The Jackson analysis and the strongest hypotheses

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Outline

- Jackson's analysis of projective sets does not extend to $L(\mathbb{R})$.
  - Missing a "global" theory of ultrafilters on ordinals.
  - The Ultrapower Axiom serves this purpose in the inner models.
  - **Question:** Does $L(\mathbb{R})$ satisfy the Ultrapower Axiom?
- Analogy between determinacy and strongest large cardinals:
  - Example: $L(\mathbb{R})$ under $\text{AD}^{L(\mathbb{R})}$ vs. $L(V_{\lambda+1})$ under the axiom I_0.
  - Mainly one understands $L(V_{\lambda+1})$ by analogy with $L(\mathbb{R})$.
    - Many properties of $L(\mathbb{R})$ (e.g., the perfect set property) generalize to $L(V_{\lambda+1})$ with completely different proofs.
- **What's new:** the analogy also makes predictions about $L(\mathbb{R})$.
  - Some of these predictions can be verified.
  - One prediction is: $L(\mathbb{R})$ satisfies the Ultrapower Axiom.
    - Some consequences of UA can be shown to hold in $L(\mathbb{R})$.

Background: a definable invariant of the continuum

- A structure $\mathcal{N}$ is *interpretable* in a structure $\mathcal{M}$ if there is a surjection $f : \mathcal{M}^k \to \mathcal{N}$ such that the f-preimage of a definable subset of $\mathcal{N}$ is definable over $\mathcal{M}$.
- δ^1_ω denotes the minimum ordinal that is not interpretable in $(\mathbb{R}, \mathbb{N}, +, \times)$; i.e., the sup of the definable prewellorders.

  Note that $\omega_1 \leq |\delta^1_\omega| \leq \mathfrak{c}$.

Blanket assumption: The Axiom of Determinacy holds in $L(\mathbb{R})$.

Theorem (Jackson)

In $L(\mathbb{R})$, $\delta^1_\omega = \aleph_{\epsilon_0}$.

Here ϵ_0 is the least ordinal α such that $\omega^\alpha = \alpha$.

Therefore in actuality, $\delta^1_\omega = (\aleph_{\epsilon_0})^{L(\mathbb{R})}$.

Background: ultrafilters under AD

The proof of Jackson's theorem requires a detailed analysis of the intricate cardinal structure of $L(\mathbb{R})$ below δ^1_ω. This ultimately reduces to a classification of the ultrafilters on ordinals below δ^1_ω.

Theorem

In $L(\mathbb{R})$, the following hold:

- *Every ultrafilter is countably complete.*
- (Solovay) $\aleph_1$ *is measurable.*
  - *The club filter is the unique normal ultrafilter on $\aleph_1$.*
- (Martin) $\aleph_2$ *is measurable.*
  - *The ω-club filter and the ω_1-club filter are the only normal ultrafilters on $\aleph_2$.*
- (Solovay) $\aleph_n$ *is singular for $3 \leq n \leq \omega$.*
- (Kunen, Martin) $\aleph_{\omega+1}$ *and* $\aleph_{\omega+2}$ *are measurable.*
- (Jackson) $\aleph_{\omega\cdot 2+1}$ *is measurable, but* $\aleph_{\omega\cdot 3+1}$ *is singular.*

Ultrafilters in inner models

Two ultrafilters U and W are *equivalent* if there exist $A \in U$ and $B \in W$ such that $(A, U \cap P(A)) \cong (B, W \cap P(B))$.

Theorem (??)

In $L(\mathbb{R})$, every ultrafilter on $\aleph_1$ is equivalent to an iterated product of the closed unbounded filter.

This calls to mind:

Theorem (Kunen)

If U is a normal ultrafilter, then in the inner model $L[U]$, every countably complete ultrafilter is isomorphic to an iterated product of $U \cap L[U]$.

Actually the first theorem can be proved using the second.

Ultrapower terminlogy

Suppose U is an ω_1-complete ultrafilter

- M_U denotes the ultrapower of the universe by U
- $j_U : V \to M_U$ denotes the associated elementary embedding

Since U is ω_1-complete, M_U is wellfounded. So without loss of generality, M_U is transitive.

If P and Q are transitive models of ZFC, $j : P \to Q$ is an *ultrapower embedding* if there is some $U \in P$ such that $Q = (M_U)^P$ and $j = (j_U)^P$.

The Ultrapower Axiom (UA)

Ultrapower Axiom (UA)

For any ultrapower embeddings $j_0 : V \to M_0$ and $j_1 : V \to M_1$, there are ultrapower embeddings $i_0 : M_0 \to N$ and $i_1 : M_1 \to N$ such that $i_0 \circ j_0 = i_1 \circ j_1$.

$$\begin{tikzcd}
 & N & \\
M_0 \arrow[ur, "i_0"] & & M_1 \arrow[ul, "i_1"'] \\
 & V \arrow[ul, "j_0"] \arrow[ur, "j_1"'] &
\end{tikzcd}$$

Why UA?

- The Ultrapower Axiom is an instance of the central *Comparison Lemma* of inner model theory, yet it can be stated without reference to fine structure.
- As a consequence, UA holds in all known canonical inner models of ZFC, and arguably in any inner model built by anything like the current methodology.
  - If there is a canonical inner model with a supercompact cardinal, then UA should be consistent with a supercompact cardinal.
  - The existence of a supercompact cardinal implies the existence of a vast array of ultrapowers, and combined with UA, provides a rich structure theory for the upper reaches of the universe of sets.
- UA is equivalent to several natural combinatorial principles.
- Seems to yield an "optimal" theory of ω_1-complete ultrafilters (in the context of the Axiom of Choice).

The Rudin-Frolík order

- U lies below W in the *Rudin-Frolík order*, denoted $U \leq_{\mathrm{RF}} W$, if $j_W = k \circ j_U$ for some ultrapower embedding $k : M_U \to M_W$.
- By definition, UA holds iff the restriction of the Rudin-Frolík order to ω_1-complete ultrafilters is directed.
- A nonprincipal ultrafilter W is *irreducible* if any nonprincipal $U \leq_{\mathrm{RF}} W$ is equivalent to W (in that $j_U = j_W$).

Theorem (UA)

- *Every ω_1-complete ultrafilter W factors as an iteration:*

$$V = M_0 \xrightarrow{j_{U_0}} M_1 \xrightarrow{j_{U_1}} \cdots \xrightarrow{j_{U_n}} M_{n+1} = M_W$$

where for all $k \leq n$, U_k is an irreducible ultrafilter of M_k.

- *In fact, an ω_1-complete ultrafilter can have only finitely many Rudin-Frolík predecessors up to equivalence.*

Some irreducible ultrafilters

An ultrafilter U on a family of nonempty sets $\mathcal{F}$ is *normal* if every choice function on $\mathcal{F}$ is constant on a set in U. If U is normal and $\lambda = \min_{A\in U} |A|$, then M_U is closed under λ-sequences.

Proposition

Normal ultrafilters are irreducible.

A uniform ultrafilter U on a cardinal κ is *Dodd sound* if the map $i : P(\kappa) \to M_U$ given by $i(A) = j_U(A) \cap [\mathrm{id}]_U$ belongs to M_U.

Proposition

Dodd sound ultrafilters are irreducible.

Theorem (UA)

Normal ultrafilters and Dodd sound ultrafilters are wellordered by the Mitchell order.

Irreducible ultrafilters and UA

Theorem (UA)

Suppose U is an irreducible ultrafilter and $\lambda = \min_{A \in U} |A|$.

- *M_U is closed under λ-sequences unless λ is inaccessible.*
- *If λ is inaccessible, then $(M_U)^{<\lambda} \subseteq M_U$ and every $A \subseteq M_U$ with $|A| \leq \lambda$ is covered by a set $B \in M_U$ with $|B|^{M_U} \leq \lambda$.*

Remark. The inaccessible case obviously raises some questions...

Corollary (UA)

A cardinal is strongly compact if and only if it is supercompact or a measurable limit of supercompacts.

By a theorem of Menas, the least measurable limit of supercompact cardinals is strongly compact but not supercompact, so the corollary cannot be improved.

The Lipschitz order

Suppose δ is an ordinal.

- A function $f : P(\delta) \to P(\delta)$ is *Lipschitz* if for all $x, y \subseteq \delta$ and $\alpha \leq \delta$, if $x \cap \alpha = y \cap \alpha$, then $f(x) \cap \alpha = f(y) \cap \alpha$.
- For $A, B \subseteq P(\delta)$, set $A \leq_L B$ if A is *Lipschitz reducible* to B; i.e., there is a Lipschitz $f : P(\delta) \to P(\delta)$ with $f^{-1}[B] = A$.

Theorem

The following hold in $L(\mathbb{R})$:

- (Wadge) *The subsets of $P(\omega)$ — i.e., "sets of reals" — are* **semi-linearly ordered** *by Lipschitz reducibility: if $A, B \subseteq \mathbb{R}$, either A is reducible to B or B is reducible to $P(\omega) \setminus A$.*
- (Martin-Monk) $\leq_L$ *is wellfounded on subsets of $P(\omega)$.*

Ketonen reducibility

Let $\beta_\kappa(X)$ denote the set of κ-complete ultrafilters on X.

Theorem (UA)

For any ordinal δ, $(\beta_{\omega_1}(\delta), \leq_L)$ is a wellorder.

- A Lipschitz $f : P(\delta) \to P(\delta)$ is *Ketonen* if for all $W \in \beta_{\omega_1}(\delta)$, $f^{-1}[W] \in \beta_{\omega_1}(\delta)$.
- U is *Ketonen reducible* to $W \in \beta_{\omega_1}(\delta)$, denoted $U \leq_{\Bbbk} W$, if there is a Ketonen $f : P(\delta) \to P(\delta)$ with $U = f^{-1}[W]$.

Theorem

For all ordinals δ, $(\beta_{\omega_1}(\delta), \leq_{\Bbbk})$ is wellfounded.

Theorem

UA *holds if and only if for all ordinals δ, $(\beta_{\omega_1}(\delta), \leq_{\Bbbk})$ is a wellorder.*

Ordinal definable ultrafilters

The linearity of Ketonen reducibility immediately yields:

Theorem (UA)

Every ω_1-complete ultrafilter on an ordinal is ordinal definable.

By a strange coincidence, it is also possible to definably wellorder the ultrafilters of $L(\mathbb{R})$, although it is not clear whether Ketonen reducibility works:

Theorem (Kunen)

In $L(\mathbb{R})$, every ultrafilter on an ordinal is ordinal definable.

The Ultrapower Axiom without choice

Question

Does the Ultrapower Axiom hold in $L(\mathbb{R})$?

- To make sense of the question, one needs an ultrapower-free formulation of the Ultrapower Axiom.

  UA₁. For all ordinals δ, $(\beta_{\omega_1}(\delta), \leq_{\Bbbk})$ is a wellorder.

- Set $U \leq_{\mathrm{RF}} W$ if for some ultrafilters $(W_x)_{x\in X}$ with pairwise disjoint underlying sets, $W = \{\bigcup_{x\in X} A_x : \forall^U x\, A_x \in W_x\}$.

  $\mathbf{UA_2}$. The Rudin-Frolík order is directed on $\bigcup_{\delta\in\mathrm{Ord}} \beta_{\omega_1}(\delta)$.

Theorem (ZF + DC)

UA_1 *and* UA_2 *are equivalent.*

The Kunen inconsistency

- The best evidence that $L(\mathbb{R})$ satisfies UA comes from one of the strongest theories known to man.
- Reinhardt proposed the principle: there is a nontrivial elementary embedding from the universe of sets to itself.

Theorem (Kunen)

There is no elementary embedding from V to V except the identity.

Kunen's proof shows that for any ordinal α, there is no elementary $j : V_{\alpha+2} \to V_{\alpha+2}$. So if $j : V_\beta \to V_\beta$ is elementary with critical point κ, $\beta < \lambda + 2$ where

$$\lambda = \sup\{\kappa, j(\kappa), j^2(\kappa), j^3(\kappa), \dots\}$$

because $j(\lambda) = \sup\{j(\kappa), j^2(\kappa), j^3(\kappa), \dots\} = \lambda$.

Strategies and embeddings

The descriptive set theory of $V_{\lambda+1}$ assuming the existence of various embeddings $j : V_{\lambda+1} \to V_{\lambda+1}$ bears a striking and unexplained resemblance to classical descriptive set theory under determinacy axioms.

Determinacy	Elementary embeddings
$\mathrm{Det}(\Delta^1_1)$	$j : V_\lambda \to V_\lambda$
$\mathrm{Det}(\Pi^1_1)$	Σ_1-elementary $j : V_{\lambda+1} \to V_{\lambda+1}$
$\mathrm{Det}(\text{Projective})$	Σ_n-elementary $j : V_{\lambda+1} \to V_{\lambda+1}$ for all n
$\mathrm{Det}(L(\mathbb{R}))$	$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with $\mathrm{crit}(j) < \lambda$

The final embedding principle above is Woodin's axiom I_0.

Going forward: λ denotes an I_0-cardinal, meaning there is an elementary $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with $\mathrm{crit}(j) < \lambda$.

The inner model $L(V_{\lambda+1})$

Theorem

In $L(V_{\lambda+1})$, the following hold:

- (Woodin) λ^+ *is measurable.*
  - (Cramer) *There is a unique normal ultrafilter on λ^+ concentrating on ordinals of countable cofinality.*
- (Cramer) *Every subset of $V_{\lambda+1}$ has the perfect set property.*

$\Theta^{L(\mathbb{R})}$ denotes the least ordinal not of the form $\{f(x) : x \in \mathbb{R}\}$ where $f \in L(\mathbb{R})$. To get $\Theta^{L(V_{\lambda+1})}$, replace $\mathbb{R}$s with $V_{\lambda+1}$s.

Theorem

- (Moschovakis) $\Theta^{L(\mathbb{R})}$ *is weakly inaccessible in $L(\mathbb{R})$.*
- (Woodin) $\Theta^{L(V_{\lambda+1})}$ *is weakly inaccessible in $L(V_{\lambda+1})$.*

Small cardinals in $L(V_{\lambda+1})$

The local theory of $L(V_{\lambda+1})$ remains a mystery in basic ways; an analysis parallel to Jackson's seems completely out of reach.

Question

In $L(V_{\lambda+1})$, do the following hold?

- λ^{++} is measurable.
- λ^{+++} is singular.
- $\lambda^+ \to (\lambda^+)^\omega$.
- Any definable binary relation on $V_{\lambda+1}$ is uniformizable.
- Every subset of λ^+ is definable over $H(\lambda^+)$ from parameters.

The global theory of $L(V_{\lambda+1})$

Theorem (Kunen)

In $L(\mathbb{R})$, every ω_1-complete filter on an ordinal below $\Theta^{L(\mathbb{R})}$ extends to an ω_1-complete ultrafilter.

The proof uses that in $L(\mathbb{R})$ there is an ω_1-complete fine ultrafilter on $P_{\omega_1}(\mathbb{R})$ induced by the Martin measure on the Turing degrees. Although no analog of this is known for $L(V_{\lambda+1})$, one can prove:

Theorem

In $L(V_{\lambda+1})$, every λ^+-complete filter on an ordinal below $\Theta^{L(V_{\lambda+1})}$ extends to a λ^+-complete ultrafilter.

The proof is by induction on λ^+-complete filters ordered by Ketonen reducibility.

The global theory of $L(V_{\lambda+1})$, continued

An *atom* of a filter F is a set S such that $F \cup \{S\}$ generates an ultrafilter; F is *atomic* if every F-positive set contains an atom.

Theorem (Kechris-Kleinberg-Moschovakis-Woodin)

If κ is a strong partition cardinal, the club filter on κ is atomic.

Extending this to arbitrary regular cardinals in $L(\mathbb{R})$ is open, arguably a reasonable test question for Jackson's analysis.

Theorem

In $L(V_{\lambda+1})$, the club filter on any regular cardinal below $\Theta^{L(V_{\lambda+1})}$ is atomic.

The Ultrapower Axiom and $L(V_{\lambda+1})$

- The *α-th level* of a wellfounded partial order $\mathbb{P}$ is the set of all $x \in \mathbb{P}$ such that $\text{rank}_{\mathbb{P}}(x) = \alpha$.
- $\mathbb{P}$ is linear iff each level of $\mathbb{P}$ has cardinality 1.

Theorem

In $L(V_{\lambda+1})$, for all ordinals δ, $(\beta_{\omega_1}(\delta), \leq_{\mathbb{k}})$ is almost linear: each of its levels has cardinality less than λ.

- As a corollary, in $L(V_{\lambda+1})$, every ω_1-complete ultrafilter on an ordinal is *almost* ordinal definable in that it belongs to an ordinal definable set of cardinality less than λ.
- Ketonen reducibility is *not* linear in $L(V_{\lambda+1})$: e.g., the normal ultrafilter extending the ω-club filter is incomparable with any normal ultrafilter extending the ω_1-club filter.

Conjecture

In $L(\mathbb{R})$, for all ordinals δ, every level of $(\beta_{\omega_1}(\delta), \leq_{\mathbb{k}})$ is finite.

The Rudin-Keisler order and $L(V_{\lambda+1})$

The *Rudin-Keisler order* is defined on ultrafilters U and W on sets X and Y by setting $U \leq_{\mathrm{RK}} W$ if there is a partition $(Y_x)_{x\in X}$ of Y such that $U = \{B \subseteq X : \bigcup_{x\in B} Y_x \in W\}$.

Theorem

In $L(V_{\lambda+1})$, no ω_1-complete ultrafilter on an ordinal has λ-many Rudin-Keisler predecessors.

- The Rudin-Keisler order extends the Rudin-Frolík order.
- Recall: under UA, no ω_1-complete ultrafilter has infinitely many Rudin-Frolík predecessors.

Conjecture

In $L(\mathbb{R})$, no ultrafilter on an ordinal has infinitely many Rudin-Keisler predecessors.

From $L(V_{\lambda+1})$ to $L(\mathbb{R})$

Until now, our insight into $L(V_{\lambda+1})$ has come from knowledge of $L(\mathbb{R})$, never the other way.

Theorem

In $L(\mathbb{R})$, no ultrafilter on an ordinal has infinitely many Rudin-Frolík predecessors.

- An ultrafilter on a regular cardinal is *seminormal* if it extends the closed unbounded filter.
- The structure of seminormal ultrafilters is a central question in extending the Jackson analysis.

Theorem

In $L(\mathbb{R})$, no ultrafilter on an ordinal has has infinitely many seminormal Rudin-Keisler predecessors.

Proofs use Steel's fine-structural analysis of $\mathrm{HOD}^{L(\mathbb{R})}$ below $\Theta^{L(\mathbb{R})}$.

Products of ultrafilters

If U and W are ultrafilters on X and Y, there are at least three natural candidates for their product:

Cartesian product: $U \times W$ is the filter on $X \times Y$ generated by sets of the form $A \times B$ where $A \in U$ and $B \in W$.
Tensor product: for $C \subseteq X \times Y$,

$$C \in U \ltimes W \iff \forall^U x\, \forall^W y\, (x, y) \in C.$$

$$C \in U \rtimes W \iff \forall^W y\, \forall^U x\, (x, y) \in C.$$

- Note: $U \times W$ is contained in both $U \ltimes W$ and $U \rtimes W$.
- Usually, $U \times W$ is not an ultrafilter and $U \ltimes W \neq U \rtimes W$, so all three products are distinct.

Products of ultrafilters, continued

In certain very special cases, however, $U \times W$ is an ultrafilter.

Theorem (Blass)

If W is $|U|$-complete, $U \times W$ is an ultrafilter

Since $U \times W$ is contained in $U \ltimes W$ and $U \rtimes W$, if $U \times W$ is an ultrafilter (i.e., is maximal), then $U \ltimes W = U \times W = U \rtimes W$.

Question

Suppose $U \ltimes W = U \rtimes W$. Must $U \times W$ be an ultrafilter?

- $U \ltimes W = U \rtimes W$ iff the ultrafilter quantifiers commute:

$$\forall^U x \, \forall^W y \, R(x,y) \iff \forall^W y \, \forall^U x \, R(x,y)$$

Products and embeddings

From an elementary embeddings perspective:

- The ultrafilters Z extending $U \times W$ represent amalgamatioins

$$M_U \xrightarrow{k_U} M_Z \xleftarrow{k_W} M_W$$

such that $k_U \circ j_U = k_W \circ j_W$.

- The tensor products correspond to the amalgamations

$$M_U \xrightarrow{j_U(j_W)} M_{U \ltimes W} \xleftarrow{j_U \restriction M_W} M_W$$
$$M_U \xrightarrow{j_W \restriction M_U} M_{U \rtimes W} \xleftarrow{j_W(j_U)} M_W$$

- Quantifiers commute iff the associated ultrapowers do:

$$\begin{aligned} U \ltimes W = U \rtimes W &\iff j_U(j_W) = j_W \restriction M_U \\ &\iff j_W(j_U) = j_U \restriction M_W \end{aligned}$$

Products of ultrafilters, continued

Theorem (UA)

$U \times W$ is an ultrafilter iff $U \ltimes W = U \rtimes W$.

Since this is such a "combinatorial" statement, it feels like the theorem must be provable in ZFC.

Theorem (GCH)

$U \times W$ is an ultrafilter iff $U \ltimes W = U \rtimes W$.

In ZF, one can prove $U \times U$ is never an ultrafilter, whereas Elliot Glazer pointed out that in $L(\mathbb{R})$, there is an ultrafilter U such that $U \ltimes U = U \rtimes U$. So the equivalence fails in $L(\mathbb{R})$. Still...

Theorem

In $L(\mathbb{R})$, if U and W are ultrafilters **on ordinals**, *$U \times W$ is an ultrafilter iff $U \ltimes W = U \rtimes W$.*

Products of ultrafilters, continued

Theorem

In $L(\mathbb{R})$, if U and W are ultrafilters on ordinals, $U \times W$ is an ultrafilter iff $U \ltimes W = U \rtimes W$.

Proof.

- Fix U and W on δ with $U \ltimes W = U \rtimes W$.
  - Fix $A \subseteq \delta \times \delta$. Must show $U \times W$ measures A.
  - For some $x \in \mathbb{R}$, $A \in \text{HOD}_x$.
  - $\bar{U} = U \cap \text{HOD}_x$ and $\bar{W} = W \cap \text{HOD}_x$ are in HOD_x.
- In HOD_x:
  - $\bar{U} \ltimes \bar{W} = \bar{U} \rtimes \bar{W}$.
  - (Steel) GCH holds! UA holds!
  - By either of the previous theorems, $\bar{U} \times \bar{W}$ is an ultrafilter.
  - So $\bar{U} \times \bar{W}$ measures A.
- This implies $U \times W$ measures A. $\square$

Conclusion

- The analogy between $L(\mathbb{R})$ and $L(V_{\lambda+1})$ cuts both ways.
- There is evidence that UA holds in $L(\mathbb{R})$.

Conjecture

The following hold in $L(\mathbb{R})$:

- *The club filter on any regular cardinal $\delta < \Theta^{L(\mathbb{R})}$ is atomic.*
- *Every level of the Ketonen order is finite.*
- *No ultrafilter on an ordinal has infinitely many Rudin-Keisler predecessors.*

Thanks

Thanks!