Outline:

1. Argue certain supposedly limitative principles are not:
   - Ground Axiom
     - resurrection principles
     - Usuba's Theorem
   - $V = HOD$
     - HOD conjecture
     - UA
2. Make a case for/against HOD conjecture

The Ground Axiom

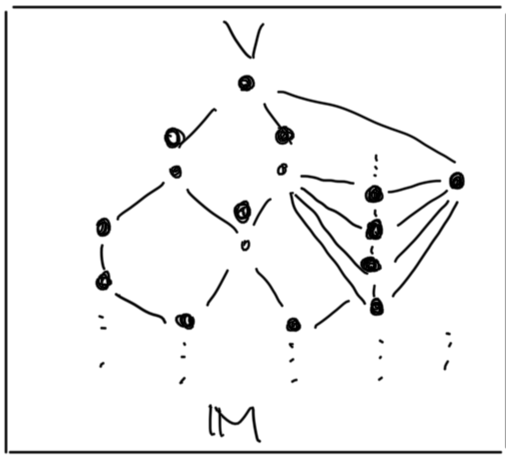
An inner model M is a ground (of V) if there is a poset $\mathbb{P} \in M$ and an M-generic filter G on $\mathbb{P}$ such that $V = M[G]$.

Ground Axiom (Hamkins-Reitz). There are no grounds of V except V itself.

- First-order expressible.
- True in L, $L[U]$, $HOD^{L(\mathbb{R})}$

The Mantle

The *mantle* $\mathbb{M}$ is the intersection of all grounds



"$V = \mathbb{M}$"

Theorem (Usuba) The mantle is a model of ZFC.

The Maximality Principle

Say φ holds *on a cone of forcing extensions* if there is a poset $\mathbb{P}$ s.t. $V^{\mathbb{P}}$ satisfies "φ holds in *all* forcing extensions."

Maximality Principle (Stavi-Väänänen, Bagaria, Chalons-Hamkins). If φ holds on a cone, then φ is true.

Consistency. Cone theory of any model is consistent and extends ZFC + MP

Boldface Maximality Principles

<u>Boldface MP</u>. For all $x \in \mathbb{R}$, if $\varphi(x)$ holds on a cone of ~~forcing~~ extensions, $\varphi(x)$ is true.

- Equiconsistent with ZFC + Ord is Mahlo
- Why real parameters? For any x on a cone, x is hereditarily countable

Necessary Maximality

<u>Necessary MP</u>. Boldface MP holds in every forcing extension.

- $\mathrm{Con}(AD) \leq \mathrm{Con}(NMP) \leq \mathrm{Con}(AD_{\mathbb{R}} + \Theta \text{ regular})$ [Woodin, unpublished]
- Implies: for all x, if $\varphi(x)$ holds on a cone then $\varphi(x)$ holds in any extension where x is countable.

Necessary maximality and the mantle

Theorem (Hamkins). Assuming the Necessary Maximality Principle, for all uncountable cardinals λ, $V_\lambda^{\mathbb{M}} \prec \mathbb{M}$.

$\Longrightarrow$ Every regular λ is inaccessible in $\mathbb{M}$

So the Necessary Maximality Principle implies ... $\neg$GA

Usuba's Theorem.

(For all γ, there is $\pi : V_\gamma \to V_{\gamma'}$)

$$\boxed{\text{Theorem (Usuba). Suppose there is an extendible cardinal. Then the mantle is a ground of } V.}$$

$\Longrightarrow$ the mantle is a model of GA.

$\Longrightarrow$ the mantle computes suff large successor cardinals.

$\Longrightarrow$ NMP is <u>false</u>.

NMP(Π_2) is a theorem

woodin showed given a proper class of Woodins

Back to the Ground Axiom

Is the Ground Axiom true?

Usuba's theorem says it is <u>almost</u> true:

— V is <u>almost</u> $\mathbb{M}$

— <u>There is at most a</u> <u>set</u> of grounds.

$$\begin{tikzcd}
\mathbb{M} \arrow[rr, bend left, "o \subseteq \mathbb{B}"] \arrow[dr] & & V = \mathbb{M}[G] \\
& N \arrow[ur] &
\end{tikzcd}$$

$N = \mathbb{M}[H]$ where $H = G \cap \mathbb{A}$

$\mathbb{A} \subseteq_c \mathbb{B}$

Part II

The axiom $V = HOD$

A set is ordinal definable if it is

definable from ordinal parameters.

V = HOD (Gödel) Every set is ordinal definable

HOD denotes the largest transitive class of ordinal definable sets.

- HOD is an inner model of ZFC &
- V = HOD means every set is in HOD.

V = HOD as a limitative principle

Cabal seminar mysticism: the reals are too wild to admit any definable wellorder

In fact, every definable set of reals is determined

Definable determinacy Any set of reals that is first-order definable is determined.

Necessary definable determinacy. Definable determinacy holds in all collapse extensions.

HOD dichotomy

Theorem (Woodin) If κ is extendible, either

① V is close to HOD
- If $A \subseteq \mathrm{HOD}$, there is $B \in \mathrm{HOD}$ w $A \subseteq B$ s.t. $|B| \leq |A| + \kappa$

$\Longrightarrow$ if $\lambda \geq \kappa$ is singular, $(\lambda^+)^{\mathrm{HOD}} = \lambda^+$

② V is far from HOD
- Every regular cardinal $\geq \kappa$ is mble $\to$ inaccessible in HOD

L Dichotomy Theorem

Theorem (Jensen). Either

① V is close to L.
- for all sets $A \subseteq L$, there is $B \in L$ with $A \subseteq B$ and $|B| \leq |A| + \aleph_1$

$\Longrightarrow$ for singular λ, $(\lambda^+)^L = \lambda^+$.

② V is far from L.
- every cardinal is inaccessible in L.

Theorem (Silver) If there is a mble, then ② holds.

HOD dichotomy vs. L dichotomy

HOD dichotomy vs L dichotomy

Large cardinal axioms <u>don't</u> imply V is far from HOD.

<u>HOD conjecture</u>. Large cardinals imply V is close to HOD.

<u>Proposition</u>. Necessary Definable Determinacy implies every regular cardinal is mbl in HOD.

$\Longrightarrow$ Either NDD or HOD conj is false

<u>Analogy</u>.

The mantle	HOD
Ground Axiom	$V = \mathrm{HOD}$
NMP $\Rightarrow$ every regular $\mathbb{M}$-inaccessible	NDD $\Rightarrow$ every regular HOD-inaccessible
Usuba's theorem	HOD conjecture

Generalized Lipschitz functions

- $f : P(\delta) \longrightarrow P(\delta)$ is <u>Lipschitz</u> if for all $A, B \subseteq \delta$ and $\alpha < \delta$, $A \cap \alpha = B \cap \alpha \implies f(A) \cap \alpha = f(B) \cap \alpha$
- If $X, Y \subseteq P(\delta)$, define long game $G_L(X, Y)$ where two players produce $x, y \in P(\delta)$ bit-by-bit and II wins if $x \in X \iff y \in Y$.
- X Lipschitz reduces to Y if II wins $G_L(X, Y)$

Countably complete ultrafilters.

$B_\kappa(X) = \kappa$-complete ufs on X.

<u>Ultrafilter Determinacy</u>: For all δ, $B_{\omega_1}(\delta)$ is linearly ordered by L-reducibility.

?

Lipschitz $f : P(\delta) \longrightarrow P(\delta)$ is <u>Ketonen</u> if for all $W \in B_{\omega_1}(\delta)$, $f^{-1}[W] \in B_{\omega_1}(\delta)$.

<u>Ultrapower Axiom</u>: $B_{\omega_1}(\delta)$ is linearly ordered by Ketonen reducibility

UF Determinacy vs. Necessary Definable Determinacy

Theorem. If there is a supercompact cardinal and Ultrafilter Determinacy holds, then V is a generic extension of HOD.

$\Rightarrow$ UF Determinacy & NDD... are incompatible

Sketch ① UF determinacy $\Rightarrow$ supercompactness measures are wellordered by the Mitchell order

② If enough supercompactness measures are OD, V is a generic extension of HOD.

Completely definable sets.

Def. A set x is completely definable if for all cardinals κ, x is definable from a κ-complete u.f. on an ordinal.

— Intuition: u.f.s are idealized ordinals

HCD is the largest transitive class of completely definable sets.

The size of HOD

Remark: consistently... $HOD \neq V$

Theorem. If there is an extendible, then V is a generic extension of HOD.

Theorem Assume the Ground Axiom + a proper class of strongly compacts. Then $V = HOD$.

Corollary Assume GA + u.f determinacy. + supercompact. Then $V = HOD$.

Elementary embeddings on HOD

Theorem (Woodin) Suppose $V = HOD$. Then for any $j_0, j_1 : V \longrightarrow M$, $j_0 = j_1$

Q (Woodin) Suppose $j_0, j_1 : V \longrightarrow M$. Does $j_0 \restriction HOD = j_1 \restriction HOD$?

- This says HOD somehow resembles the core model.
- Follows from HOD conjecture [Woodin, unpublished]

<u>Uniqueness of elementary embeddings</u>

$j_0, j_1 : V_\gamma \longrightarrow M$ are δ-similar if $j_0(\delta) = j_1(\delta)$ and $\lim\limits_{\alpha \to \delta} j_0(\alpha) = \lim\limits_{\alpha \to \delta} j_1(\delta)$.

<u>Theorem</u>. If κ is extendible, TFAE:

1. For all regular δ, all suff large $\gamma > \delta$, and all δ-similar $j_0, j_1 : V_\gamma \to M$, $j_0 \restriction V_\delta^{HOD} = j_1 \restriction V_\delta^{HOD}$
2. V is close to HOD.

<u>Global uniqueness theorem</u>

The theorem on δ-similar embeddings is compelling evidence for HOD Conjecture given a positive answer to Woodin's qst:

<u>Theorem</u>. If $j_0, j_1 : V \longrightarrow M$ are elementary, $j_0 \restriction HOD = j_1 \restriction HOD$.

$$\begin{tikzcd} V \arrow[r, "j_0"] \arrow[rr, bend left, "j_0"] & H \arrow[r, "k"] & M \end{tikzcd}$$

$H \subseteq H$

π_1 j_1

More uniqueness theorems

Theorem. Suppose κ is extendible and $j_0, j_1 : V \longrightarrow M$ are elementary w/ critical points above κ.

Theorem. Assume Ground Axiom holds and there is a proper class of strong cardinals. Then for any inner model M,

Is $V = \text{HOD}$ like $V = L$?

Theorem (Kunen) V is far from L iff there is an elementary $j : L \longrightarrow L$.

Theorem. Assume there is an extendible cardinal.

Q. Assume there is an extendible and V is far from HOD. For suff large

$\delta, \ldots$

Reinhardt's principle

Reinhardt proposed: there is a nontrivial elementary $j : V \longrightarrow V$.

Theorem (Kunen). ① Reinhardt's principle

② There is no nontrivial

Proof of ①.

$\heartsuit$.

Large cardinals beyond choice.

Theorem (Schlutzenberg). The following are equiconsistent:

① $ZF + \lambda\text{-}DC +$ an elementary $j : V_{\lambda+2} \longrightarrow V_{\lambda+2}$

②

Beyond $j : V_{\lambda+2} \longrightarrow V_{\lambda+2}$ lies a new consistency hierarchy "beyond AC"

Failure of HOD conjecture

Theorem (Woodin) The consistency of "ZF + a proper class of extendibles + $j : V \to V$" implies that of

So either the choiceless cardinals...
OR the HOD conjecture...

Evidence of consistency.

Theorem Assume there is $j : V \longrightarrow V$.
For a club class of κ:

1. κ or κ^+ is measurable
2. For regular $\delta \geq \kappa$, the club filter on δ is κ-complete and atomic

—

3. Every κ-complete filter...

Every ...

Back to HOD.

Observation. Assume $V_\gamma \prec_{\Sigma_2} V$ and there exist δ-similar $j_0, j_1 : V_\gamma \longrightarrow M$ s.t. $j_0[\delta] \subsetneq j_1[\delta]$.

If one could get the reversal (HOD small $\Longrightarrow$ $j : V_\delta^{HOD} \longrightarrow V_\delta^{HOD}$), one would have a strong argument against the HOD conjecture.