda+1}) \to L(V_{\lambda+1}) \quad \text{crit}(j) < \lambda}$

$j : V_{\lambda+1} \to V_{\lambda+1} \quad \Sigma_0$

$j : V_{\lambda+1} \to V_{\lambda+1} \quad \Sigma_1 \quad \Sigma_2$

Σ_3

I_0 is equiconsistent w/

for all $T \subseteq V_{\lambda+1}$ master coder $j : (V_{\lambda+1}, T) \to (V_{\lambda+1}, T)$

(Kunen) There is no $j : V_{\lambda+2} \to V_{\lambda+2}$

(Schlutzenberg) The following are equiconsistent:

① ZFC + I_0

② ZF + λ-DC + $j : V_{\lambda+2} \to V_{\lambda+2}$

$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$

$\mathrm{crit}(j) < \lambda$

$L(V_{\lambda+1})[j \restriction V_{\lambda+2}^{L(V_{\lambda+1})}]$

$$L(V_{\lambda+1})$$

$$\cup\!|$$

$$M \models j : V_{\lambda^*+2} \to V_{\lambda^*+2}$$

<u>Fact</u> $L(V_{\lambda+1})$ is a forcing extension.

<u>Q</u> <u>$L(\mathbb{R})$</u> is not ??

$AD^{L(\mathbb{R})}$

<u>Jensen</u>: Exactly one of the following holds:

① V is close to L: for all singular λ, λ is singular in L and <u>$\lambda^{+L} = \lambda^+$</u>

② V is far from L: every cardinal is inaccessible in L.

<u>Woodin</u> If there is an extendible κ

one holds

① V is close to HOD: for all singular $\lambda \geq \kappa$ λ is singular in HOD and $\lambda^{+HOD} = \lambda^+$.

② V is far from HOD: for all regular $\delta \geq \kappa$, δ is measurable in HOD.

ω-strongly measurable

$$j : HOD(V_{\lambda+1}) \longrightarrow HOD(V_{\lambda+1})$$

HOD conjecture. Large cardinals imply V is close to HOD.

Theorem (Woodin) If the HOD conjecture is true, then a Reinhardt cardinal + proper class of extendibles is inconsistent.

Rank Berkeley cardinals.

Def. λ is rank Berkeley if for all $\alpha < \lambda \leq \beta$, there is $j: V_\beta \to V_\beta$ w/ $\mathrm{crit}(j) > \alpha$ (and $\mathrm{crit}(j) < \lambda$).

Open Is a Reinhardt equiconsistent w/ a rank Berkeley?

Theorem (Cutolo). If λ is ~~a~~ singular ~~...~~ Berkeley limit of extendibles, then λ^+ is measurable.

Gitik: it is consistent w/ ZF that every uncountable cardinal is singular.

Q (Aspero) Is it consistent w/ λ being rank Berkeley that every cardinal above λ is singular?

Thm it is consistent w/ ZF that $\exists$ rank Berkeley and λ^+ is singular.
(Reinhardt)

Theorem If λ is rank Berkeley then there is a proper class of regulars.

- for all γ, for all suff large δ, $cf(\delta^+) \geq \gamma$.

- for all γ, for all suff large regular δ, the club filter on δ is γ-complete.

$$\gamma \longmapsto \delta_\gamma \longleftarrow \text{(rank Berkeley)}$$

Theorem. For a club of δ the club filter on δ or δ^+

is δ-complete.

If F is a filter on X, a set $A \subseteq X$ is an atom of F if

$$\{S \cap A : S \in F\}$$

is an ultrafilter.

(AD) F = club filter on ω_2

$\{\alpha < \omega_2 : cf(\alpha) = \omega\}$ is an atom

$\{\alpha < \omega_1 : cf(\alpha) = \omega_1\}$ is an atoms

A filter is atomic if every positive set contains an atom.

Theorem If there is a card Berkeley, then all large regular cardinals $\delta,$

then for suff [illegible] δ^{+}
the club filter on δ is atomic.

Cor. If there's a rk Berkeley, then there is a club of δ s.t. δ or δ^{+} is measurable.

Measurable cardinals

Theorem. (rk Berkeley) For a club class of cardinals δ, every δ-complete filter on an ordinal extends to a δ-complete ultrafilter.

Theorem (Kunen) Under AD + DC, every ω_1-complete filter on $\alpha < \Theta$ extends to an ω_1-complete ultrafilter.

Ketonen order.

If ν is an ordinal and X is a set, $\beta_\nu(X)$ = ν-complete u.fs on X,

Ketonen order is an order on u.fs on ordinals.

Fix an ordinal δ.

A function $f: P(\delta) \longrightarrow P(\delta)$ is Ketonen if

① f is Lipschitz

if $x \subseteq \delta$ and $\alpha \leq \delta$,

$f(x) \cap \alpha$ depends only on $x \cap \alpha$.

② If $W \in \beta_{\omega_1}(\delta)$, then $f^{-1}[W] \in \beta_{\omega_1}(\delta)$.

$x \in \underline{U} \iff f(x) \in \underline{W}$ $\in \beta_{\omega_1}(\delta)$

Ketonen reducibility: $U \leq_{\mathbb{K}} W$

if $\exists$ Ketonen $f : P(\delta) \to P(\delta)$

s.t. $f^{-1}[W] = U$.

Provably wellfounded (DC)

(ZFC)

Ultrapower Axiom $\Longrightarrow$ for all δ, $\mathcal{B}_{\omega_1}(\delta)$ is linearly orded by Ketonen reducibility

$$\begin{tikzcd} & N & \\ M_0 \arrow[ur] & & M_1 \arrow[ul] \\ & V \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Open Does UA imply linearity of Ketonen reducibility? semi-linearity

Rank Berkeley cardinals imply "semilinearity" of the Ketonen order.

Theorem. If λ is rank Berkeley,

... for any δ

then for some

the Kelorey order on $B_\nu(S)$ is
almost linear: every level has
cardinality $< \lambda_0$ and every set of
incomparables has size $\leq \lambda$.

Pseudo large cardinals

<u>Def.</u> κ is <u>(ν, ∞)-supercompact</u>
if for all $\lambda \geq \kappa$, there is
$\bar{\lambda} < \kappa$ and $\pi : V_{\bar{\lambda}} \longrightarrow V_\lambda$
s.t. $\pi(\nu)$.

κ is <u>almost supercompact</u>
if it is (ν, ∞)-supercompact for all
$\nu < \kappa$.

Pro

~~Thm~~ If the least (ν, ∞)-supercompact is $<$ the least rank Berkeley, then it is supercompact

Fact. If there is a rank Berkeley, then there is a proper class of almost supercompacts.

Theorem. (wellordered collection)
If κ is almost supercompact, then for all $\eta < \kappa$, if $\langle A_\alpha : \alpha < \eta \rangle$ is a sequence of nonempty sets, then there is a set σ s.t. $\sigma \cap A_\alpha \neq \emptyset$ for all $\alpha < \eta$ and σ is the surjective image of V_κ.

least supercompact

Cor. If κ is closed

and $\gamma \geq \kappa$, then γ^+ has cofinality at least κ.

Proof. Suppose not. Then take $\eta < \kappa$ and $\langle \alpha_\xi : \xi < \eta \rangle$ converging to γ^+.

By wellordered collection, there is a set σ that is the surjective image of V_κ and for each $\xi < \eta$, there is a wellorder of γ in σ of ordertype α_ξ.

$$\gamma^+ = \sup(\alpha_\xi) = \sup\{\mathrm{rank}(s) : s \in \sigma\}$$

$$f : \gamma \times \sigma \longrightarrow \gamma^+$$

$$f(\alpha, s) = \mathrm{rank}_s(\alpha) \qquad |f[\{\alpha\} \times \sigma]| < \kappa$$

$\operatorname{ran}(f) = \gamma^+.$

<u>Proof of wellordered collection lemma.</u>

Suppose for all $\beta < \eta$, the wellordered collection lemma holds.

Fix $\langle A_\xi : \xi < \eta \rangle$. For each $\beta < \eta$, let $B_\beta = \{ g : \begin{array}{l} \operatorname{dom}(g) = V_\kappa \\ \operatorname{ran}(g) \cap A_\xi \\ \text{for all } \xi < \beta \end{array} \}$

Now let $j : V_{\bar\lambda} \to V_\lambda$ be elementary $\bar\lambda < \alpha$ and $j(\eta) = \eta$. ~~Consider~~

$\operatorname{ran}(j)$ is cofinal in η so for cofinally many $\beta < \eta$, there's some $g \in B_\beta \cap \operatorname{ran}(j)$. (if $\beta \in \operatorname{ran}(j)$)

$\mathrm{ran}(j) \subseteq V_\lambda < V_\kappa$

Let $\sigma = \{\mathrm{ran}(f) : f \in B_\beta \cap \mathrm{ran}(j)\}$

For any $\beta \in \mathrm{ran}(j)$, for all $\xi < \beta$,

$\sigma \cap A_\xi$ is nonempty

Define $h : V_\kappa \times V_\lambda \twoheadrightarrow \sigma$

$h(x, f) = j(f)(x)$